

WEIL–PETERSSON CURVES AND DIRICHLET FINITE HARMONIC FUNCTIONS ON RIEMANN SURFACES

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ABSTRACT. On two subsurfaces of a Riemann surface divided by an essential p -Weil–Petersson curve, we consider the spaces of harmonic functions whose p -Dirichlet integrals are finite. By ensuring the coincidence of the boundary values on the curve, we establish the correspondence of harmonic functions in these spaces. We examine the operator between them, defined in this manner, through the space of p -Besov functions on the p -Weil–Petersson curve.

1. INTRODUCTION

Suppose a simple closed curve γ on a Riemann surface R divides it into two subsurfaces R^+ and R^- that are not simply connected. In this situation, we can consider the correspondence between harmonic functions on R^\pm with the same boundary values on γ , provided γ has reasonable regularity. In his recent work [6], Shiga studied this problem inspired by Schippers and Staubach [5], focusing on the case where γ is a quasicircle and harmonic functions have finite Dirichlet integrals. When γ is a quasicircle, however, the notion of coinciding boundary values on γ must be interpreted indirectly through conformal mappings, without essentially using a space of boundary functions on γ . In this paper, we consider this problem in a slightly different setting.

First, we assume better regularity for γ . A Weil–Petersson curve in the plane is a rectifiable Jordan curve that is the image of a special embedding of the unit circle. This embedding can be described using the theory of the universal Teichmüller space, particularly its subspace defined by integrable Beltrami coefficients (integrable Teichmüller space). Bishop [2] has comprehensively studied Weil–Petersson curves and explored their connections to other concepts in various mathematical fields. A Weil–Petersson curve can also be defined on a Riemann surface. Due to its analytic properties, we can consider a certain function space on γ , which allows us to describe the relationship between Dirichlet finite harmonic functions on the complementary domains R^\pm of γ on a Riemann surface R .

Second, we generalize the exponent for the finite Dirichlet integral from 2 to any $p > 1$. This generalization aligns with the extension of the integrability exponent of Beltrami coefficients for the integrable Teichmüller space. The original theory addresses the case $p = 2$, and a 2-Weil–Petersson curve relates to the 2-integrable Teichmüller space T_2 . By introducing the p -integrable Teichmüller space T_p for $p > 1$, we can define a p -Weil–Petersson curve, as detailed in [7]. For a p -Weil–Petersson curve γ , it is natural

to consider p -Dirichlet finite harmonic functions on R^\pm . In this setting, the space of boundary functions on γ corresponds to the homogeneous p -Besov space $B_p(\mathbb{S})$ on the circle, which is well-studied in real analysis, providing more scales $B_{p,q}^s$. Let $\text{HD}_p(R^\pm)$ denote the space of p -Dirichlet finite harmonic functions on R^\pm . Under the condition that $\text{HD}_p(R^\pm)$ is isomorphic to $B_p(\mathbb{S})$ as a complex Banach space, the transmission operator $\Theta_\gamma : \text{HD}_p(R^+) \rightarrow \text{HD}_p(R^-)$, corresponding to having the same boundary function, can be represented as an automorphism of $B_p(\mathbb{S})$.

Although we assume that γ is a Weil–Petersson curve, most results stated in this paper remain valid even if γ is a quasicircle, as studied by Shiga [6]. In this sense, our results generalize some of his theorems to p -Dirichlet finite harmonic functions by involving the p -Besov space of boundary functions. An exception is discussed in the last subsection, where we examine the variation of the transmission operators when deforming R by suitable quasiconformal homeomorphisms determined in the p -integrable Teichmüller space. Suppose a sequence of pairs (R_n, γ_n) converges to (R_0, γ_0) in the Teichmüller space. We represent the corresponding sequence of transmission operators $\Theta_{\gamma_n} : \text{HD}_p(R_n^+) \rightarrow \text{HD}_p(R_n^-)$ as a sequence of Banach automorphisms $\theta(R_n, \gamma_n)$ of $B_p(\mathbb{S})$. Our claim is that $\theta(R_n, \gamma_n)$ converges to $\theta(R_0, \gamma_0)$ strongly as $n \rightarrow \infty$.

2. WEIL–PETERSSON CURVES IN THE PLANE

In this section, we introduce p -integrable Teichmüller spaces as the parameter space for Weil–Petersson curves in the plane. Weil–Petersson embeddings of the unit circle \mathbb{S} are defined, and Weil–Petersson curves in the plane are the images of these embeddings. The results developed in this section, along with a portion of the next section, are explained in a survey article [4] and the references therein.

2.1. p -integrable Teichmüller space. For $p > 1$, the p -integrable Teichmüller space T_p is defined in the universal Teichmüller space T . We assume that several ingredients for the definition of T such as quasiconformal mappings and quasimetric homeomorphisms are well-known. Basic definitions and results can be found in a textbook [3]. In the sequel, the unit disk \mathbb{D} stands for either $\mathbb{D}^+ = \{|z| < 1\}$ or $\mathbb{D}^- = \{|z| > 1\} \cup \{\infty\}$.

Definition 1. For $p > 1$, a Beltrami coefficient $\mu \in M(\mathbb{D})$, a measurable function on the unit disk \mathbb{D} with $\|\mu\|_\infty < 1$, is p -integrable with respect to the hyperbolic metric $\rho_{\mathbb{D}}(z)|dz|$ if it satisfies

$$\|\mu\|_p = \left(\int_{\mathbb{D}} |\mu(z)|^p \rho_{\mathbb{D}}^2(z) dx dy \right)^{1/p} < \infty.$$

The set of all such p -integrable Beltrami coefficients is denoted by $M_p(\mathbb{D})$. The topology of $M_p(\mathbb{D})$ is defined by the norm $\|\mu\|_p + \|\mu\|_\infty$.

For $p > 1$, the p -integrable Teichmüller space T_p is the set of all normalized (fixing 1, i , and -1) quasimetric homeomorphisms $h : \mathbb{S} \rightarrow \mathbb{S}$ that can be extended to quasiconformal homeomorphisms $H : \mathbb{D} \rightarrow \mathbb{D}$ whose complex dilatations belong to $M_p(\mathbb{D})$. By this correspondence from Beltrami coefficients μ to quasimetric homeomorphisms

h_μ via quasiconformal ones H_μ , we have the Teichmüller projection $\pi : M_p(\mathbb{D}) \rightarrow T_p$. An element h_μ of T_p can be represented by a Teichmüller equivalence class $[\mu]$ for $\mu \in M_p(\mathbb{D})$. We provide the quotient topology induced by π for T_p .

Let F^μ denote the normalized quasiconformal homeomorphism of the extended complex plane $\widehat{\mathbb{C}}$ whose complex dilatation is $\mu \in M_p(\mathbb{D}^-)$ on \mathbb{D}^- and 0 on \mathbb{D}^+ . Usually, we consider the Schwarzian derivative of $F^\mu|_{\mathbb{D}^+}$, but here we take a different approach. The pre-Schwarzian derivative map $L : M_p(\mathbb{D}^-) \rightarrow \mathcal{B}_p(\mathbb{D}^+)$ is defined by the correspondence $\mu \mapsto \log(F^\mu|_{\mathbb{D}^+})'$, where $\mathcal{B}_p(\mathbb{D})$ is the space of holomorphic functions Φ on \mathbb{D} with

$$\|\Phi\|_{\mathcal{B}_p} = \left(\int_{\mathbb{D}} (|\Phi'(z)|\rho_{\mathbb{D}}^{-1}(z))^p \rho_{\mathbb{D}}(z) dx dy \right)^{1/p} < \infty.$$

This is called the *p-analytic Besov space*. See [8, Section 5]. By ignoring the difference in constant functions, $\mathcal{B}_p(\mathbb{D})$ is a complex Banach space with norm $\|\Phi\|_{\mathcal{B}_p}$.

The pre-Schwarzian derivative map L induces a well-defined injection $\beta : T_p \rightarrow \mathcal{B}_p(\mathbb{D})$ such that $\beta \circ \pi = L$. We call β the *pre-Bers embedding*. This provides the complex Banach structure for T_p .

Proposition 2.1. *The pre-Bers embedding $\beta : T_p \rightarrow \mathcal{B}_p(\mathbb{D})$ is a homeomorphism onto a contractible domain $\beta(T_p) = L(M_p(\mathbb{D}))$ in $\mathcal{B}_p(\mathbb{D})$.*

The integrable Teichmüller space T_p possesses the group structure as a subgroup of the universal Teichmüller space T , where T is the group of all normalized quasimetric homeomorphisms of \mathbb{S} . For $\mu_1, \mu_2 \in M(\mathbb{D})$ with $\pi(\mu_i) = h_i$ ($i = 1, 2$), the group operation in T is defined and denoted by $[\mu_1] * [\mu_2] = \pi(\mu_1 * \mu_2) = h_1 \circ h_2$. For every $[\nu] \in T_p$, the right translation $r_{[\nu]} : T_p \rightarrow T_p$ of the group elements of T_p is defined by $[\mu] \mapsto [\mu] * [\nu]$.

Proposition 2.2. *T_p is a topological group. Moreover, every right translation $r_{[\nu]}$ is a biholomorphic automorphism of T_p .*

2.2. Weil–Petersson curve. We regard Weil–Petersson curves as the images of certain embeddings $g : \mathbb{S} \rightarrow \widehat{\mathbb{C}}$, and we use Teichmüller spaces to coordinate these embeddings.

Definition 2. For $\mu^+ \in M_p(\mathbb{D}^+)$ and $\mu^- \in M_p(\mathbb{D}^-)$, the Weil-Petersson homeomorphism $G(\mu^+, \mu^-)$ is the normalized quasiconformal self-homeomorphism of $\widehat{\mathbb{C}}$ whose complex dilatation on \mathbb{D}^+ is μ^+ and on \mathbb{D}^- is μ^- . The restriction of $G(\mu^+, \mu^-)$ to \mathbb{S} is called a *p-Weil–Petersson embedding*. Its image is referred to as a *p-Weil–Petersson curve*.

The following claim guarantees that *p-Weil–Petersson embeddings* are parametrized by the product of Teichmüller spaces.

Proposition 2.3. *$G(\mu^+, \mu^-)|_{\mathbb{S}} = G(\nu^+, \nu^-)|_{\mathbb{S}}$ if and only if $[\mu^+] = [\nu^+]$ and $[\mu^-] = [\nu^-]$.*

Hence, a *p-Weil–Petersson embedding* $G(\mu^+, \mu^-)|_{\mathbb{S}}$ is determined by the pair of the Teichmüller equivalence classes, which is denoted by $g([\mu^+], [\mu^-])$. This is absolutely continuous, and its image Γ is rectifiable.

Let $T_p^+ = \pi(M_p(\mathbb{D}^+))$ and $T_p^- = \pi(M_p(\mathbb{D}^-))$. Then, $T_p^+ \times T_p^-$ becomes the coordinates of the space of p -Weil–Petersson embeddings. For the space of quasi-Fuchsian groups, this method is called *simultaneous uniformization*, which is originally due to Bers. The entire space of quasi-Fuchsian groups can be coordinated as a product of Teichmüller spaces. Similarly, the entire space of p -Weil–Petersson embeddings can be coordinated as $T_p^+ \times T_p^-$. We refer to this as *the Bers coordinates*.

3. DIRICHLET FINITE HARMONIC FUNCTIONS ON THE UNIT DISK

Harmonic functions on the unit disk with finite Dirichlet energy are well-studied. In this section, we generalize the exponent of the integrability of the derivative from 2 to any $p > 1$, and lay the foundation for this space.

3.1. Dirichlet integral and Besov functions. Let Φ be a complex-valued harmonic function on the unit disk \mathbb{D} . For $p > 1$, let

$$\|\Phi\|_{\text{HD}_p} = \left(\int_{\mathbb{D}} (\|\nabla\Phi(z)\|_{\rho_{\mathbb{D}}^{-1}})^p \rho_{\mathbb{D}}^2(z) dx dy \right)^{1/p},$$

where $\rho_{\mathbb{D}}$ is the hyperbolic density on \mathbb{D} . The set of all those Φ with $\|\Phi\|_{\text{HD}_p} < \infty$ is denoted by $\text{HD}_p(\mathbb{D})$. By ignoring the difference in constant functions, $\text{HD}_p(\mathbb{D})$ is a complex Banach space with norm $\|\Phi\|_p$. For $p = 2$, this is the space of Dirichlet finite harmonic functions on \mathbb{D} .

Definition 3. For an integrable function ϕ on the unit circle \mathbb{S} and for $p > 1$, let

$$\|\phi\|_{B_p} = \left(\frac{1}{4\pi^2} \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^2} dy dx \right)^{1/p},$$

and the set of all those ϕ with $\|\phi\|_{B_p} < \infty$ is denoted by $B_p(\mathbb{S})$ and called the (homogeneous critical) p -Besov space.

By ignoring the difference in constant functions, $B_p(\mathbb{S})$ is a complex Banach space with norm $\|\phi\|_{B_p}$.

3.2. Boundary functions and the Hilbert transform. Every harmonic function Φ in $\text{HD}_p(\mathbb{D})$ has non-tangential limits almost everywhere on \mathbb{S} and we denote this boundary function by $E(\Phi)$. Then, it is known that $E(\Phi)$ belongs to $B_p(\mathbb{S})$. For $p = 2$, this is in [1, Theorem 2-5] as the Douglas formula. Conversely, for any $\phi \in B_p(\mathbb{S})$, its Poisson integral

$$P(\phi)(z) = \frac{1}{2\pi} \int_{\mathbb{S}} \frac{1 - |\zeta|^2}{|\zeta - z|^2} \phi(\zeta) |d\zeta| \quad (z \in \mathbb{D})$$

belongs to $\text{HD}_p(\mathbb{D})$.

Proposition 3.1. $P : B_p(\mathbb{S}) \rightarrow \text{HD}_p(\mathbb{D})$ is a Banach isomorphism with $P^{-1} = E$. When $p = 2$, this is an isometry up to a constant multiple.

For $\phi \in B_p(\mathbb{S})$, the *Hilbert transform* is defined by the singular integral

$$\mathcal{H}(\phi)(x) = \frac{1}{2\pi i} \text{p.v.} \int_{\mathbb{S}} \frac{\phi(\zeta)}{\zeta - x} d\zeta \quad (x \in \mathbb{S}).$$

Then, $\mathcal{H}(\phi)$ belongs to $B_p(\mathbb{S})$. Moreover, the *Szegő projection* is defined by the Cauchy integral

$$S^\pm(\phi)(z) = \frac{1}{2\pi i} \int_{\mathbb{S}} \frac{\phi(\zeta)}{\zeta - z} d\zeta \quad (z \in \mathbb{D}^\pm),$$

where the orientation of the line integral on \mathbb{S} is taken counter-clockwise in both cases. Then, $S^\pm(\phi)$ belongs to the p -analytic Besov space $\mathcal{B}_p(\mathbb{D}^\pm)$.

Proposition 3.2. *The following are satisfied:*

- (1) $\mathcal{H} : B_p(\mathbb{S}) \rightarrow B_p(\mathbb{S})$ is a Banach automorphism with $\mathcal{H} \circ \mathcal{H} = \frac{1}{4}I$;
- (2) $S^\pm : B_p(\mathbb{S}) \rightarrow \mathcal{B}_p(\mathbb{D}^\pm)$ is a bounded linear map such that $E^\pm \circ S^\pm : B_p(\mathbb{S}) \rightarrow B_p(\mathbb{S})$ is a bounded projection satisfying $E^\pm \circ S^\pm = \frac{1}{2}I \pm \mathcal{H}$ and $E^+ \circ S^+ + E^- \circ S^- = I$.

In particular, by the identification under E , we have

$$\text{HD}_p(\mathbb{D}) \cong B_p(\mathbb{S}) \cong \mathcal{B}_p(\mathbb{D}^+) \oplus \mathcal{B}_p(\mathbb{D}^-).$$

For $z \in \mathbb{D}^+$, $z^* = 1/\bar{z}$ denotes the reflection with respect to \mathbb{S} . For $\Phi \in \mathcal{B}_p(\mathbb{D}^-)$, let $\bar{\Phi}(z) = \Phi(z^*)$ be the anti-holomorphic function defined on \mathbb{D}^+ , and let $\overline{\mathcal{B}_p(\mathbb{D}^+)}$ be the set of all such functions $\bar{\Phi}$. Then, $\mathcal{B}_p(\mathbb{D}^-)$ and $\overline{\mathcal{B}_p(\mathbb{D}^+)}$ are identified by this reflection, and the above identification is nothing but the decomposition

$$\text{HD}_p(\mathbb{D}) \cong \mathcal{B}_p(\mathbb{D}) \oplus \overline{\mathcal{B}_p(\mathbb{D})}$$

of a harmonic function into holomorphic and anti-holomorphic parts.

3.3. Composition operator. Let $h : \mathbb{S} \rightarrow \mathbb{S}$ be a quasimetric homeomorphism in general. For $\phi \in B_p(\mathbb{S})$, let $C_h(\phi) = \phi \circ h$. We define C_h as the *composition operator* induced by h and acting on $B_p(\mathbb{S})$. It is known that $C_h(\phi)$ also belongs to $B_p(\mathbb{S})$.

Proposition 3.3. *For any quasimetric homeomorphism h , the composition operator $C_h : B_p(\mathbb{S}) \rightarrow B_p(\mathbb{S})$ is a Banach automorphism whose operator norm is depending only on the quasimetric constant of h .*

We handle only quasimetric homeomorphisms h that are induced from $M_p(\mathbb{D})$, namely, those in $\pi(M_p(\mathbb{D}))$. The set of all such normalized quasimetric homeomorphisms h is identified with the p -integrable Teichmüller space T_p . Then, we consider a family of composition operators C_h as h moves in T_p .

Proposition 3.4. *The map $B_p(\mathbb{S}) \times T_p \rightarrow B_p(\mathbb{S})$ defined by $(\phi, h) \mapsto C_h(\phi)$ is continuous.*

3.4. Deformation of function spaces. Let $g = g([\mu^+], [\mu^-])$ be a Weil–Petersson embedding for $([\mu^+], [\mu^-]) \in T_p^+ \times T_p^-$ and let $\Gamma = g(\mathbb{S})$ be the Weil–Petersson curve. Let Ω^+ and Ω^- be the complementary domains of $\widehat{\mathbb{C}}$ divided by Γ .

By fixing the parametrization $g : \mathbb{S} \rightarrow \Gamma$, we define $B_p(\Gamma)$ as the set of all integrable functions ψ on Γ that are given by $g_*\phi = \phi \circ g^{-1}$ for some $\phi \in B_p(\mathbb{S})$. By setting $\|\psi\|_{B_p} = \|\phi\|_{B_p}$, we regard $B_p(\Gamma)$ as the Banach space with this norm, which is isometric to $B_p(\mathbb{S})$.

Let $F^\pm : \mathbb{D}^\pm \rightarrow \Omega^\pm$ be the normalized Riemann mapping. Let $\text{HD}_p(\Omega^\pm)$ be the Banach space of p -Dirichlet finite harmonic functions on Ω^\pm . Namely, a harmonic function Ψ on Ω^\pm belongs to $\text{HD}_p(\Omega^\pm)$ if the norm

$$\|\Psi\|_{\text{HD}_p} = \left(\int_{\Omega^\pm} (\|\nabla\Psi(z)\| \rho_{\Omega^\pm}^{-1}(z))^p \rho_{\Omega^\pm}^2(z) dx dy \right)^{1/p}$$

is finite, where ρ_{Ω^\pm} is the hyperbolic density on Ω^\pm . The Riemann mapping F^\pm induces the isometric isomorphism

$$(F^\pm)_* : \text{HD}_p(\mathbb{D}^\pm) \rightarrow \text{HD}_p(\Omega^\pm)$$

by $(F^\pm)_*\phi = \phi \circ (F^\pm)^{-1}$.

For any $\Psi \in \text{HD}_p(\Omega^\pm)$, its boundary extension by non-tangential limits defines a function $E_\Gamma^\pm(\Psi)$ in $B_p(\Gamma)$. Conversely, for any $\psi \in B_p(\Gamma)$, its integral by the harmonic measure on Ω^\pm gives a harmonic function $P_\Gamma^\pm(\psi)$ in $\text{HD}_p(\Omega^\pm)$.

Proposition 3.5. $P_\Gamma^\pm : B_p(\Gamma) \rightarrow \text{HD}_p(\Omega^\pm)$ is a Banach isomorphism with $(P_\Gamma^\pm)^{-1} = E_\Gamma^\pm$. The operator norm $\|P_\Gamma^\pm\|$ is bounded by $\|C_{h^\pm}\|$ and $\|E_\Gamma^\pm\|$ is bounded by $\|C_{h^\pm}^{-1}\|$, where h^\pm is the quasimetric homeomorphism of \mathbb{S} given by $\pi(\mu^\pm) = h^\pm$.

Proof. This follows from the equation $g = F^\pm \circ h^\pm$. □

To any $\Psi \in \text{HD}_p(\Omega^+)$, there corresponds a harmonic function $\Theta_\Gamma(\Psi)$ in $\text{HD}_p(\Omega^-)$ that has the same boundary values a.e. on Γ as Ψ . This map $\Theta_\Gamma : \text{HD}_p(\Omega^+) \rightarrow \text{HD}_p(\Omega^-)$ is called the *transmission operator* through Γ .

Lemma 3.6. The transmission operator $\Theta_\Gamma : \text{HD}_p(\Omega^+) \rightarrow \text{HD}_p(\Omega^-)$ is given by $\Theta_\Gamma = P_\Gamma^- \circ E_\Gamma^+$. The operator norm of Θ_Γ is bounded by $\|C_{h^-}^{-1}\| \cdot \|C_{h^+}\|$.

4. ON A RIEMANN SURFACE WITH A FUNNEL

In this section, we present several facts about the space of Dirichlet finite harmonic functions on a hyperbolic Riemann surface with a single circular boundary at infinity. In the final section, this will correspond to the portions of a Riemann surface divided by a simple closed curve on it.

4.1. **p -Dirichlet finite harmonic functions.** Let Φ be a complex-valued harmonic function on a Riemann surface R° equipped with a hyperbolic metric $\rho_{R^\circ}(z)|dz|$. For $p > 1$, let

$$\|\Phi\|_{\text{HD}_p} = \left(\int_{R^\circ} (\|\nabla\Phi(z)\| \rho_{R^\circ}^{-1}(z))^p \rho_{R^\circ}^2(z) dx dy \right)^{1/p},$$

and the set of all those Φ with $\|\Phi\|_{\text{HD}_p} < \infty$ is denoted by $\text{HD}_p(R^\circ)$. By ignoring the difference in constant functions, $\text{HD}_p(R^\circ)$ is a complex Banach space with norm $\|\Phi\|_p$. For $p = 2$, this is the space of Dirichlet finite harmonic functions on R° .

4.2. **Annular cover of a surface with a funnel.** Let R° be a hyperbolic Riemann surface with a single funnel. The boundary at infinity of the funnel is denoted by ∂R° . We take its holomorphic annular cover $\alpha : \widetilde{R}^\circ \rightarrow R^\circ$ with respect to ∂R° . Then, we realize this annulus \widetilde{R}° in \mathbb{D} so that $\widetilde{R}^\circ \subset \mathbb{D}$ and $\mathbb{S} \subset \partial \widetilde{R}^\circ$, and fix this holomorphic embedding $\widetilde{R}^\circ \hookrightarrow \mathbb{D}$. For a closed annular neighborhood U of ∂R° , let $\widetilde{U} \subset \widetilde{R}^\circ$ be the inverse image of U under α . Then $\alpha|_{\widetilde{U}}$ is a conformal homeomorphism of \widetilde{U} onto U . By replacing U , we may assume that \widetilde{U} is a round annulus in \mathbb{D} .

Let $\text{HD}_p(\widetilde{U})$ be the set of all p -Dirichlet finite harmonic functions on \widetilde{U} , where the hyperbolic density $\rho_{\widetilde{R}^\circ}$ on \widetilde{R}° is applied for the definition of the Dirichlet norm. Because $\rho_{\widetilde{R}^\circ}$ and $\rho_{\mathbb{D}}$ are comparable on \widetilde{U} , we may also use the hyperbolic density $\rho_{\mathbb{D}}$ on \mathbb{D} for this norm.

For any $\Phi \in \text{HD}_p(R^\circ)$, we take its lift $\widetilde{\Phi}$ on \widetilde{R}° . Namely, $\widetilde{\Phi} = \Phi \circ \alpha = \alpha^*(\Phi)$. Then, $\widetilde{\Phi}|_{\widetilde{U}}$ belongs to $\text{HD}_p(\widetilde{U})$. Moreover, $\|\widetilde{\Phi}|_{\widetilde{U}}\|_{\text{HD}_p} \leq \|\Phi\|_{\text{HD}_p}$.

4.3. **Boundary extension in the annular cover.** For $\widetilde{\Phi}_{\widetilde{U}} \in \text{HD}_p(\widetilde{U})$, we consider its non-tangential limits on $\partial \widetilde{R}^\circ = \mathbb{S}$. We denote this boundary extension by $E_{\partial \widetilde{R}^\circ}(\widetilde{\Phi}_{\widetilde{U}})$.

Lemma 4.1. *For any $\widetilde{\Phi}_{\widetilde{U}} \in \text{HD}_p(\widetilde{U})$, its boundary extension $E_{\partial \widetilde{R}^\circ}(\widetilde{\Phi}_{\widetilde{U}})$ belongs to $B_p(\mathbb{S})$. Moreover, $E_{\partial \widetilde{R}^\circ} : \text{HD}_p(\widetilde{U}) \rightarrow B_p(\mathbb{S})$ is a surjective bounded linear operator.*

Proof. For a given $\widetilde{\Phi}_{\widetilde{U}} \in \text{HD}_p(\widetilde{U})$, we define a continuous function $\widetilde{\Phi}_\#$ on \mathbb{D} as follows: $\widetilde{\Phi}_\#$ coincides with $\widetilde{\Phi}_{\widetilde{U}}$ on \widetilde{U} , and the extension to $\mathbb{D} \setminus \widetilde{U}$ is given by the Poisson integral of $\widetilde{\Phi}_{\widetilde{U}}$ on $\partial \widetilde{U} \setminus \mathbb{S}$. This function $\widetilde{\Phi}_\#$ defined on \mathbb{D} has the same boundary extension $E_{\partial \widetilde{R}^\circ}(\widetilde{\Phi}_{\widetilde{U}})$ to \mathbb{S} . Moreover, its p -Dirichlet integral on \mathbb{D} , denoted by the same notation $\|\widetilde{\Phi}_\#\|_{\text{HD}_p}$, satisfies

$$\|\widetilde{\Phi}_\#\|_{\text{HD}_p} \leq C \|\widetilde{\Phi}_{\widetilde{U}}\|_{\text{HD}_p},$$

where $C > 0$ is a constant depending only on r and p .

We will extract the harmonic part from $\widetilde{\Phi}_\#$ keeping the p -integrability and the boundary values. In the case $p = 2$, this is the Dirichlet principle. Indeed, $\widetilde{\Phi}_\#$ is in the space $\mathcal{D}_2(\mathbb{D})$ of Tonelli functions on \mathbb{D} , which has the orthogonal decomposition

$$\mathcal{D}_2(\mathbb{D}) = \text{HD}_2(\mathbb{D}) \oplus \mathcal{D}_2^0(\mathbb{D})$$

with respect to the inner product in $\mathcal{D}_2(\mathbb{D})$. Here, $\mathcal{D}_2^0(\mathbb{D})$ is the space of Tonelli functions vanishing at the boundary \mathbb{S} . Then, by applying the orthogonal projection $\mathcal{D}_2(\mathbb{D}) \rightarrow \text{HD}_2(\mathbb{D})$ to $\tilde{\Phi}_\#$, we have $\Phi \in \text{HD}_2(\mathbb{D})$ with $\|\Phi\|_{\text{HD}_2} \leq \|\tilde{\Phi}_\#\|_{\text{HD}_2}$. Moreover, since $E(\Phi) = E_{\partial\tilde{R}^\circ}(\tilde{\Phi}_{\tilde{U}})$, Proposition 3.1 implies that $E_{\partial\tilde{R}^\circ}(\tilde{\Phi}_{\tilde{U}})$ belongs to $B_2(\mathbb{S})$ and

$$\|E_{\partial\tilde{R}^\circ}(\tilde{\Phi}_{\tilde{U}})\|_{B_2} \asymp \|\Phi\|_{\text{HD}_2} \leq C\|\tilde{\Phi}_{\tilde{U}}\|_{\text{HD}_2}.$$

This shows that the linear operator $E_{\partial\tilde{R}^\circ}$ is bounded.

We generalize the Dirichlet principle to any $p > 1$. Namely, we define the space $\mathcal{D}_p(\mathbb{D})$ of p -Tonelli functions and show the topological direct sum decomposition $\mathcal{D}_p(\mathbb{D}) = \text{HD}_p(\mathbb{D}) \oplus \mathcal{D}_p^0(\mathbb{D})$ though the orthogonality does not make sense in this general case. However, the boundedness of the projection $\mathcal{D}_p(\mathbb{D}) \rightarrow \text{HD}_p(\mathbb{D})$ still holds true and the rest of the proof is the same as above.

Conversely, for any $\phi \in B_p(\mathbb{S})$, Proposition 3.1 asserts that the Poisson integral $P(\phi)$ is in $\text{HD}_p(\mathbb{D})$. Let $\tilde{\Phi}_{\tilde{U}} = P(\phi)|_{\tilde{U}}$ be the restriction of $P(\phi)$ to \tilde{U} . Since the hyperbolic densities $\rho_{\tilde{R}^\circ}$ and $\rho_{\mathbb{D}}$ are comparable on \tilde{U} , we see that $\tilde{\Phi}_{\tilde{U}} \in \text{HD}_p(\tilde{U})$. Then, $E_{\partial\tilde{R}^\circ}(\tilde{\Phi}_{\tilde{U}}) = \phi$, and hence $E_{\partial\tilde{R}^\circ}$ is surjective. \square

4.4. Surjectivity of the boundary extension map. We define the boundary extension map $E_{\partial R^\circ}$ for any $\Phi \in \text{HD}_p(R^\circ)$ so that

$$E_{\partial R^\circ}(\Phi) = E_{\partial\tilde{R}^\circ}(\alpha^*(\Phi)|_{\tilde{U}}) \in B_p(\mathbb{S}).$$

Lemma 4.2. *The boundary extension map $E_{\partial R^\circ} : \text{HD}_p(R^\circ) \rightarrow B_p(\mathbb{S})$ is a surjective bounded linear operator.*

Proof. For any $\Phi \in \text{HD}_p(R^\circ)$, we have $\|\tilde{\Phi}|_{\tilde{U}}\|_{\text{HD}_p} \leq \|\Phi\|_{\text{HD}_p}$ for $\tilde{\Phi} = \alpha^*(\Phi)$, and Lemma 4.1 then yields $\|E_{\partial\tilde{R}^\circ}(\tilde{\Phi}|_{\tilde{U}})\|_{B_p} \lesssim \|\tilde{\Phi}|_{\tilde{U}}\|_{\text{HD}_p}$. Hence, $E_{\partial R^\circ}$ is a bounded linear operator. Regarding the surjectivity, we take $\tilde{\Phi}_{\tilde{U}} = P(\phi)|_{\tilde{U}}$ for any $\phi \in B_p(\mathbb{S})$ as in the last part of the proof of Lemma 4.1. Let $\Phi_U = \alpha_*(\tilde{\Phi}_{\tilde{U}}) = \tilde{\Phi}_{\tilde{U}} \circ \alpha^{-1}$. Then, $\|\Phi_U\|_{\text{HD}_p} = \|\tilde{\Phi}_{\tilde{U}}\|_{\text{HD}_p} < \infty$ and $E_{\partial\tilde{R}^\circ}(\tilde{\Phi}_{\tilde{U}}) = \phi$. By using this Φ_U , we can obtain a p -Dirichlet finite harmonic function Φ on R° so that its boundary extension to ∂R° coincides with that of Φ_U . See [6, Proposition 3.3] in the case $p = 2$. Then, $E_{\partial R^\circ}(\Phi) = \phi$, and $E_{\partial R^\circ}$ is surjective. \square

5. VARIATION OF THE SPACE OF HARMONIC FUNCTIONS

In this section, we deform the Riemann surface with a funnel quasiconformally and observe its effect on the space of Dirichlet finite harmonic functions.

5.1. Deformation of Riemann surfaces. Let $M_p(R^\circ)$ be the space of all Beltrami coefficients (differentials) $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on R° that are p -integrable with respect to the hyperbolic metric on R° ;

$$\|\mu\|_p = \left(\int_{R^\circ} |\mu(z)|^p \rho_{R^\circ}^2(z) dx dy \right)^{1/p} < \infty.$$

The topology on $M_p(R^\circ)$ is defined by the norm $\|\mu\|_p + \|\mu\|_\infty$.

For any $\mu \in M_p(R^\circ)$, we take the quasiconformal homeomorphism H_μ of R° and denote its image by R_μ° . We lift H_μ to a quasiconformal homeomorphism $\tilde{H}_\mu : \tilde{R}^\circ \rightarrow \tilde{R}_\mu^\circ$ between their annular covers sitting inside of \mathbb{D} with $\partial\tilde{R}^\circ = \partial\tilde{R}_\mu^\circ = \mathbb{S}$. Let $\tilde{\mu}$ be a Beltrami coefficient on \mathbb{D} defined by the complex dilatation of \tilde{H}_μ on \tilde{R}° and 0 on $\mathbb{D} \setminus \tilde{R}^\circ$. We define this correspondence $\mu \mapsto \tilde{\mu}$ by J .

Lemma 5.1. *For any $\mu \in M_p(R^\circ)$, $J(\mu) = \tilde{\mu}$ belongs to $M_p(\mathbb{D})$. Moreover, $J : M_p(R^\circ) \rightarrow M_p(\mathbb{D})$ is continuous.*

Proof. We divide the integral of $\tilde{\mu}$ on \tilde{U} and on $\tilde{R}^\circ \setminus \tilde{U}$. The second one is bounded because the hyperbolic density $\rho_{\mathbb{D}}$ is bounded there. For the first one, by the monotone principle of hyperbolic densities, we have

$$\begin{aligned} \int_{\tilde{U}} |\tilde{\mu}(z)|^p \rho_{\mathbb{D}}^2(z) dx dy &\leq \int_{\tilde{U}} |\tilde{\mu}(z)|^p \rho_{\tilde{R}^\circ}^2(z) dx dy \\ &= \int_U |\mu(z)|^p \rho_{R^\circ}^2(z) dx dy \leq \int_{R^\circ} |\mu(z)|^p \rho_{R^\circ}^2(z) dx dy. \end{aligned}$$

Hence, $\tilde{\mu}$ belongs to $M_p(\mathbb{D})$.

Suppose that a sequence μ_n converges to μ in $M_p(R^\circ)$. This implies that $\|\mu - \mu_n\|_p \rightarrow 0$ and $\|\mu - \mu_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then, it is easy to see that $\|\tilde{\mu} - \tilde{\mu}_n\|_\infty \rightarrow 0$. For the convergence on the p -norm, we use the division of \tilde{R}° into \tilde{U} and $\tilde{R}^\circ \setminus \tilde{U}$. Then, on \tilde{U} , the above estimate can be applied. On $\tilde{R}^\circ \setminus \tilde{U}$, the hyperbolic density $\rho_{\mathbb{D}}$ is bounded. Hence, the dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_{\tilde{R}^\circ \setminus \tilde{U}} |\tilde{\mu}(z) - \tilde{\mu}_n(z)|^p \rho_{\mathbb{D}}^2(z) dx dy = 0.$$

Hence, J is continuous. □

5.2. The correspondence of Dirichlet finite harmonic functions. For any $\mu \in M_p(R^\circ)$, we consider the quasiconformal deformation $H_\mu : R^\circ \rightarrow R_\mu^\circ$ whose complex dilatation is μ . Let $\tilde{\mu} = J(\mu) \in M_p(\mathbb{D})$. Then, the quasiconformal self-homeomorphism $\tilde{H}_{\tilde{\mu}} : \mathbb{D} \rightarrow \mathbb{D}$ whose complex dilatation is $\tilde{\mu}$ induces the quasiconformal deformation $\tilde{R}^\circ \rightarrow \tilde{R}_\mu^\circ$ of the annular covers. The extension of $\tilde{H}_{\tilde{\mu}}$ to \mathbb{S} yields a quasiasymmetric homeomorphism $h_{\tilde{\mu}} : \mathbb{S} \rightarrow \mathbb{S}$.

In the sequel, we assume that the boundary extension map $E_{\partial R^\circ} : \text{HD}_p(R^\circ) \rightarrow B_p(\mathbb{S})$ in Lemma 4.2 is injective. This means that a p -Dirichlet finite harmonic function on R° is uniquely determined by its boundary values on ∂R° . We also assume this for any quasiconformal deformation R_μ° by $\mu \in M_p(R^\circ)$. Under this assumption, $E_{\partial R^\circ} : \text{HD}_p(R^\circ) \rightarrow B_p(\mathbb{S})$ is a Banach isomorphism.

Proposition 5.2. *For any $\mu \in M_p(R^\circ)$, let $H_\mu : R^\circ \rightarrow R_\mu^\circ$ be the quasiconformal homeomorphism and $h_{\tilde{\mu}} : \mathbb{S} \rightarrow \mathbb{S}$ the associated quasiasymmetric homeomorphism. Suppose*

that $E_{\partial R^\circ}$ and $E_{\partial R_\mu^\circ}$ are injective. Then, there exists a Banach isomorphism $(H_\mu)_* : \text{HD}_p(R^\circ) \rightarrow \text{HD}_p(R_\mu^\circ)$ such that

$$(H_\mu)_* = E_{\partial R_\mu^\circ}^{-1} \circ (h_{\tilde{\mu}})_* \circ E_{\partial R^\circ} = E_{\partial R_\mu^\circ}^{-1} \circ C_{h_{\tilde{\mu}}}^{-1} \circ E_{\partial R^\circ}.$$

If we assume the injectivity of $E_{\partial R^\circ}$ and $E_{\partial R_\mu^\circ}$, the above statement is straightforward. In the case $p = 2$, $(H_\mu)_*(\Phi) \in \text{HD}_p(R_\mu^\circ)$ for $\Phi \in \text{HD}_2(R^\circ)$ is the orthogonal projection of a Tonelli function $\Phi \circ H_\mu^{-1}$ on R_μ° by the Dirichlet principle. For any $p > 1$, we expect that this can be generalized.

6. THE TRANSMISSION OPERATORS

In Subsection 3.4, we dealt with the correspondence between the spaces of Dirichlet finite harmonic functions on domains separated by a Weil–Petersson curve in the plane. In this section, we consider this problem for a Weil–Petersson curve on a Riemann surface. We define the transmission operator in this setting and represent it using operators we have defined in simpler situations.

6.1. Weil–Petersson curves on a Riemann surface. A Weil–Petersson curve on a Riemann surface can be naturally defined as follows.

Definition 4. A simple closed curve γ on a Riemann surface R is a p -Weil–Petersson curve if there exists an annular neighborhood W of γ and a conformal embedding $\varphi : W \rightarrow \widehat{\mathbb{C}}$ such that $\varphi(\gamma)$ is a p -Weil–Petersson curve on $\widehat{\mathbb{C}}$.

This is well-defined, that is, it is independent of the choice of W and φ .

Proposition 6.1. *Let γ be an essential dividing simple closed curve on a Riemann surface R . Then, γ is a p -Weil–Petersson curve if and only if there exists a conformal embedding $\tilde{\varphi} : \tilde{R}_\gamma \rightarrow \widehat{\mathbb{C}}$ of the holomorphic annular cover \tilde{R}_γ of R with respect to γ such that $\tilde{\varphi}(\gamma)$ is a p -Weil–Petersson curve on $\widehat{\mathbb{C}}$.*

Proof. Suppose that γ is a p -Weil–Petersson curve. Then, there exists an annular neighborhood W of γ and a conformal embedding φ of W into $\widehat{\mathbb{C}}$ such that $\varphi(\gamma)$ is the image of \mathbb{S} under a p -Weil–Petersson quasiconformal self-homeomorphism G of $\widehat{\mathbb{C}}$. We take a conformal embedding $\tilde{\varphi}$ of the holomorphic annular cover \tilde{R}_γ into $\widehat{\mathbb{C}}$. We may assume that $W \subset \tilde{R}_\gamma$. Then, $\tilde{\varphi} \circ \varphi^{-1}$ on $\varphi(W)$ is conformal. We extend this map to a quasiconformal self-homeomorphism G' of $\widehat{\mathbb{C}}$. Because $G' \circ G$ is a p -Weil–Petersson quasiconformal self-homeomorphism of $\widehat{\mathbb{C}}$, we see that $\tilde{\varphi}(\gamma)$ is a p -Weil–Petersson curve. The converse is obvious. \square

6.2. Transmission through a Weil–Petersson curve. Let γ be a p -Weil–Petersson curve on a Riemann surface R such that $R \setminus \gamma$ consists of two connected components R^+ and R^- that are not simply connected. We regard R^\pm as a hyperbolic Riemann surface with a single funnel and the boundary at infinity is $\partial R^\pm = \gamma$. Let $\text{HD}_p(R^\pm)$ be the complex Banach space of p -Dirichlet finite harmonic functions on R^\pm .

Definition 5. Suppose that for any $\Psi^+ \in \text{HD}_p(R^+)$, there exists uniquely a harmonic function Ψ^- in $\text{HD}_p(R^-)$ that has the same boundary values a.e. on the p -Weil–Petersson curve γ as Ψ^+ has. Then, this correspondence $\Theta_\gamma : \text{HD}_p(R^+) \rightarrow \text{HD}_p(R^-)$ is called the *transmission operator* through γ on R .

By Proposition 6.1, there is a holomorphic embedding $\varphi : \tilde{R}_\gamma \rightarrow \hat{\mathbb{C}}$, and let $\Gamma = \varphi(\gamma)$ be the p -Weil–Petersson curve. Let $g = g([\mu^+], [\mu^-]) : \mathbb{S} \rightarrow \hat{\mathbb{C}}$ be the p -Weil–Petersson embedding with its image Γ . For $\Psi^\pm \in \text{HD}_p(R^\pm)$, we can regard its boundary function on $\partial R^\pm = \gamma$ as a function on Γ through φ . We can also regard it as the image of the boundary extension map $E_{\partial R^\pm} : \text{HD}_p(R^\pm) \rightarrow B_p(\mathbb{S})$ in Lemma 4.2, which we assume to be injective (and hence isomorphic). Then, the transmission operator $\Theta_\gamma : \text{HD}_p(R^+) \rightarrow \text{HD}_p(R^-)$ can be described in terms of an automorphism of $B_p(\mathbb{S})$.

Theorem 6.2. *The transmission operator $\Theta_\gamma : \text{HD}_p(R^+) \rightarrow \text{HD}_p(R^-)$ coincides with $E_{\partial R^-}^{-1} \circ C_{h^-}^{-1} \circ C_{h^+} \circ E_{\partial R^+}$, where $h^\pm = \pi(\mu^\pm)$ is a quasimetric homeomorphism of \mathbb{S} . In particular, Θ_γ is a bounded linear operator.*

Proof. Let Ω^\pm be the complementary domains of $\hat{\mathbb{C}}$ divided by the p -Weil–Petersson curve $\Gamma = \varphi(\gamma)$. Let $g = g([\mu^+], [\mu^-]) : \mathbb{S} \rightarrow \Gamma$ be its p -Weil–Petersson embedding. Let F_{μ^\pm} be the normalized quasiconformal self-homeomorphism of $\hat{\mathbb{C}}$ with complex dilatation μ^\pm on \mathbb{D}^\pm and 0 on \mathbb{D}^\mp . Then, considering the boundary values of $\text{HD}_p(R^\pm)$ on Γ , we see that the boundary extension map $E_{\partial R^\pm} : \text{HD}_p(R^\pm) \rightarrow B_p(\mathbb{S})$ is given by the pull-back of the functions on Γ by $F_{\mu^\mp}|_\mathbb{S}$. Since $\gamma = F_{\mu^\mp} \circ h^\pm$, the map $B_p(\mathbb{S}) \rightarrow B_p(\mathbb{S})$ caused by taking the same boundary values on Γ is given by $C_{h^-}^{-1} \circ C_{h^+}$. \square

6.3. Strong convergence of the transmission operators. We investigate the variation of the transmission operator Θ_γ when we deform a Riemann surface (R, γ) with a Weil–Petersson curve γ under quasiconformal homeomorphisms. To this end, we regularize Θ_γ by extracting the action on $B_p(\mathbb{S})$. Namely, we define $\theta(R, \gamma) = C_{h^-}^{-1} \circ C_{h^+}$ in the notation of Theorem 6.2. Then, $\theta(R, \gamma)$ is a Banach automorphism of $B_p(\mathbb{S})$ such that $\Theta_\gamma = E_{\partial R^-}^{-1} \circ \theta(R, \gamma) \circ E_{\partial R^+}$.

We fix a base surface (R_0, γ_0) and a conformal embedding $\varphi_0 : \tilde{R}_0 \rightarrow \hat{\mathbb{C}}$ with a p -Weil–Petersson embedding $g_0 = G(\mu_0^+, \mu_0^-)|_\mathbb{S}$ of \mathbb{S} onto $\varphi_0(\gamma_0)$. Let Ω_0^\pm be the complementary domains of $\hat{\mathbb{C}}$ divided by Γ_0 . For any Beltrami differentials $\nu^\pm = \nu^\pm(z)d\bar{z}/dz$ in $M_p(R_0^\pm)$, let $f = f(\nu^+, \nu^-)$ be the quasiconformal homeomorphism of R_0 whose complex dilatation is ν^\pm on R_0^\pm . Then, for $(f(R_0), f(\gamma_0))$, we consider a conformal embedding of the annular cover of $f(R_0)$ with respect to $f(\gamma_0)$. We note that $(f(R_0), f(\gamma_0))$ is in fact determined by the pair of the Teichmüller equivalence classes $([\nu^+], [\nu^-]) \in T_p(R_0^+) \times T_p(R_0^-)$.

By the continuous map $J^\pm : M_p(R_0^\pm) \rightarrow M_p(\mathbb{D}^\pm) \cong M_p(\Omega_0^\pm)$ in Lemma 5.1, we take $\tilde{\nu}^\pm = J^\pm(\nu^\pm)$ in $B_p(\Omega_0^\pm)$. Here, under the normalized Riemann mapping $\mathbb{D}^\pm \rightarrow \Omega_0^\pm$, the spaces of Beltrami coefficients $M_p(\mathbb{D}^\pm)$ and $M_p(\Omega_0^\pm)$ are identified. Let $G = G(\tilde{\nu}^+, \tilde{\nu}^-)$ be the normalized quasiconformal self-homeomorphism of $\hat{\mathbb{C}}$ whose complex dilatation is $\tilde{\nu}^\pm$ on Ω_0^\pm .

Lemma 6.3. *For the quasiconformal deformation $(f(R_0), f(\gamma_0))$ of (R_0, γ_0) by $f = f(\nu^+, \nu^-)$ with $\nu^\pm \in M_p(R_0^\pm)$, the annular cover of $f(R_0)$ with respect to $f(\gamma_0)$ is conformally equivalent to $G(\tilde{R}_0)$ for $G = G(\tilde{\nu}^+, \tilde{\nu}^-)$ under some conformal embedding φ . Moreover, it holds that*

$$\varphi(f(\gamma_0)) = G(\tilde{\nu}^+, \tilde{\nu}^-) \circ G(\mu_0^+, \mu_0^-)(\mathbb{S}).$$

*In particular, $f(\gamma_0)$ is a p -Weil–Petersson curve on $f(R_0)$ and it is realized as $\varphi(f(\gamma_0))$ on $\widehat{\mathbb{C}}$ by the p -Weil–Petersson embedding $g([\tilde{\nu}^+] * [\mu_0^+], [\tilde{\nu}^-] * [\mu_0^-])$ of \mathbb{S} .*

Proof. We only note that $[\tilde{\nu}^+] * [\mu_0^+] \in T_p^+$ and $[\tilde{\nu}^-] * [\mu_0^-] \in T_p^-$ by the group structure of T_p as in Proposition 2.2. Other statements follow straightforward from definition. \square

We choose a sequence $([\nu_n^+], [\nu_n^-])$ in $T_p(R_0^+) \times T_p(R_0^-)$ converging to $([0], [0])$. For $f_n = f(\nu_n^+, \nu_n^-)$, let $R_n = f_n(R_0)$ and $\gamma_n = f_n(\gamma_0)$. By the regularization of the transmission operators Θ_{γ_n} , we obtain a sequence of Banach automorphisms $\theta(R_n, \gamma_n)$ of $B_p(\mathbb{S})$.

Theorem 6.4. *The sequence $\theta(R_n, \gamma_n)$ converges strongly to $\theta(R_0, \gamma_0)$. That is, for any $\phi \in B_p(\mathbb{S})$, $\theta(R_n, \gamma_n)(\phi)$ converges to $\theta(R_0, \gamma_0)(\phi)$ in $B_p(\mathbb{S})$.*

Proof. Let $\tilde{\nu}_n^\pm = J(\nu_n^\pm)$ and $h_n^\pm = \pi(\tilde{\nu}_n^\pm * \mu_0^\pm)$, $h_0^\pm = \pi(\mu_0^\pm)$. Then, by Lemma 5.1, h_n^\pm converges to h_0^\pm in T_p^\pm . Since $\theta(R_n, \gamma_n) = C_{h_n^\pm}^{-1} \circ C_{h_n^\pm}$ by Lemma 6.3 and $\theta(R_0, \gamma_0) = C_{h_0^\pm}^{-1} \circ C_{h_0^\pm}$, we see that $\theta(R_n, \gamma_n)$ converges strongly to $\theta(R_0, \gamma_0)$ by Propositions 2.2 and 3.4. \square

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