CONJUGATION OF A GROUP OF SYMMETRIC HOMEOMORPHISMS OF THE CIRCLE

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0. Introduction

Arguments in this note are based on the following result. Terminology used in its statement will be defined later.

Theorem 1. *If a uniformly quasisymmetric group G contained in* $\mathrm{Sym}^2(\mathbb{S}) \subset \mathrm{QS}(\mathbb{S})$ *satisfies a uniform boundedness condition $c_2(g) \leq c$ for all $g \in G$ and for some $c > 0$, then there exists $f \in \mathrm{Sym}^2(\mathbb{S})$ such that $f^{-1}Gf \subset \mathrm{M\ddot{o}b}(\mathbb{S})$.*

Note that $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ for $\alpha > 1/2$ is contained in $\mathrm{Sym}^2(\mathbb{S})$. We will see how the uniform boundedness condition for $\mathrm{Sym}^2(\mathbb{S})$ is given in terms of constants defined for $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$. Then, in the case where $G \subset \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$, we consider whether the conjugating map $f \in \mathrm{Sym}^2(\mathbb{S})$ belongs to $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$.

Here are some known results related to Theorem 1.

Theorem 2 (Markovic [13]). *A uniformly quasisymmetric group $G \subset \mathrm{QS}(\mathbb{S})$ is conjugate into $\mathrm{M\ddot{o}b}(\mathbb{S})$ by a quasisymmetric automorphism $f \in \mathrm{QS}(\mathbb{S})$.*

Theorem 3 (Navas [16]). *If $G \subset \mathrm{Diff}_+^3(\mathbb{S})$ satisfies a certain uniform boundedness condition, then G is conjugate into $\mathrm{M\ddot{o}b}(\mathbb{S})$ by $f \in \mathrm{Diff}_+^3(\mathbb{S})$.*

Theorem 4 ([14]). *A uniformly symmetric group $G \subset \mathrm{Sym}(\mathbb{S})$ is not necessarily conjugate into $\mathrm{M\ddot{o}b}(\mathbb{S})$ by any symmetric automorphism $f \in \mathrm{Sym}(\mathbb{S})$.*

The groups $\mathrm{QS}(\mathbb{S})$ and $\mathrm{Sym}(\mathbb{S})$ are defined as follows.

Definition. The *quasisymmetric quotient* of $g \in \mathrm{Homeo}_+(\mathbb{S})$ is defined to be

$$m_g(x,t) := \frac{g(x+t) - g(x)}{g(x) - g(x-t)}$$

for $x, t \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. If $1/M \leq m_g(x,t) \leq M$ for some $M \geq 1$, then g is *quasisymmetric.* Moreover, if

$$\epsilon_g(t) := \sup_{x \in \mathbb{S}} |m_g(x,t) - 1| \to 0 \quad (t \to 0),$$

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then g is called *symmetric*. We denote the groups of quasisymmetric and symmetric automorphisms of $\mathbb{S}$ by

$$\begin{aligned} \mathrm{QS}(\mathbb{S}) &= \{g \in \mathrm{Homeo}_+(\mathbb{S}) \mid g : \text{quasisymmetric}\}; \\ \mathrm{Sym}(\mathbb{S}) &= \{g \in \mathrm{Homeo}_+(\mathbb{S}) \mid g : \text{symmetric}\}. \end{aligned}$$

A subgroup $G \subset \mathrm{QS}(\mathbb{S})$ is called *uniformly quasisymmetric* if there is $M \geq 1$ such that $1/M \leq m_g(x,t) \leq M$ for every $g \in G$. In addition, if G is contained in $\mathrm{Sym}(\mathbb{S})$, G is called *uniformly symmetric*.

To explain related problems around Theorem 1, this note is composed by the following topics, each of which is discussed in a separate section.

(1) Relation between diffeomorphisms of the circle $\mathbb{S}$ and asymptotically conformal automorphisms of the disk $\mathbb{D}$:

$$\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}) \longleftrightarrow \mathrm{Bel}^\alpha(\mathbb{D})$$

(2) The isometric action of G on the universal Teichmüller space T. Existence of a fixed point of G on a suitable subspace T^2.

(3) Use of a subspace B^α in the Hilbert space A^2 of the square integrable holomorphic functions containing the Bers embedding of T^2 and limitation of the place of the fixed point:

$$\mathrm{Bel}^\alpha(\mathbb{D}) \longleftrightarrow B^\alpha, \quad T^2 \longleftrightarrow A^2$$

1. Diffeomorphisms of $\mathbb{S}$ and asymptotically conformal maps on $\mathbb{D}$

Definition. A Beltrami coefficient

$$\mu \in \mathrm{Bel}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\}$$

on the unit disk $\mathbb{D} = \{|z| < 1\}$ (or a quasiconformal homeomorphism w of $\mathbb{D}$ with its complex dilatation $\mu_w(z) = \bar\partial w(z)/\partial w(z)$ equal to $\mu(z)$) is called *asymptotically conformal* if

$$\kappa_\mu(t) = \kappa_w(t) := \sup_{1-t \leq |z| < 1} |\mu(z)| \to 0 \quad (t \to 0).$$

Then we set

$$\mathrm{Bel}_0(\mathbb{D}) = \{\mu \in \mathrm{Bel}(\mathbb{D}) \mid \mu : \text{ asymptotically conformal}\}.$$

The fundamental relation between differentiability on $\mathbb{S}$ and asymptotic conformality in $\mathbb{D}$ can be described as follows.

Theorem 5. *$g \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ $(0 < \alpha < 1)$ if and only if g extends to an asymptotically conformal automorphism $\tilde g : \mathbb{D} \to \mathbb{D}$ with $\kappa_{\tilde g}(t) = O(t^\alpha)$.*

This result originates from Carleson [7]. In fact, the "only if" part has been proved there. He showed that

$$g \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S}^1) \Longleftrightarrow \epsilon_g(t) = O(t^\alpha).$$

(The definition of $\epsilon_g(t)$ is in Section 0.) Then the (modified) Beurling-Ahlfors extension $\tilde{g} : \mathbb{D} \to \mathbb{D}$ of $g : \mathbb{S} \to \mathbb{S}$ satisfies $\kappa_{\tilde{g}}(t) = O(t^\alpha)$.

The "if" part of the statement of Theorem 5 will be proved by using theories of conformal mappings of $\mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$ that admit asymptotically conformal extension to $\mathbb{D}$. This is based on works by Pommerenke, Becker, Warschawski, Anderson, Lesley among others in references [1], [3], [4] and [17].

Remark. For the "if" part, only a weaker estimate was shown by Carleson [7] as follows:

$$\kappa_{\tilde{g}}(t) = O(t^\alpha) \quad \Longrightarrow \quad \epsilon_g(t) = O(t^{\alpha/2}) \quad \Longleftrightarrow \quad g \in \mathrm{Diff}_+^{1+\alpha/2}(\mathbb{S}).$$

As usual, we extend $\mu(z)$ to $\hat{\mathbb{C}}$ by setting $\mu(z) \equiv 0$ for $z \in \mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$ and take a quasiconformal automorphism $w = w^\mu : \mathbb{C} \to \mathbb{C}$ with $\mu_w = \mu$. Then $w^\mu|_{\mathbb{D}^*}$ is conformal (univalent). We consider

$$T_{w^\mu|_{\mathbb{D}^*}}(\zeta) = \frac{(w^\mu)''(\zeta)}{(w^\mu)'(\zeta)} \ : \ \text{pre-Schwarzian derivative}$$

$$S_{w^\mu|_{\mathbb{D}^*}}(\zeta) = \left(T_{w^\mu|_{\mathbb{D}^*}}\right)'(\zeta) - \frac{1}{2}\left\{T_{w^\mu|_{\mathbb{D}^*}}(\zeta)\right\}^2 \ : \ \text{Schwarzian derivative}$$

It is known ([4]) that the condition that μ is asymptotically conformal is equivalent to each of the following conditions:

$$\lim_{|\zeta|\to 1+0} \rho_{\mathbb{D}^*}^{-1}(\zeta)|T_{w^\mu|_{\mathbb{D}^*}}(\zeta)| = 0; \qquad \lim_{|\zeta|\to 1+0} \rho_{\mathbb{D}^*}^{-2}(\zeta)|S_{w^\mu|_{\mathbb{D}^*}}(\zeta)| = 0.$$

Here $\rho_{\mathbb{D}^*}(\zeta)$ denotes the hyperbolic density on $\mathbb{D}^*$.

To estimate this decay order quantitatively in terms of that for μ, the following functions have been introduced:

$$\beta_\mu(t) = \beta_w(t) = \sup_{1<|\zeta|\le 1+t} \rho_{\mathbb{D}^*}^{-1}(\zeta)|T_{w^\mu|_{\mathbb{D}^*}}(\zeta)|;$$

$$\sigma_\mu(t) = \sigma_w(t) = \sup_{1<|\zeta|\le 1+t} \rho_{\mathbb{D}^*}^{-2}(\zeta)|S_{w^\mu|_{\mathbb{D}^*}}(\zeta)|.$$

Improving an estimate given by Becker [3], we can obtain an expected result, which is the first step of our arguments.

Lemma 6. *The following conditions are equivalent:*

$$\kappa_\mu(t) = O(t^\alpha); \qquad \beta_\mu(t) = O(t^\alpha); \qquad \sigma_\mu(t) = O(t^\alpha).$$

The assumption $\kappa_\mu(t) = O(t^\alpha)$ of Theorem 5 implies $\beta_\mu(t) = O(t^\alpha)$ by Lemma 6. We represent g by conformal welding: there is a quasiconformal automorphism $\tilde{w}$ of $\hat{\mathbb{C}}$ that is conformal on $\mathbb{D}$ with $\tilde{w}(\mathbb{D}) = w^\mu(\mathbb{D})$ such that $g = \tilde{w}^{-1} \circ w^\mu$ on $\mathbb{S}$. By a quantitative version of the three point property of quasicircle due to Pommerenke and Warschawski [17], we see that $\tilde{\beta}(t)$ for the pre-Schwarzian derivative of $\tilde{w}|_{\mathbb{D}}(z)$ also satisfies $\tilde{\beta}(t) = O(t^\alpha)$.

The modulus of continuity

$$I(t; g') := \sup_{|x-y|\leq t} |g'(x) - g'(y)|.$$

of $g'(x)$ can be estimated in terms of $\beta_\mu(t)$ and $\tilde{\beta}(t)$ according to Anderson, Becker and Lesley [1]. We first see that g is continuously differentiable and

$$g'(x) = (w^\mu)'(x)/\tilde{w}'(g(x)).$$

Since $|g'(x)|$ is bounded by some L, $|g(x) - g(y)| \leq L|x - y|$ holds. Then, we have

$$\begin{aligned} \frac{1}{L} I(t; g') \leq I(t; \log g') &\leq I(t, \log(w^\mu)') + I(t, \log \tilde{w}' \circ g) \\ &\leq I(t, \log(w^\mu)') + I(Lt, \log \tilde{w}'). \end{aligned}$$

Since $(\log(w^\mu)')'(\zeta) = T_{w^\mu|_{\mathbb{D}^*}}(\zeta)$ and μ is asymptotically conformal, we see that

$$(\log(w^\mu)')'(x + t) = O(\beta_\mu(t)/t)$$

uniformly for all $x \in \mathbb{S}$. The same holds for $\tilde{w}$ and $\tilde{\beta}$. Hence $I(t; g') = O(t^\alpha)$, that is, $g \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$.

2. Fixed point theorem on the universal Teichmüller space

A quasiconformal automorphism of $\mathbb{D}$ extends to $\mathbb{S}$ as a quasisymmetric homeomorphism, and the converse is also true. We can define the universal Teichmüller space T as the space of quasisymmetric automorphism on $\mathbb{S}$ modulo post-composition of Möbius transformations:

$$T = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \mathrm{QS}(\mathbb{S}) = \mathrm{Bel}(\mathbb{D})/\sim.$$

The Teichmüller equivalence $\sim$ is just given by the coincidence of boundary values of quasiconformal automorphisms. The norm on $\mathrm{Bel}(\mathbb{D})$ provides a metric on T, which is called the Teichmüller metric.

The group $\mathrm{QS}(\mathbb{S})$ of all quasisymmetric automorphisms of $\mathbb{S}$ acts on $T = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \mathrm{QS}(\mathbb{S})$:

$$([f], g) \in T \times \mathrm{QS}(\mathbb{S}) \mapsto g_*[f] := [f \circ g^{-1}] \in T.$$

This can be regarded as the *mapping class group* of the universal Teichmüller space T. The action is faithful, transitive and isometric with respect to the Teichmüller metric on T.

We consider a uniformly quasisymmetric or symmetric group G as a subgroup of the mapping class group $\mathrm{QS}(\mathbb{S})$. The condition that $g \in G$ fixes $[f] \in T$, that is, $g_*[f] = [f]$, can be written as $[fgf^{-1}] = [\mathrm{id}]$, and this is equivalent to that $fgf^{-1} \in \mathrm{M\ddot{o}b}(\mathbb{S})$. This observation yields the following reformulation of the theorems mentioned in the introduction.

Corollary 7 (to Theorem 2)**.** *A uniformly quasisymmetric group G has a common fixed point $[f]$ in T.*

Corollary 8 (to Theorem 4)**.** *There is a uniformly symmetric subgroup $G \subset \mathrm{Sym}(\mathbb{S}) \subset \mathrm{QS}(\mathbb{S})$ such that G preserves the subspace*

$$T_0 = \text{Möb}(\mathbb{S})\backslash \mathrm{Sym}(\mathbb{S}) = \mathrm{Bel}_0(\mathbb{D})/\sim$$

of T invariant but has no fixed point in T_0.

Note that the condition that G is a *uniformly* quasisymmetric group implies that the orbit $G(o) = \{g_*(o)\}_{g\in G}$ of the origin $o = [\mathrm{id}] \in T$ is a *bounded* set. However, different from the situation discussed later, this does not automatically guarantee the existence of a fixed point of the isometric action of G on T. Nevertheless, Theorem 2 finds a fixed point by hand.

The proof of Theorem 3 (Navas) utilizes a unitary representation of $\mathrm{Diff}_+(\mathbb{S})$ in $L^2(\mathbb{S}\times\mathbb{S})$ and shows that the action of G on it has bounded orbit. Then the fixed point corresponds to the projective structure on $\mathbb{S}$ invariant under G.

Theorem 4 says that, even if G acts on T_0 isometrically with bounded orbit, it does not necessarily gives a fixed point on T_0. Our problem is to find a subspace of T_0 where the boundedness of the orbit implies the existence of a fixed point on it.

To this end, we define the space of square integrable symmetric automorphisms. Let

$$L^2(\mathbb{D},\rho) = \left\{\mu \mid \|\mu\|_2^2 := \int_{\mathbb{D}} |\mu(z)|^2\rho(z)^2|dz\wedge d\bar{z}| < \infty\right\},$$

where $\rho(z)$ denotes the hyperbolic density on $\mathbb{D}$.

Definition. A symmetric automorphism $g \in \mathrm{Sym}(\mathbb{S})$ is called a *square integrable symmetric automorphism* if there is an asymptotically conformal extension $\tilde{g} : \mathbb{D} \to \mathbb{D}$ such that $\mu_{\tilde{g}} \in L^2(\mathbb{D},\rho) \cap \mathrm{Bel}(\mathbb{D}) =: \mathrm{Ael}^2(\mathbb{D})$. A constant $c_2(g)$ is defined by the infimum of $\|\mu_{\tilde{g}}\|_2$ taken over all such extensions $\tilde{g} : \mathbb{D} \to \mathbb{D}$ of g.

All square integrable symmetric automorphisms of $\mathbb{S}$ constitutes a subgroup of $\mathrm{Sym}(\mathbb{S})$, which we denote by $\mathrm{Sym}^2(\mathbb{S})$. The corresponding subspace of $T_0 = \text{Möb}(\mathbb{S})\backslash \mathrm{Sym}(\mathbb{S})$ is

$$T^2 = \text{Möb}(\mathbb{S})\backslash \mathrm{Sym}^2(\mathbb{S}) = \mathrm{Ael}^2(\mathbb{D})/\sim .$$

Cui [8] introduced the *Weil-Petersson metric* d_{WP} on T^2; the inner product on the tangent space at the origin $o = [\mathrm{id}] \in T^2$ is defined by

$$\langle[\mu_1],[\mu_2]\rangle_{WP} = \int_{\mathbb{D}} h(\mu_1)(z)\overline{h(\mu_2)(z)}\rho(z)^2|dz\wedge d\bar{z}|$$

with the harmonic Beltrami differential $h(\mu)$ for an infinitesimal Teichmüller class $[\mu]$. For an arbitrary point $[\nu] \in T^2$, we use the base point change $R_{\nu^{-1}} : \mathrm{Bel}(\mathbb{D}) \to \mathrm{Bel}(\mathbb{D})$ that induces an automorphism of T^2 sending $[\nu]$ to the origin. The topology on T^2 induced by this metric is weaker than the Teichmüller topology. Later, Takhtajan and Teo [18] defined a (foliated) Hilbert manifold structure on T by using the subspace T^2.

Concerning this metric on T^2, the following properties are important for us.

Theorem 9 (Cui [8]). *T^2 is complete and contractible with respect to d_{WP}.*

Theorem 10 (Takhtajan-Teo [18]). *The Weil-Petersson metric on T^2 is negatively curved.*

We explain the Bruhat and Tits method of finding a fixed point for an isometric group action. The following formulation is in Ballmann [2]. An *Hadamard space* is a complete, simply connected, geodesic metric space that is locally a CAT(0) space. For example, the Euclidean space $\mathbb{E}$, the hyperbolic space $\mathbb{H}$ and any Hilbert space are Hadamard spaces. A *circumcenter* (Chebyshev center) of a bounded subset E in a metric space X is the center of the smallest ball that contains E. The existence and uniqueness of the circumcenter is guaranteed by the following result due to Serre. See [2] and [6].

Theorem 11. *A bounded subset of an Hadamard space has the unique circumcenter.*

Remark. There are other metric spaces besides Hadamard spaces that hold the existence and uniqueness of circumcenter for any bounded subset. For instance, uniformly convex Banach spaces such as L^p $(1 < p < \infty)$ belong to this category. See [12], [5] and [15].

Corollary 12 (to the above two theorems). *T^2 is an Hadamard space with respect to d_{WP}.*

Assume that a group G acts on an Hadamard space X isometrically having a bounded orbit $E = \{g(x)\}_{g \in G}$. Let x_0 be the unique circumcenter of E. Then $g(x_0)$ for any $g \in G$ is the circumcenter of $g(E)$. However, since $g(E) = E$, we have $g(x_0) = x_0$, that is, x_0 is a fixed point of G.

If $G \subset \mathrm{Sym}^2(\mathbb{S})$, it acts on T^2 isometrically. We will see in the next section that our group G as in Theorem 1 satisfies this condition. If the orbit $G(o)$ in T^2 is bounded, then its unique circumcenter $[f] \in T^2$ is fixed by G. Here we consider how to check the orbit is bounded. An estimate of the distance on T^2 is given as follows. From this, we have Theorem 1 in the introduction.

Lemma 13. *For any $\mu \in \mathrm{Ael}^2(\mathbb{D})$, the Weil-Petersson distance satisfies*

$$d_{WP}([0], [\mu]) \leq C \|\mu\|_2,$$

where C is a constant depending only on $\|\mu\|_\infty$.

The proof of Lemma 13 becomes simpler if we use a certain estimate of the Jacobian of the conformal barycentric extension (see Douady and Earle [9]) of a square integrable symmetric automorphism by Yanagishita [19].

3. Banach spaces of holomorphic quadratic differentials

We consider the following Banach spaces of holomorphic quadratic differentials, which are actually the spaces of holomorphic functions:

$$B = \{\varphi \in \mathrm{Hol}(\mathbb{D}^*) \mid \|\varphi\|_\infty = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| < \infty\};$$
$$B_0 = \{\varphi \in B \mid \lim_{|z|\to 1} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = 0\};$$
$$A^2 = \{\varphi \in B \mid \|\varphi\|_2^2 = \int_{\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)|^2|dz \wedge d\bar{z}| < \infty\};$$
$$B^\alpha = \{\varphi \in B \mid \|\varphi\|_{\infty,\alpha} = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2+\alpha}(z)|\varphi(z)| < \infty\} \quad (\alpha > 0).$$

There are inclusion relations $A^2 \subset B_0 \subset B$ and $B^\alpha \subset A^2$ if $\alpha > 1/2$. The corresponding spaces of Beltrami coefficients

$$\mathrm{Bel}(\mathbb{D}), \quad \mathrm{Bel}_0(\mathbb{D}), \quad \mathrm{Ael}^2(\mathbb{D})$$

have been defined; in addition,

$$\mathrm{Bel}^\alpha(\mathbb{D}) = \{\mu \in \mathrm{Bel}(\mathbb{D}) \mid \|\mu\|_{\infty,\alpha} = \mathrm{ess.\,sup}_{z\in\mathbb{D}}\, \rho^\alpha(z)|\mu(z)| < \infty\}.$$

The correspondence between these spaces are given as follows. Here the Teichmüller projection is denoted by

$$\Phi : \mathrm{Bel}(\mathbb{D}) \to T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS}(\mathbb{S}) = \mathrm{Bel}(\mathbb{D})/\sim$$

and the Bers embedding is denoted by

$$\beta : T \hookrightarrow B; \quad [\mu] \mapsto S_{w^\mu|_{\mathbb{D}^*}}.$$

- $\mathrm{Bel}(\mathbb{D}) \xrightarrow{\Phi} T \xrightarrow{\beta} B$
- $\mathrm{Bel}_0(\mathbb{D}) \to T_0 \hookrightarrow B_0$ (Gardiner and Sullivan [11])
- $\mathrm{Ael}^2(\mathbb{D}) \to T^2 \hookrightarrow A^2$ (Cui [8], Takhtajan and Teo [18])
- $\mathrm{Bel}^\alpha(\mathbb{D}) \to (*) \hookrightarrow B^\alpha$ (Lemma 6)

If a uniformly quasisymmetric group G is in $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ for $\alpha > 1/2$, then the extension $\tilde{g} : \mathbb{D} \to \mathbb{D}$ for every $g \in G$ satisfies $\kappa_{\tilde{g}}(t) = O(t^\alpha)$ by Theorem 5 (Section 1). Hence $\mu_{\tilde{g}} \in \mathrm{Bel}^\alpha(\mathbb{D})$. Similar to the above, $\mathrm{Bel}^\alpha(\mathbb{D}) \subset \mathrm{Ael}^2(\mathbb{D})$ if $\alpha > 1/2$.

By choosing the base point at this fixed point, we consider the Bers embedding $\beta : T^2 \to A^2$. Then G acts linear isometrically on A^2 through β. The original base point is moved to $[\mu_0]$ for some $\mu_0 \in \mathrm{Ael}^2(\mathbb{D})$.

For the moment, we assume that the following estimate is true:

$$(\#) \qquad \|\beta \circ \Phi(\nu * \mu_0) - \beta \circ \Phi(\mu_0)\|_{\infty,\alpha} \le C\|\nu\|_{\infty,\alpha}$$

for every $\nu \in \mathrm{Bel}^\alpha(\mathbb{D})$. Here $\nu * \mu$ is the Beltrami coefficient of $w^\nu \circ w^\mu$. Take $\nu = \mu_{\tilde{g}}$ for every $g \in G$. The orbit $G([\mu_0])$ is contained in $\beta \circ \Phi(\mu_0) + B^\alpha$, which is an affine subspace of A^2. We want to know whether the unique circumcenter of $G([\mu_0])$ in A^2 also lies in this affine subspace. Since the circumcenter is a fixed point of G and the fixed point of G in A^2 is unique, which is the origin 0, we have $\beta \circ \Phi(\mu_0) \in B^\alpha$. This implies that $\mu_0 \in \mathrm{Bel}^\alpha(\mathbb{D})$.

We exchange the rules of the fixed point [id] and the base point $[\mu_0]$. Using Theorem 5 in the other direction, we would have that $h \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ corresponding to $\mu_0 \in \mathrm{Bel}^\alpha(\mathbb{D})$ gives the conjugation of G into $\mathrm{Möb}(\mathbb{S})$.

More rigorous arguments proceed as follows. Actually, we see that the previous estimate (#) is true if $\mu_0 \in \mathrm{Bel}^{\alpha'}(\mathbb{D})$ for some $\alpha' > 0$. We choose $\varepsilon > 0$ so that $\alpha' = \alpha - \varepsilon > 1/2$. By the same argument but based on the following correct statement instead of (#), we first see that $\mu_0 \in \mathrm{Bel}^{\alpha'}(\mathbb{D})$.

Lemma 14. *For every $\mu \in \mathrm{Ael}^2(\mathbb{D})$ and every $\varepsilon > 0$, there is a constant $C = C(\alpha, \varepsilon, \mu_0) > 0$ such that*

$$\|\beta \circ \Phi(\nu * \mu) - \beta \circ \Phi(\mu)\|_{\infty,\alpha-\varepsilon} \leq C\|\nu\|_{\infty,\alpha}$$

for every $\nu \in \mathrm{Bel}^\alpha(\mathbb{D})$.

Remark. Results similar to Lemma 14 have been obtained for T_0 in Earle, Markovic and Saric [10] and for T_2 in Takhtajan and Teo [18]. Our proof of Lemma 14 is in a similar line to that in [18] but several additional estimates are inserted from univalent function theory such as in [17].

4. Some extension

Finally, we state a certain generalization of Theorem 1 in the case where we do not put the limitation $\alpha > 1/2$ to the order α for $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}) \subset \mathrm{Sym}^p(\mathbb{S})$ with $\alpha p > 1$.

Theorem 15. *If a uniformly quasisymmetric group G contained in $\mathrm{Sym}^p(\mathbb{S}) \subset \mathrm{QS}(\mathbb{S})$ satisfies a uniform boundedness condition $c_2(g) \leq c$ for all $g \in G$ and for some sufficiently small $c > 0$, then there exists $f \in \mathrm{Sym}^p(\mathbb{S})$ such that $f^{-1}Gf \subset \mathrm{Möb}(\mathbb{S})$.*

For the proof, we utilize the Banach space A^p of p-integrable holomorphic quadratic differentials instead of A^2. We also introduce T^p, Sym^p and Ael^p for the corresponding Teichmüller spaces, the space of integrable symmetric automorphisms and the space of Beltrami coefficients. In the previous case, we find a fixed point in T^2 directly due to the fact that T^2 is an Hadamard space. However, in this generalized case, we find a fixed point in A^p and must show that it is in the image of the Bers embedding of T^p. At this point, we need the assumption that the uniform constant c is sufficiently small in Theorem 15.

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