Tarski monsters and infinite dimensional Teichmüller spaces

Katsuhiko Matsuzaki

Department of Mathematics, Ochanomizu University

§1. The action of isotropy subgroups of the modular groups on infinite dimensional Teichmüller spaces

For a compact Riemann surface R of genus greater than one, it is well known that the Teichmüller modular group (or mapping class group) $\mathrm{Mod}(R)$ acts on the finite dimensional Teichmüller space $T(R)$ isometrically and properly discontinuously. In more details, although $\mathrm{Mod}(R)$ has fixed points on $T(R)$, the isotropy subgroup $\mathrm{Stab}(p)$ at any $p \in R$ is a finite group. However, this is not always the case for non-compact Riemann surfaces such as R of infinite genus or of the infinite number of punctures, for which the Teichmüller space $T(R)$ is infinite dimensional. In this case, the orbit of a point in $T(R)$ under $\mathrm{Mod}(R)$ may be non-discrete and the isotropy subgroup $\mathrm{Stab}(p)$ may be infinite. In this section, we consider the action of isotropy subgroups more closely. Teichmüller spaces are always assumed to be infinite dimensional hereafter.

Let R be a Riemann surface and $\mathrm{Aut}(R)$ the group of all conformal automorphisms of R. The isotropy subgroup at the origin of the Teichmüller space $T(R)$ is identified with $\mathrm{Aut}(R)$. Let $B(R)$ be the complex Banach space of the holomorphic quadratic differentials φ on R with the hyperbolic L^∞-norm $\|\varphi\|$ finite. By the Bers embedding, the Teichmüller space $T(R)$ can be identified with a bounded contractible domain in $B(R)$. Then the action of $\mathrm{Aut}(R)$ on $T(R)$ is nothing but the restriction of the action on $B(R)$ to $T(R)$, which is defined by $\varphi \mapsto g^*(\varphi) := \varphi(g)\cdot(g')^2$ for $\varphi \in B(R)$ and $g \in \mathrm{Aut}(R)$. For a subgroup G of $\mathrm{Aut}(R)$, we set

$$B(R/G) = \{\varphi \in B(R) \mid g^*(\varphi) = \varphi \ for \ \forall g \in G\}.$$

This is a Banach subspace of $B(R)$, whose intersection with $T(R)$ corresponds to the Bers embedding of the Teichmüller space of the orbifold R/G.

For a subset X of $B(R)$, the limit set of X is defined as $L(X) := \overline{X} - X$. For a subgroup $G \subset \mathrm{Aut}(R)$ and a point $\varphi \in B(R)$, the orbit of φ under G is defined as

$$G(\varphi) := \{g^*(\varphi) \in B(R) \mid g \in G\}.$$

We say that the orbit $G(\varphi)$ is discrete if it has no accumulation points in $B(R)$.

Proposition. *Let G be a subgroup of* $\mathrm{Aut}(R)$ *and φ a point in $B(R)$. The orbit $G(\varphi)$ is discrete if and only if the limit set of the orbit $L(G(\varphi))$ is empty.*

Proof. If the orbit $G(\varphi)$ is discrete, then $G(\varphi)$ is closed and hence the limit set $L(G(\varphi))$ is empty. Conversely, suppose that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to some point in $B(R)$. We may assume that this point is φ itself by replacing g_n with $g_{n+1}^{-1} \circ g_n$. Moreover, for each point $g^*(\varphi)$ in $G(\varphi)$, a sequence $\{(g \circ g_n)^*(\varphi)\} \subset G(\varphi)$ converges to $g^*(\varphi)$. If $G(\varphi)$ is closed, then this implies that $G(\varphi)$ is a closed perfect set. In a complete metric space in general, every closed perfect set contains uncountably many points. However this contradicts the fact that $G(\varphi)$ is countable. Hence $G(\varphi)$ is not closed, that is, $L(G(\varphi))$ is not empty. □

We prove the following two results in this section. These are prototypes of our further investigation of the action of the modular groups on infinite dimensional Teichmüller spaces, which will be seen in the next section.

Theorem 1. *If φ belongs to the limit set $L(\cup B(R/G_n))$ for some infinite sequence of subgroups $\{G_n\}_{n=1}^{\infty}$ of $G = \mathrm{Aut}(R)$, then the orbit $G(\varphi)$ is not discrete. Such an orbit always exists whenever G contains an element of infinite order.*

Proof. Take a sequence $\{\varphi_n\}$ such that $\varphi_n \in B(R/G_n)$ and $\|\varphi_n - \varphi\| \to 0$. Take an element $g_n \in G_n$ for each n and consider a sequence $\{g_n^*(\varphi)\}$. Since $g_n^*(\varphi_n) = \varphi_n$, we have

$$\begin{aligned}\|g_n^*(\varphi) - \varphi\| &= \|g_n^*(\varphi) - g_n^*(\varphi_n)\| + \|\varphi_n - \varphi\| \\ &= 2\|\varphi_n - \varphi\| \to 0,\end{aligned}$$

which means that $g_n^*(\varphi)$ converge to φ. Here $g_n^*(\varphi) \neq \varphi$ for every n because φ does not belong to any $B(R/G_n)$. Hence the orbit $G(\varphi)$ is not discrete.

Next suppose that G contains an element g of infinite order and set $G_n = \langle g^{2^{(n-1)}} \rangle$. Consider the normal covering $R/G_{n+1} \to R/G_n$ for each n. Then $G_n/G_{n+1} \cong \mathbb{Z}_2$ acts on R/G_{n+1} as the covering transformation group and thus acts on $B(R/G_{n+1})$ with the fixed point set $B(R/G_n)$. Excluding a few exceptional surfaces which do not appear in our present case, we know that the action of the Teichmüller modular group is faithful. (This was first proved in [1]. Another proof was given in [2].) This implies that the containment $B(R/G_n) \subset B(R/G_{n+1})$ is proper. Therefore we have a strictly increasing sequence of closed subspaces

$$B(R/G_1) \subsetneqq B(R/G_2) \subsetneqq \cdots \subsetneqq B(R/G_n) \subsetneqq \cdots \subset B(R).$$

Then $L(\cup B(R/G_n))$ is not empty by the Baire category theorem. □

Theorem 2. *Suppose that the orders of the elements of $G = \mathrm{Aut}(R)$ is uniformly bounded. If φ does not belong to the limit set $L(\cup B(R/G_n))$ for any infinite sequence of subgroups $\{G_n\}_{n=1}^{\infty}$ of G, then $G(\varphi)$ is discrete.*

Proof. Assume that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to φ as in the proof of Proposition. Also we may assume that none of $\{g_n\}$ fixes φ. For $G_n = \langle g_n \rangle$, this means that φ does not belong to $\cup B(R/G_n)$. Let $k(n)$ be the order of g_n. The average of the orbit of φ under G_n is defined as

$$P_{G_n}(\varphi) := \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} (g_n^i)^*(\varphi).$$

Then $\psi_n = P_{G_n}(\varphi)$ satisfies $g_n^*(\psi_n) = \psi_n$, which means that $\psi_n \in B(R/G_n)$.

We prove that ψ_n converge to φ. The difference is estimated by

$$\begin{aligned} \|\psi_n - \varphi\| &\leq \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \|(g_n^i)^*(\varphi) - \varphi\| \\ &\leq \frac{\sum_{i=0}^{k(n)-1} i}{k(n)} \|(g_n)^*(\varphi) - \varphi\| \\ &= \frac{k(n)-1}{2} \|(g_n)^*(\varphi) - \varphi\|. \end{aligned}$$

Since $(g_n)^*(\varphi)$ converge to φ and since $k(n)$ is uniformly bounded, we see that this converges to 0 as $n \to \infty$. This implies that φ belongs to $L(\cup B(R/G_n))$. $\square$

Concrete examples of the point φ for which the orbit $G(\varphi)$ is not discrete was given in [3]. Theorem 1 asserts that such points always exist if G has an element of infinite order.

An infinite group the orders of whose elements are bounded is known to exist as a counterexample to the Burnside problem in the group theory. Hence, due to the uniformization theorem, we can see that there exists a Riemann surface R such that $G = \mathrm{Aut}(R)$ satisfies the assumption of Theorem 2.

The remaining case where the orders of the elements of G are finite but not bounded seems more difficult to treat.

Remark. In the proof of Theorem 1, we have used the fact that if a holomorphic normal covering of non-exceptional Riemann surfaces $R \to R'$ is not trivial, then the containment $B(R) \supset B(R')$ is proper. In [4], this result is extended to any covering $R \to R'$, not necessarily normal.

§2. The Fréchet axiom for the moduli spaces of Riemann surfaces of infinite type

Weakening proper discontinuity, we make a new criterion for the action of $\mathrm{Mod}(R)$ to be discontinuous, which should be suitable for infinite dimensional cases:

Definition. We say that the Teichmüller modular group $\mathrm{Mod}(R)$ acts on $T(R)$ *weakly discontinuously* if, for any point $p \in T(R)$, there exists an open ball U centered at p such that U is equivariant under the isotropy group $\mathrm{Stab}(p)$, meaning that $g(U) = U$ for any $g \in \mathrm{Stab}(p)$ and $g(U) \cap U = \emptyset$ for any $g \in \mathrm{Mod}(R) - \mathrm{Stab}(p)$.

We consider moduli spaces of non-compact Riemann surfaces. No matter how the action of $\mathrm{Mod}(R)$ is far from proper discontinuity, the moduli space is a topological space by the quotient topology induced by the projection

$$\pi : T(R) \to M(R) = T(R)/\mathrm{Mod}(R).$$

Moreover a pseudo-metric d_M on $M(R)$ is always defined as

$$d_M(\pi(p), \pi(q)) = \inf\{d_T(g(p), q) \mid g \in \mathrm{Mod}(R)\}.$$

We investigate necessary and sufficient conditions on the action of $\mathrm{Mod}(R)$ under which the moduli space $M(R)$ becomes a complete metric space with the metric d_M, like moduli spaces of compact Riemann surfaces do. However, if we deal with non-compact Riemann surfaces in general, we may encounter a situation that a neighborhood of $p \in T(R)$ splits into two directions according to whether the action of $\mathrm{Mod}(R)$ is discontinuous or not, and hence d_M becomes a metric even if $\mathrm{Mod}(R)$ acts properly discontinuously nowhere on $T(R)$. In order to avoid that and make our situation simpler, we need to impose a certain geometric assumption on the base Riemann surface R.

A hyperbolic Riemann surfaces R is of *bounded geometry* if the injectivity radius at any point of R is uniformly bounded away from zero except in cusp neighborhoods and if there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the homotopy classes of simple closed curves in R^* carry the fundamental group of R.

We are ready to state our result in the following way:

Theorem 3. *Let $T(R)$ be the Teichmüller space of a Riemann surface R of bounded geometry,* $\mathrm{Mod}(R)$ *the Teichmüller modular group and $M(R) = T(R)/\mathrm{Mod}(R)$ the moduli space. Then the following conditions are equivalent:*

(1) $\mathrm{Mod}(R)$ *acts on $T(R)$ weakly discontinuously;*
(2) *the orbit of any point $p \in T(R)$ under* $\mathrm{Mod}(R)$ *is a discrete set in $T(R)$;*
(3) *the pseudo-metric d_M on $M(R)$ is a metric;*
(4) *$M(R)$ satisfies the first (Fréchet) separation axiom.*

However, as we can see in the next theorem, the difference between the two discontinuities appears in a very restricted case.

Theorem 4. *If* $\mathrm{Mod}(R)$ *acts on $T(R)$ weakly discontinuously but not properly discontinuously, then in an isotropy group* $\mathrm{Stab}(p)$*, there exists a finitely generated infinite group T whose proper subgroups are all finite.*

A group T that has the property stated in Theorem 4 is called *Tarski monster.*

References

1. C. Earle, F. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence* (preprint).
2. A. Epstein, *Effectiveness of Teichmüller modular groups*, In the tradition of Ahlfors and Bers, Contemporary Math. 256, AMS, 2000, pp. 69–74.
3. E. Fujikawa, H. Shiga and M. Taniguchi, *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan (to appear).
4. K. Matsuzaki, *Isomorphisms between the Bers embeddings of infinite dimensional Teichmüller spaces* (preprint).

Ochanomizu University, Otsuka 2-1-1, Bunkyo-ku, Tokyo 112-8610, JAPAN
E-mail address: matsuzak@math.ocha.ac.jp