Chapter 15

# Infinite-dimensional Teichmüller spaces and modular groups

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## 1 Introduction

The theory of Teichmüller space is studied in various fields of mathematics but the complex-analytic approach has one advantage in a sense that it can deal with Teichmüller spaces of finite and infinite type Riemann surfaces in parallel and simultaneously. However that may be, Teichmüller spaces of analytically infinite Riemann surfaces are infinite-dimensional and they display several aspects and phenomena different from those of the finite-dimensional cases, and some results involve much more difficult and complicated arguments for their proofs. On the other hand, through these generalization and unification of theories, arguments given for finite-dimensional cases become clearer and more transparent in some occasions.

One of the recent developments of the infinite-dimensional Teichmüller theory is brought by the fact that the biholomorphic automorphism group of the Teichmüller space is completely determined. Namely, it is proved that every biholomorphic automorphism is induced by a quasiconformal mapping class of the base Riemann surface, which is called a Teichmüller modular transformation. This result was first proved by Royden [44] for Teichmüller spaces of compact Riemann surfaces, and through succeeding researches due to Kra, Earle, Gardiner and Lakic (see [3] and [9]), it has been proved in full generality by Markovic [31]. See the report in Volume II of this Handbook [12]. After this, it should be natural that we investigate the moduli space of an infinite type Riemann surface, which is the quotient space of the Teichmüller space by the Teichmüller modular group (now known to be the biholomorphic automorphism group). In fact, compared with the finite-dimensional theory, the study of moduli spaces has not been developed yet in the infinite-dimensional case.

In this chapter, we survey recent results concerning the dynamics of modular groups of infinite-dimensional Teichmüller spaces and their quotient spaces. As we mentioned above, the Teichmüller modular group is the automorphism group of the Teichmüller space induced geometrically by the quasiconformal mapping class group. Although they can be identified with each other in almost all cases, we prefer to use "Teichmüller modular group" since we consider its dynamics on the Teichmüller space. Occasionally "mapping class group" is still used when its property as a surface automorphism is a matter in question.

Unlike in the finite-dimensional case, the action of the Teichmüller modular group is not necessarily discontinuous in our case. In general, we say that a group $\Gamma$ acts on a Hausdorff space $X$ discontinuously if, for every point $x \in X$, there is a neighborhood $U$ of $x$ such that the number of elements $\gamma \in \Gamma$ satisfying $\gamma(U) \cap U \neq \emptyset$ is finite. If $X$ is locally compact, this is equivalent to saying that $\Gamma$ acts on $X$ properly discontinuously. However, since our Teichmüller spaces are not locally compact, we use the term "discontinuous" instead of "properly discontinuous". Because of such non-discontinuous action, the topological moduli space obtained simply by taking the quotient of the Teichmüller space by the modular group does not have a nice geometric structure, which might be a reason why this subject matter has not been so attractive. However, assuming these facts in a positive way, conversely, we can specify the set of points where the Teichmüller modular group does not act discontinuously, and observe some properties of this set, called the limit set. As is well known, limit sets play an important role in the theory of Kleinian groups and the iteration of rational maps. We import this concept for the study of dynamics of Teichmüller modular groups. Then, the non-homogeneity of Teichmüller space appears to be tangible and in particular it provides an interesting research subject, which is to understand the interaction between the hyperbolic structure of an infinite type Riemann surface and the behavior of the orbit of the corresponding point in the infinite-dimensional Teichmüller space.

The region of discontinuity is the complement of the limit set and the quotient restricted to this set inherits a geometric structure from the Teichmüller space. However, another problem is caused by the fact that this region is not always dense in the Teichmüller space. To overcome this difficulty, we introduce the concept of region of stability, which is the set of points where the Teichmüller modular group acts in a stable way. Stability is defined by closedness of the orbit. Then the region of stability sits in the Teichmüller space as an open dense subset and the metric completion of the quotient of the region of stability defines the stable moduli space. As a generalization of the moduli space for finite type Riemann surfaces, we expect that the stable moduli space should be an object we have to work with.

In the actual arguments developed in this theory, a comparison between countability and uncountability, such as the Baire category theorem, appears at several places. Uncountability is expressed in the non-separability of the Teichmüller space of an infinite type Riemann surface and by the cardinality of its Teichmüller modular group. In various situations, ideas of our arguments lie in how to pick out countability in the presence of these uncountable circumstances and how to use it in each specific case. Typically, countability comes from $\sigma$-compactness of Riemann surfaces and from compactness of a family of normalized quasiconformal homeomorphisms with bounded dilatation in the compact-open topology. Further, by considering a fiber of the projection of the Teichmüller space onto the asymptotic Teichmüller space, we are able to extract countability in an implicit manner.

The asymptotic Teichmüller space is a new concept for infinite-dimensional Teichmüller spaces introduced by Gardiner and Sullivan [26] and developed by Earle, Gardiner and Lakic [4], [5], [6]. It parametrizes the deformation of complex structures on arbitrarily small neighborhoods of the topological ends of an infinite type Riemann surface. Therefore, the projection from the Teichmüller space means ignoring the deformation of the complex structures on any compact regions, and hence, in each fiber of this projection, all these ignored deformations constitute a separable closed subspace in the Teichmüller space. The Teichmüller modular group acts preserving the fiber structure on the Teichmüller space and it induces a group of biholomorphic automorphisms of the asymptotic Teichmüller space, which is called the asymptotic Teichmüller modular group. In this way, we can divide the action of the Teichmüller modular group into that on the fibers and that on the asymptotic Teichmüller space. Between these actions, a study of the action of a stabilizer subgroup of a fiber, which is called an asymptotically elliptic subgroup, has already been developed to some extent. In this chapter we review the dynamics of asymptotically elliptic subgroups. The dynamics of the asymptotic Teichmüller modular group will be an interesting future research project.

In the next section (Section 2), we survey fundamental results on the dynamics of the Teichmüller modular group, without considering asymptotic Teichmüller spaces. We summarize results contained in a series of papers by Fujikawa [13], [14] and [15], in particular the concept of limit set of Teichmüller modular groups and the bounded geometry condition on hyperbolic Riemann surfaces. Concerning the stability of Teichmüller modular groups and several criteria for stable actions of particular subgroups, especially those for closed subgroups in the compact-open topology, we extract the arguments from [42] and [41] and edit them in a new way.

Then, in Section 3, we add the consideration of asymptotic Teichmüller spaces. In particular, the action on a fiber over the asymptotic Teichmüller space is discussed in detail. We survey several results on asymptotically elliptic subgroups obtained in

[36], [37], [38], [39] and [40]. The topological characterization of a quasiconformal mapping class that acts trivially on the asymptotic Teichmüller space is excerpted from [16] and [19]. As an application of this result, we explain a version of the Nielsen realization theorem for the asymptotic Teichmüller modular group, which is obtained in [20].

Finally, in Section 4, we construct moduli spaces and several quotient spaces by subgroups of Teichmüller modular groups. The stable moduli space is introduced here, which is one of the aims persued in [42]. As a quotient space by the subgroup consisting of all mapping classes acting trivially on the asymptotic Teichmüller space, we obtain an intermediate Teichmüller space. When $R$ satisfies the bounded geometry condition, this coincides with the enlarged moduli space, which is the quotient by the stable quasiconformal mapping class group. This is a subgroup of the mapping class group given by the exhaustion of mapping class groups of topologically finite subsurfaces. Then the asymptotic Teichmüller modular group is canonically realized as the automorphism group of the intermediate Teichmüller space. These arguments are demonstrated in [19].

Throughout this chapter, our original work [19] and [42] is frequently cited as basic references. The research announcement of [42] appeared in [33].

## 2 Dynamics of Teichmüller modular groups

In this section, we develop the theory of dynamics of Teichmüller modular groups acting on infinite-dimensional Teichmüller spaces. For an *analytically finite* Riemann surface, which is a Riemann surface obtained from a compact Riemann surface by removing at most a finite number of points, the mapping class group and its action on the Teichmüller space are well known and broadly studied. For an *analytically infinite* Riemann surface whose Teichmüller space is infinite-dimensional, we also consider mapping classes in the quasiconformal category. Their action on the infinite-dimensional Teichmüller space induces Teichmüller modular transformations just like in the finite-dimensional cases. Especially in this case, non-homogeneity of the Teichmüller space indicates an interesting interaction between the dynamics of orbits and hyperbolic structures on the base Riemann surface. In the first part of this section, we give basic concepts on the dynamics of Teichmüller modular groups. Then we discuss fundamental techniques for treating various kinds of subgroups of these groups. We also show some application of these theories to infinite-dimensional Teichmüller spaces.

### 2.1 Teichmüller spaces and modular groups

Throughout this chapter, we assume that a Riemann surface $R$ is *hyperbolic*, that is, it is represented as a quotient space $\mathbb{D}/H$ of the unit disk $\mathbb{D}$ endowed with the hyperbolic metric by a torsion-free Fuchsian group $H$. Without specific mention, we always regard $R$ to have the hyperbolic structure, but when a hyperbolic geometrical aspect of $R$ is a matter in question, we sometimes call $R$ a hyperbolic surface. If the limit set $\Lambda(H)$ of the Fuchsian group $H$ is a proper subset of the unit circle $\partial\mathbb{D}$, then $H$ acts properly discontinuously on $\overline{\mathbb{D}} - \Lambda(H)$ and a bordered Riemann surface $(\overline{\mathbb{D}} - \Lambda(H))/H$ is obtained, which contains $R$ as its interior. In this case, $(\partial\mathbb{D} - \Lambda(H))/H$ is called the *boundary at infinity* of $R$ and denoted by $\partial_\infty R$. We are mainly interested in the case where the fundamental group $\pi_1(R) \cong H$ is infinitely generated, namely, $R$ is of *infinite topological type*. (Conversely, if $\pi_1(R)$ is finitely generated, then $R$ is said to be of *finite topological type*. Furthermore, if $\pi_1(R)$ is cyclic, we call $R$ *elementary*. ) We now define the Teichmüller space for $R$ and its Teichmüller modular group.

### Teichmüller spaces in general

The *Teichmüller space* $\mathcal{T}(R)$ of an arbitrary Riemann surface $R$ is the set of all equivalence classes of quasiconformal homeomorphisms $f$ of $R$ onto another Riemann surface. Two quasiconformal homeomorphisms $f_1$ and $f_2$ are defined to be equivalent if there is a conformal homeomorphism $h\colon f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on $R$. Here the homotopy is considered to be relative to the boundary at infinity $\partial_\infty R$ when $\partial_\infty R$ is not empty. It is proved in Earle and McMullen [10] that the existence of a homotopy is equivalent to saying that there is an isotopy to the identity of $R$ through uniformly quasiconformal automorphisms (relative to $\partial_\infty R$ if $\partial_\infty R \neq \emptyset$). The equivalence class of $f$ is called its Teichmüller class and denoted by $[f]$.

The Teichmüller space $\mathcal{T}(R)$ has a complex Banach manifold structure. When $R$ is analytically finite, $\mathcal{T}(R)$ is finite-dimensional, and otherwise $\mathcal{T}(R)$ is infinite-dimensional. A distance between $p_1 = [f_1]$ and $p_2 = [f_2]$ in $\mathcal{T}(R)$ is defined by $d_T(p_1, p_2) = \frac{1}{2}\log K(f)$, where $f$ is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$ (relative to the boundary at infinity if it is not empty). This is called the *Teichmüller distance*. In virtue of a compactness property of quasiconformal maps, the Teichmüller distance $d_T$ is complete on $\mathcal{T}(R)$. This coincides with the Kobayashi distance on $\mathcal{T}(R)$ with respect to the complex Banach manifold structure. Consult [12], [24], [25], [28], [30] and [43] for the theory of Teichmüller space.

### Quasiconformal mapping class groups

For an arbitrary Riemann surface $R$, the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ is the group of all homotopy classes $[g]$ of quasiconformal automorphisms $g$ of $R$ (relative to $\partial_\infty R$ if $\partial_\infty R \neq \emptyset$). Each element $[g]$ is called a mapping class and it acts on $\mathcal{T}(R)$ from the left in such a way that $[g]_*\colon [f] \mapsto [f \circ g^{-1}]$. It is evident from the definition that $\mathrm{MCG}(R)$ acts on $\mathcal{T}(R)$ isometrically with respect to the Teichmüller distance. It also acts biholomorphically on $\mathcal{T}(R)$.

**Definition 2.1.** Let $\iota\colon \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{T}(R))$ be the homomorphism defined by $[g] \mapsto [g]_*$, where $\mathrm{Aut}(\mathcal{T}(R))$ denotes the group of all isometric biholomorphic automorphisms of $\mathcal{T}(R)$. The image $\operatorname{Im}\iota \subset \operatorname{Aut}(\mathcal{T}(R))$ is called the *Teichmüller modular group* and is denoted by $\operatorname{Mod}(R)$.

Except for a few low-dimensional cases, $\iota$ is injective. In particular, if $R$ is analytically infinite, then $\iota$ is always injective. This was first proved by Earle, Gardiner and Lakic [4] and another proof was given by Epstein [11]. We will discuss again this proof in Section 2.4 later. The map $\iota$ is also surjective except in the one-dimensional case. This was finally proved by Markovic [31] after a series of pioneering works. We refer to the account [12] in Volume II of this Handbook. Hence, when there is no risk of confusion, we sometimes identify $\operatorname{MCG}(R)$ with $\operatorname{Mod}(R)$.

**Bounded geometry condition**

We often put some moderate assumptions concerning the geometry of hyperbolic Riemann surfaces which make the analysis of Teichmüller modular groups easier.

**Definition 2.2.** We say that a hyperbolic surface $R$ satisfies the *lower boundedness condition* if the injectivity radius at every point of $R$ is uniformly bounded away from zero except in horocyclic cusp neighborhoods of area 1. We say that $R$ satisfies the *upper boundedness condition* if the injectivity radius at every point of $R^*$ is uniformly bounded from above, where $R^*$ is some connected subsurface of $R$ such that the inclusion map $R^* \to R$ induces a surjective homomorphism $\pi_1(R^*) \to \pi_1(R)$. Then $R$ satisfies the *bounded geometry condition* if both the lower and upper boundedness conditions are satisfied and if the boundary at infinity $\partial_\infty R$ is empty.

These conditions are quasiconformally invariant and hence we may regard them as conditions for the Teichmüller space $\mathcal{T}(R)$. For example, an arbitrary non-universal normal cover of an analytically finite Riemann surface satisfies the bounded geometry condition (see [13]).

A pair of pants is a hyperbolic surface with three geodesic boundary components and zero genus, where geodesic boundaries can degenerate to punctures. When a hyperbolic surface $R$ can be decomposed into the union of pairs of pants, we say that $R$ has a *pants decomposition*. If $R$ has a pants decomposition such that all the lengths of boundary geodesics of the pairs of pants are uniformly bounded from above and from below, then $R$ satisfies the bounded geometry condition. However, the converse is not true. Counter-examples can be easily obtained by considering a planar non-universal normal cover of an analytically finite Riemann surface with a puncture (see [17]).

## 2.2 Orbits of Teichmüller modular groups

Except for the universal Teichmüller space $\mathcal{T}(\mathbb{D})$, which is the Teichmüller space of the unit disk $\mathbb{D}$, and for the Teichmüller spaces $\mathcal{T}(R)$ of the punctured disk or the three-punctured sphere $R$, no Teichmüller space $\mathcal{T}(R)$ is homogeneous in a sense that the

Teichmüller modular group $\mathrm{Mod}(R)$ acts transitively on $\mathcal{T}(R)$. Actually, these are the only hyperbolic Riemann surfaces which have no moduli. In the non-homogeneous case, the aspects of the dynamics of $\mathrm{Mod}(R)$ are different depending on the points $p \in \mathcal{T}(R)$, and the geometric structure of the Riemann surface corresponding to $p$ reflects the action of $\mathrm{Mod}(R)$ at $p$.

**Limit sets**

For an analytically finite Riemann surface $R$, the Teichmüller modular group $\mathrm{Mod}(R)$ acts on $\mathcal{T}(R)$ properly discontinuously. Although $\mathrm{Mod}(R)$ has fixed points on $\mathcal{T}(R)$, each orbit is discrete and each stabilizer subgroup is finite. Hence an orbifold structure on the moduli space $M(R)$ is induced from $\mathcal{T}(R)$ as the quotient space by $\mathrm{Mod}(R)$. However, this is not always satisfied for analytically infinite Riemann surfaces. Hence, we introduce the concept of limit set for the Teichmüller modular group $\mathrm{Mod}(R)$.

For a subgroup $\Gamma \subset \mathrm{Mod}(R)$, the orbit of $p \in \mathcal{T}(R)$ under $\Gamma$ is denoted by $\Gamma(p)$ and the stabilizer subgroup of $p$ in $\Gamma$ is denoted by $\mathrm{Stab}_\Gamma(p)$. In the case where $\Gamma = \mathrm{Mod}(R)$, $\mathrm{Stab}_\Gamma(p)$ is denoted by $\mathrm{Stab}(p)$. For an element $\gamma \in \Gamma$, the set of all fixed points of $\gamma$ is denoted by $\mathrm{Fix}(\gamma)$. The set of all common fixed points of the elements of $\Gamma$ is denoted by $\mathrm{Fix}(\Gamma)$.

**Definition 2.3.** For a subgroup $\Gamma \subset \mathrm{Mod}(R)$ and for a point $p \in \mathcal{T}(R)$, we say that $q \in \mathcal{T}(R)$ is a *limit point* of $p$ if there exists a sequence $\{\gamma_n\}$ of distinct elements of $\Gamma$ such that $\gamma_n(p)$ converge to $q$. The set of all limit points of $p$ is denoted by $\Lambda(\Gamma, p)$. The *limit set* for $\Gamma$ is defined by $\Lambda(\Gamma) = \bigcup_{p \in \mathcal{T}(R)} \Lambda(\Gamma, p)$ and the elements of $\Lambda(\Gamma)$ are called the limit points of $\Gamma$. A point $p \in \mathcal{T}(R)$ is a *recurrent point* of $\Gamma$ if $p \in \Lambda(\Gamma, p)$ and the set of all recurrent points of $\Gamma$ is denoted by $\mathrm{Rec}(\Gamma)$.

It follows from the definition that $\mathrm{Rec}(\Gamma) \subset \Lambda(\Gamma)$ and these sets are $\Gamma$-invariant. In fact, we have the following fact.

**Proposition 2.4** ([13], [18]). *For a subgroup $\Gamma \subset \mathrm{Mod}(R)$, the limit set $\Lambda(\Gamma)$ coincides with $\mathrm{Rec}(\Gamma)$ and it is a closed set. Moreover, $p \in \mathcal{T}(R)$ is a limit point of $\Gamma$ if and only if either the orbit $\Gamma(p)$ is not a discrete set or the stabilizer subgroup $\mathrm{Stab}_\Gamma(p)$ consists of infinitely many elements.*

The notion of limit set was originally defined for a Kleinian group and it was also defined for the iteration of a holomorphic function as the Julia set. Some properties of our limit set are common to the original settings but some are not. For instance, the limit set is the smallest invariant closed subset in the original setting, but this is not true for the case of Teichmüller modular groups.

**Discontinuity and stability**

The complement of the limit set should be defined as the region of discontinuity. Hence we define discontinuity of the action at a point $p \in \mathcal{T}(R)$ as the complementary condition for $p$ to be a limit point. By weakening the property of discreteness of the orbit, we also introduce another concept of manageable action, stability, which will be important for our arguments later.

**Definition 2.5.** Let $\Gamma$ be a subgroup of $\mathrm{Mod}(R)$. We say that $\Gamma$ acts at $p \in \mathcal{T}(R)$

- *discontinuously* if $\Gamma(p)$ is discrete and $\mathrm{Stab}_\Gamma(p)$ is finite;
- *weakly discontinuously* if $\Gamma(p)$ is discrete;
- *stably* if $\Gamma(p)$ is closed and $\mathrm{Stab}_\Gamma(p)$ is finite;
- *weakly stably* if $\Gamma(p)$ is closed.

If $\Gamma$ acts at every point $p$ in $\mathcal{T}(R)$ (weakly) discontinuously or stably, then we say that $\Gamma$ acts on $\mathcal{T}(R)$ (weakly) discontinuously or stably, respectively. The set of points $p \in \mathcal{T}(R)$ where $\Gamma$ acts discontinuously is denoted by $\Omega(\Gamma)$ and called the *region of discontinuity* for $\Gamma$. The set of points $p \in \mathcal{T}(R)$ where $\Gamma$ acts stably is denoted by $\Phi(\Gamma)$ and called the *region of stability* for $\Gamma$.

Note that $\Gamma \subset \mathrm{Mod}(R)$ acts at $p \in \mathcal{T}(R)$ discontinuously if and only if there exists a neighborhood $U$ of $p$ such that the number of elements $\gamma \in \Gamma$ satisfying $\gamma(U) \cap U \neq \emptyset$ is finite. When $\mathcal{T}(R)$ is locally compact (i.e., finite-dimensional), this condition is the same as proper discontinuity.

The discontinuity and stability criteria mentioned above have obvious inclusion relations that immediately follow from their definitions. The following theorem says that the converse assertion holds under a certain countability assumption. This fact is based on the Baire category theorem and the uncountability of perfect closed sets in a complete metric space.

**Lemma 2.6** ([42]). *Assume that $\Gamma \subset \mathrm{Mod}(R)$ contains a subgroup $\Gamma_0$ of countable index (that is, the cardinality of the cosets $\Gamma/\Gamma_0$ is countable) such that $\Gamma_0$ acts at $p \in \mathcal{T}(R)$ weakly discontinuously. If $\Gamma$ acts at $p$ (weakly) stably, then $\Gamma$ acts at $p$ (weakly, resp.) discontinuously.*

The region of discontinuity $\Omega(\Gamma)$ is always an open set because it is the complement of the limit set $\Lambda(\Gamma)$ as it follows from Proposition 2.4. However, we only see that the region of stability $\Phi(\Gamma)$ becomes an open set under a certain condition upon $\Gamma$. This is also obtained by an argument based on the Baire category theorem.

**Lemma 2.7** ([42]). *If $\Gamma \subset \mathrm{Mod}(R)$ contains a subgroup $\Gamma_0$ of countable index such that $\Gamma_0$ acts on $\mathcal{T}(R)$ stably, then the region of stability $\Phi(\Gamma)$ is open.*

We regard these two lemmata as fundamental principles of our arguments on the dynamics of Teichmüller modular groups and we utilize them in later discussion.

If $\Gamma$ acts discontinuously, then every subgroup of $\Gamma$ acts discontinuously. However, this property is not necessarily satisfied for stability. This is because any subset of a discrete set is always discrete whereas any subset of a closed set is not always closed. By this evidence, we have the following claim.

**Proposition 2.8.** *Let* $\{\Gamma_i\}_{i \in I}$ *be a family of subgroups of* $\mathrm{Mod}(R)$ *such that each* $\Gamma_i$ *acts stably at* $p \in \mathcal{T}(R)$. *Then the intersection* $\Gamma = \bigcap_{i \in I} \Gamma_i$ *acts stably at* $p \in \mathcal{T}(R)$.

### Boundedness and divergence

Now we consider another aspect of the dynamics of $\mathrm{Mod}(R)$. We classify the action of subgroups of $\mathrm{Mod}(R)$ according to the global behavior of their orbits.

**Definition 2.9.** Let $\Gamma$ be a subgroup of $\mathrm{Mod}(R)$. If the orbit $\Gamma(p)$ of some $p \in \mathcal{T}(R)$ is a bounded set in $\mathcal{T}(R)$, we say that $\Gamma$ is of *bounded type*. On the other hand, if the orbit $\Gamma(p)$ is divergent to the infinity of $\mathcal{T}(R)$, meaning that $\Gamma(p)$ is infinite and each bounded subset of $\mathcal{T}(R)$ contains only finitely many of $\Gamma(p)$, we say that $\Gamma$ is of *divergent type*.

Note that the notions of bounded type and of divergent type are well-defined for $\Gamma$ since these properties of the orbit are independent from the choice of $p \in \mathcal{T}(R)$.

When $R$ is analytically finite, $\mathcal{T}(R)$ is locally compact and $\mathrm{Mod}(R)$ acts properly discontinuously on $\mathcal{T}(R)$. Hence every infinite subgroup of $\mathrm{Mod}(R)$ is of divergent type and of course every finite subgroup of $\mathrm{Mod}(R)$ is of bounded type. However, for a general $R$, there are subgroups of $\mathrm{Mod}(R)$ neither of bounded type nor of divergent type even for infinite cyclic subgroups. See [36], where we have tried to give a classification of the Teichmüller modular transformations. If we restrict subgroups of $\mathrm{Mod}(R)$ to certain classes, then they have the dichotomy of boundedness and divergence. We will discuss later these classes having a nature similar to the finite-dimensional cases.

**Example 2.10.** Here we give an example of a Teichmüller modular transformation $\gamma \in \mathrm{Mod}(R)$ such that $\langle \gamma \rangle$ acts discontinuously on $\mathcal{T}(R)$ and it is neither of bounded type nor of divergent type.

Let $S$ be a closed hyperbolic surface of genus 3 and take three mutually disjoint non-dividing simple closed geodesics $a$, $b$ and $c$ on $S$. Cut $S$ along $a$ and $b$ to make a totally geodesic surface $S'$ of genus 1 with four boundary components, and give a pants decomposition for $S'$ having $c$ and the copies of $a$ and $b$ as boundary geodesics. We prepare copies of $S'$ and paste them to make an abelian covering surface $R_0$ of $S$ with the covering transformation group isomorphic to $\mathbb{Z}^2$. Then we index all the lifts of $c$ to $R_0$ in such a way that $c_{nk}$ is the image of some fixed lift $c_{00}$ under the covering transformation corresponding to $(n,k) \in \mathbb{Z}^2$. We extend the pants decomposition of $S'$ to $R_0$ such that the action of the covering transformation group $\mathbb{Z}^2$ preserves this decomposition.

By assigning the geodesic lengths $\ell(c_{nk})$ to each $c_{nk}$ and keeping the lengths of the other boundary curves of the pants decomposition invariant, we can obtain various hyperbolic Riemann surfaces $R$. This is performed by a locally quasiconformal deformation but it is not necessarily globally quasiconformal. A suitable choice of $\{\ell(c_{nk})\}$ gives an interesting example of a mapping class $[g]$ corresponding to the element of the covering transformation $n \mapsto n+1$, $k \mapsto k$ in $\mathbb{Z}^2$. For our purpose, we define

$$\ell(c_{nk}) = \exp\{-2^{|k|}\, h(2^{-|k|(|k|+1)/2} n)\},$$

where $h$ is a periodic function of period one defined on $\mathbb{R}$ such that $h(x) = x$ for $0 \le x \le 1/2$ and $h(x) = 1 - x$ for $1/2 \le x \le 1$. Then, we see that the mapping class $[g]$ gives a Teichmüller modular transformation $\gamma \in \mathrm{Mod}(R)$ such that $\langle \gamma \rangle$ acts discontinuously on $\mathcal{T}(R)$. Furthermore, we can find subsequences $\{n_i\}$ and $\{n_j\}$ such that $\{\gamma^{n_i}(p)\}$ is bounded and $\{\gamma^{n_j}(p)\}$ is divergent for any $p \in \mathcal{T}(R)$. See [21].

The classification by divergence and boundedness is more restrictive than that by discontinuity and instability.

**Proposition 2.11.** *If a subgroup $\Gamma \subset \mathrm{Mod}(R)$ is of divergent type, then $\Gamma$ acts discontinuously on $\mathcal{T}(R)$. On the contrary, if an infinite subgroup $\Gamma$ is of bounded type, then $\Gamma$ does not act stably on $\mathcal{T}(R)$.*

*Proof.* The first statement is obvious from the definition. The second statement will be seen later on by the arguments on elliptic subgroups (Theorem 2.34). □

## 2.3 Subgroups of Teichmüller modular groups

In this subsection, we list up several subgroups of the Teichmüller modular group, which have special properties with respect to their action on Teichmüller space. We intend to summarize a glossary of basic facts on their dynamics.

### Countable index subgroups

The following subgroup of countable index in $\mathrm{Mod}(R)$ plays an important role in our arguments for the application of Lemmata 2.6 and 2.7.

**Definition 2.12.** For a homotopically non-trivial simple closed curve $c$, we define $\mathrm{MCG}_c(R)$ to be a subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes that preserve $c$:

$$\mathrm{MCG}_c(R) = \{[g] \in \mathrm{MCG}(R) \mid g(c) \sim c\},$$

where $\sim$ means free homotopy equivalence. We denote the image of $\mathrm{MCG}_c(R)$ under the representation $\iota\colon \mathrm{MCG}(R) \to \mathrm{Mod}(R)$ by $\mathrm{Mod}_c(R)$.

**Proposition 2.13.** *For any non-trivial simple closed curve $c$ in $R$, the subgroup $\mathrm{MCG}_c(R)$ is of countable index in $\mathrm{MCG}(R)$. For an arbitrary subgroup $G \subset \mathrm{MCG}(R)$, there is a subgroup $G' \subset \mathrm{MCG}_c(R)$ of countable index in $G$.*

*Proof.* The countability of the indices comes from the fact that the number of free homotopy classes of non-trivial simple closed curves on $R$ is countable. The latter statement is obtained by taking the intersection of $G$ with $\mathrm{MCG}_c(R)$. □

As this proposition shows, we can say that $\sigma$-compactness of Riemann surfaces is at the basis of the countability involved in the dynamics of Teichmüller modular groups.

**Countable subgroups**

It is well known that the mapping class group of an analytically finite Riemann surface is finitely generated and in particular countable. However, for almost all analytically infinite Riemann surfaces, the quasiconformal mapping class groups are uncountable.

We consider countable subgroups of $\mathrm{Mod}(R)$. The fundamental lemma 2.6 includes the following claims in particular if $\Gamma \subset \mathrm{Mod}(R)$ itself is countable.

**Theorem 2.14.** *Let* $\Gamma$ *be a countable subgroup of* $\mathrm{Mod}(R)$. *Then* $\Gamma$ *acts* (*weakly*) *discontinuously at* $p \in \mathcal{T}(R)$ *if and only if* $\Gamma$ *acts* (*weakly, resp.*) *stably at* $p$. *In particular* $\Omega(\Gamma) = \Phi(\Gamma)$.

The following subgroup of $\mathrm{MCG}(R)$ can be regarded as the exhaustion of mapping class groups of topologically finite subsurfaces of $R$.

**Definition 2.15.** A mapping class $[g] \in \mathrm{MCG}(R)$ is called *trivial near infinity* (or *essentially trivial*) if some representative $g\colon R \to R$ is the identity outside some topologically finite bordered subsurface $V \subset R$ possibly having punctures such that $V$ is closed in $R \cup \partial_\infty R$. Let $\mathrm{MCG}_\infty(R)$ be the subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes trivial near infinity. Then we call it the *stable mapping class group*. The image of $\mathrm{MCG}_\infty(R)$ under the representation $\iota\colon \mathrm{MCG}(R) \to \mathrm{Mod}(R)$ is denoted by $\mathrm{Mod}_\infty(R)$.

Since $\mathrm{MCG}_\infty(R)$ admits an exhaustion by countable groups, it is countable. Moreover, $\mathrm{MCG}_\infty(R)$ is a normal subgroup of $\mathrm{MCG}(R)$. This group plays an important role when we consider the action of $\mathrm{MCG}(R)$ on the asymptotic Teichmüller space later on.

Under the assumption that $R$ satisfies the bounded geometry condition, we see that $\mathrm{MCG}_\infty(R)$ acts nicely on $\mathcal{T}(R)$ as the following theorem asserts.

**Theorem 2.16** ([16]). *If $R$ satisfies the bounded geometry condition, then* $\mathrm{Mod}_\infty(R)$ *acts discontinuously on* $\mathcal{T}(R)$. *Moreover, whenever $R$ is of infinite topological type, this action is fixed-point free.*

If $\mathrm{MCG}(R)$ itself is countable for a hyperbolic Riemann surface $R$ of infinite topological type, then the geometry of $R$ is much more restricted (in the opposite direction to the boundedness) by this assumption and we have the following stronger result. Note that the existence of such a Riemann surface $R$ is also known. To all appearances, this theorem is a generalization of the case where $R$ is analytically finite.

**Theorem 2.17** ([35]). *If* $\mathrm{Mod}(R)$ *is countable, then it acts discontinuously on* $\mathcal{T}(R)$.

In [37], an example of a Riemann surface $R$ of infinite topological type satisfying $\mathrm{MCG}_\infty(R) = \mathrm{MCG}(R)$ is given.

**Closed subgroups**

The compact-open topology on the space of all homeomorphic automorphisms of $R$ induces a topology on the quasiconformal mapping class group $\mathrm{MCG}(R)$. More precisely, we say that a sequence of mapping classes $[g_n] \in \mathrm{MCG}(R)$ converges to a mapping class $[g] \in \mathrm{MCG}(R)$ in the *compact-open topology* if we can choose representatives $g_n \in [g_n]$ and $g \in [g]$ such that $g_n$ converge to $g$ locally uniformly on $R$. When $R$ has boundary at infinity $\partial_\infty R$, assuming that the quasiconformal automorphisms $g_n$ and $g$ extend to $\partial_\infty R$, we further require that these extensions converge locally uniformly on $R \cup \partial_\infty R$. If $[g_n]$ converge to $[g]$ in the compact-open topology, then there are quasisymmetric automorphisms $\tilde{g}_n$ and $\tilde{g}$ of the unit circle $\partial\mathbb{D}$ corresponding to $[g_n]$ and $[g]$ respectively such that $\tilde{g}_n$ converge uniformly to $\tilde{g}$.

**Definition 2.18.** We say that a subgroup $G$ of $\mathrm{MCG}(R)$ is *discrete* if it is discrete in the compact-open topology on $\mathrm{MCG}(R)$, and *closed* if it is closed. For a subgroup $G$, we denote by $\overline{G}$ the closure of $G$ in $\mathrm{MCG}(R)$. We also use the same terminology for the corresponding subgroup $\Gamma = \iota(G)$ of $\mathrm{Mod}(R)$ and define the closure $\overline{\Gamma}$ by $\iota(\overline{G})$.

It is clear that the intersection $\bigcap_{i \in I} G_i$ of closed subgroups $G_i$ of $\mathrm{MCG}(R)$ is also closed.

For a point $p \in \mathcal{T}(R)$ such that the stabilizer $\mathrm{Stab}(p)$ in $\mathrm{Mod}(R)$ is trivial (see Theorem 2.37 in Section 2.4), the orbit of $p$ defines a topology on $\mathrm{Mod}(R)$ by using the Teichmüller distance on $\mathcal{T}(R)$. However, this topology does not coincide with the compact-open topology introduced above. In fact, the orbit $\overline{\Gamma}(p)$ for $p \in \mathcal{T}(R)$ does not necessarily coincide with the closure of the orbit $\Gamma(p)$ in the topology on $\mathcal{T}(R)$.

Closed subgroups have preferable properties. The following theorem provides an algebraic condition on $G$ for being closed.

**Theorem 2.19** ([41]). *Assume that $\partial_\infty R = \emptyset$. If $G \subset \mathrm{MCG}(R)$ is a finitely generated abelian group, then $G$ is discrete, and in particular closed.*

This result is no more valid for a countable group in general. The stable mapping class group $\mathrm{MCG}_\infty(R)$, which is countable, is not closed in almost all cases. In what follows, we will give a closed subgroup that contains $\mathrm{MCG}_\infty(R)$.

We consider the *end compactification* $R^*$ of a Riemann surface $R$ by adding all the ends of $R$ and by providing this union with the canonical topology. Here an *end* means a topological end if the boundary at infinity $\partial_\infty R$ is empty. However, if $\partial_\infty R \neq \emptyset$, we first consider the double $\widehat{R}$ of $R$ with respect to $\partial_\infty R$ and then take the closure of $R$ in the end compactification $\widehat{R}^*$ of $\widehat{R}$, which we define to be $R^*$. This has been introduced in [22]. Every quasiconformal automorphism of $R$ extends to a homeomorphic automorphism of $R^*$. Furthermore this extension preserves the cuspidal ends. The extension restricted to the ideal boundary $R^* - R$ is determined by the mapping class of the quasiconformal automorphism. Clearly, every mapping class trivial near infinity fixes all the ends except cuspidal ends.

**Definition 2.20.** The subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes that fix all the ends except the cuspidal ends is called the *pure mapping class group* and denoted by $\mathrm{MCG}_\partial(R)$.

**Proposition 2.21.** *The pure mapping class group* $\mathrm{MCG}_\partial(R)$ *is a closed normal subgroup of* $\mathrm{MCG}(G)$ *which contains* $\mathrm{MCG}_\infty(R)$.

*Proof.* The property of fixing the ends is preserved under convergence in the compact-open topology. □

**Stationary subgroups I: with closedness**

To generalize certain properties shared by the mapping class group of an analytically finite Riemann surface, we will consider a subgroup of $\mathrm{MCG}(R)$ that keeps the images of some compact bordered subsurface bounded.

**Definition 2.22.** We call a subgroup $G$ of $\mathrm{MCG}(R)$ *stationary* if there exists a compact bordered subsurface $V$ of $R$ such that every representative $g$ of every mapping class $[g] \in G$ satisfies $g(V) \cap V \neq \emptyset$. The corresponding subgroup $\Gamma = \iota(G)$ of $\mathrm{Mod}(R)$ is also called stationary.

It is clear from the definition that if $G$ is stationary, then so is the closure $\overline{G}$ in the compact-open topology.

A basic feature of stationary subgroups in connection with their closedness and discreteness can be summarized as the following theorem.

**Theorem 2.23.** *Let* $\Gamma$ *be a stationary subgroup of* $\mathrm{Mod}(R)$. *If* $\Gamma$ *is closed then it acts stably on* $\mathcal{T}(R)$. *If* $\Gamma$ *is infinite and discrete then it is of divergent type, and in particular it acts discontinuously on* $\mathcal{T}(R)$.

*Proof.* Compactness of a family of stationary quasiconformal automorphisms with uniformly bounded dilatations yields that if there is a sequence $[g_n]$ in $\mathrm{MCG}(R)$ such that $\gamma_n(p)$ is bounded in $\mathcal{T}(R)$ for $\gamma_n = [g_n]_*$ and for $p \in \mathcal{T}(R)$, then a subsequence of $[g_n]$ converges to some $[g] \in \mathrm{MCG}(R)$ in the compact-open topology.

Suppose that a stationary subgroup $\Gamma$ is closed. If $\gamma_n(p)$ converges to $q \in \mathcal{T}(R)$ for some sequence $\gamma_n \in \Gamma$ and for $p \in \mathcal{T}(R)$, then we see that $\gamma = [g]_*$ belongs to $\Gamma$ and $\gamma(p) = q$. This implies that the orbit $\Gamma(p)$ is closed for every $p \in \mathcal{T}(R)$. Since the stabilizer is finite for any stationary subgroup, $\Gamma$ acts stably on $\mathcal{T}(R)$. Suppose that $\Gamma$ is infinite and discrete. In this case, we see that there is no subsequence $\gamma_n \in \Gamma$ such that $\gamma_n(p)$ is bounded in $\mathcal{T}(R)$. This implies that $\Gamma$ is of divergent type. □

**Corollary 2.24.** *If a subgroup* $\Gamma$ *of* $\mathrm{Mod}(R)$ *is stationary, then* $\overline{\Gamma}$ *acts stably on* $\mathcal{T}(R)$ *and* $\overline{\Gamma}(p) = \overline{\Gamma(p)}$ *for every* $p \in \mathcal{T}(R)$.

*Proof.* Since $\overline{\Gamma}$ is stationary and closed, it acts stably on $\mathcal{T}(R)$ by Theorem 2.23. Then $\overline{\Gamma}(p)$ is closed for every $p \in \mathcal{T}(R)$, which gives $\overline{\Gamma}(p) \subset \overline{\Gamma(p)}$. To prove the converse inclusion, we take an arbitrary point $q \in \overline{\Gamma(p)}$ and consider a sequence $\gamma_n \in \Gamma$ such that $\gamma_n(p) \to q$ as $n \to \infty$. As in the proof of Theorem 2.23, since $\Gamma$ is stationary, we have $\gamma \in \overline{\Gamma}$ such that $\gamma(p) = q$. This implies $q \in \overline{\Gamma}(p)$. □

By imposing an algebraic condition on $\Gamma$ as before, we have another corollary obtained from Theorems 2.19 and 2.23.

**Corollary 2.25.** *Assume that* $\partial_\infty R = \emptyset$. *If a finitely generated infinite abelian group* $\Gamma \subset \mathrm{Mod}(R)$ *is stationary, then* $\Gamma$ *is of divergent type.*

Note that for an infinite cyclic group $\Gamma$ this has been proved in [36] without imposing the condition $\partial_\infty R = \emptyset$. We expect that the statement of the corollary is always valid without this assumption.

As an example of a stationary subgroup, we have the pure mapping class group $\mathrm{MCG}_\partial(R)$ in many cases as the following proposition states. See [16] for details. Recall that $\mathrm{MCG}_\partial(R)$ is also closed by Proposition 2.21.

**Proposition 2.26.** *Assume that* $R$ *has at least three non-cuspidal topological ends. Then the pure mapping class group* $\mathrm{MCG}_\partial(R)$ *is stationary. In this case,* $\mathrm{Mod}_\partial(R) = \iota(\mathrm{MCG}_\partial(R))$ *acts stably on* $\mathcal{T}(R)$.

*Proof.* This is because a pair of pants that divides three non-cuspidal topological ends has nonempty intersection with its image under every element of $\mathrm{MCG}_\partial(R)$. The latter statement is a consequence of Theorem 2.23. □

The subgroup $\mathrm{Mod}_c(R)$ defined before is closed and it is stationary if $R$ is non-elementary. Hence $\mathrm{Mod}_c(R)$ acts stably on $\mathcal{T}(R)$ by Theorem 2.23. More generally, we have the following.

**Lemma 2.27.** *Assume that* $R$ *is non-elementary. Each subgroup* $\Gamma$ *of* $\mathrm{Mod}(R)$ *contains a stationary subgroup* $\Gamma'$ *of countable index in* $\Gamma$. *In addition, if* $\Gamma$ *is closed, then* $\Gamma'$ *can be taken to be closed and hence acting stably on* $\mathcal{T}(R)$.

*Proof.* Set $\Gamma' = \Gamma \cap \mathrm{Mod}_c(R)$. Then this is stationary since so is $\mathrm{Mod}_c(R)$, and it is of countable index in $\Gamma$ by Proposition 2.13. Furthermore if $\Gamma$ is closed, then $\Gamma'$ is closed since $\mathrm{Mod}_c(R)$ is closed. □

We mentioned that the region of stability $\Phi(\Gamma)$ for a subgroup $\Gamma \subset \mathrm{Mod}(R)$ is not necessarily open. However, by applying Lemmata 2.7 and 2.27, we can now recognize a sufficient condition for the region of stability to be open.

**Theorem 2.28.** *For a closed subgroup* $\Gamma$ *of* $\mathrm{Mod}(R)$*, the region of stability* $\Phi(\Gamma)$ *is an open subset of* $\mathcal{T}(R)$*. In particular,* $\Phi(\mathrm{Mod}(R))$ *is open.*

### Stationary subgroups II: with bounded geometry

Another feature of a stationary subgroup is that under the bounded geometry condition it acts discontinuously on $\mathcal{T}(R)$. Note that we cannot drop any of the three assumptions in the bounded geometry condition (lower boundedness, upper boundedness and $\partial_\infty R = \emptyset$) for the validity of this claim.

**Theorem 2.29** ([23], [14]). *Let* $\Gamma$ *be a stationary subgroup of* $\mathrm{Mod}(R)$*. If* $R$ *satisfies the bounded geometry condition, then* $\Gamma$ *acts discontinuously on* $\mathcal{T}(R)$*.*

We apply this theorem to $\mathrm{Mod}_c(R)$. Then Lemma 2.6 implies the following result.

**Theorem 2.30.** *Assume that* $R$ *satisfies the bounded geometry condition. Then a subgroup* $\Gamma$ *of* $\mathrm{Mod}(R)$ *acts (weakly) discontinuously at* $p \in \mathcal{T}(R)$ *if and only if* $\Gamma$ *acts (weakly, resp.) stably at* $p$*. In particular* $\Omega(\Gamma) = \Phi(\Gamma)$*.*

We also see in the following theorem that $\Phi(\Gamma) = \Omega(\Gamma)$ is non-empty in this case, which has been proved in [13]. Later we will see a stronger assertion that $\Phi(\mathrm{Mod}(R))$ is always dense in $\mathcal{T}(R)$ without any assumption on the geometry of $R$.

**Theorem 2.31.** *If* $R$ *satisfies the bounded geometry condition, then, for any subgroup* $\Gamma$ *of* $\mathrm{Mod}(R)$*,* $\Omega(\Gamma)$ *is non-empty.*

*Proof.* We only have to show the statement for $\Gamma = \mathrm{Mod}(R)$. Since $R$ satisfies the lower boundedness condition, we choose an arbitrary simple closed geodesic $c$ on $R$ and give a deformation of the hyperbolic structure by pinching $c$ such that it becomes the unique shortest simple closed geodesic with respect to the new hyperbolic structure. Let $p$ be the corresponding point on $\mathcal{T}(R)$.

For a neighborhood $U$ of $p$, we consider the smallest subgroup $\Gamma_0$ of $\mathrm{Mod}(R)$ that contains $\{\gamma \in \mathrm{Mod}(R) \mid \gamma(p) \in U\}$. If $U$ is sufficiently small, $\Gamma_0$ is contained in $\mathrm{Mod}_c(R)$. Since $\mathrm{Mod}_c(R)$ acts discontinuously on $\mathcal{T}(R)$ by Theorem 2.29, so does $\Gamma_0$ and hence $\mathrm{Mod}(R)$ acts discontinuously at $p$. □

If we do not assume a geometric condition on $R$, this result is not satisfied any more. For instance, if $R$ does not satisfy the lower boundedness condition or if the boundary at infinity $\partial_\infty R$ is not empty, then $\Omega(\mathrm{Mod}(R)) = \emptyset$. As a conjecture, we expect that the converse of this claim is also true.

**Conjecture 2.32.** *The region of discontinuity* $\Omega(\mathrm{Mod}(R))$ *is not empty if and only if* $R$ *satisfies the lower boundedness condition together with* $\partial_\infty R = \emptyset$*.*

**Elliptic subgroups**

If a Teichmüller modular transformations $[g]_* \in \mathrm{Mod}(R)$ has a fixed point on $\mathcal{T}(R)$, then it is called *elliptic* according to Bers [1]. This is equivalent to a condition that the mapping class $[g] \in \mathrm{MCG}(R)$ is realized as a conformal automorphism of the Riemann surface $f(R)$ corresponding to $p = [f]$, that is, $fgf^{-1}$ is homotopic to a conformal automorphism of $f(R)$ (relative to the boundary at infinity if it is not empty). Such a mapping class $[g]$ is called a *conformal mapping class*. When $R$ is an analytically finite Riemann surface, $[g]_* \in \mathrm{Mod}(R)$ is elliptic if and only if it is periodic (of finite order). We extend the concept of ellipticity to the case where $R$ is not necessarily analytically finite. In this case, elliptic modular transformations can be of infinite order.

**Definition 2.33.** A subgroup $\Gamma$ of $\mathrm{Mod}(R)$ is called elliptic if it has a common fixed point on $\mathcal{T}(R)$.

If $\Gamma = \iota(G)$ is an elliptic subgroup of $\mathrm{Mod}(R)$ fixing $p = [f] \in \mathcal{T}(R)$, then the subgroup $G$ of $\mathrm{MCG}(R)$ is realized as a group of conformal automorphisms of $f(R)$. Hence $G$ is a countable group. Furthermore $G$ is discrete in the compact-open topology, and if $G$ is an infinite group, then it is not stationary.

We characterize elliptic subgroups by their orbits. It is clear that any orbit $\Gamma(q)$ of an elliptic subgroup $\Gamma$ is bounded since $d_T(\gamma(q), p) = d_T(q, p)$ for a common fixed point $p$ and for all $\gamma \in \Gamma$. The following theorem says that the converse is also true.

**Theorem 2.34.** *A subgroup $\Gamma$ of $\mathrm{Mod}(R)$ is elliptic if and only if $\Gamma$ is of bounded type, that is, the orbit $\Gamma(p)$ for some $p \in \mathcal{T}(R)$ is bounded.*

In the case where $R$ is analytically finite, the *Nielsen realization theorem*, which was finally proved by Kerckhoff [29], is equivalent to saying that every finite subgroup of $\mathrm{Mod}(R)$ is elliptic. The realization theorem says that every finite subgroup of $\mathrm{MCG}(R)$ is realized as a group of conformal automorphisms of a Riemann surface corresponding to the fixed point. Since a finite subgroup has a bounded orbit, Theorem 2.34 can be regarded as a generalization of the realization theorem. The proof is essentially based on a theorem due to Markovic [32], which asserts that a uniformly quasisymmetric group on the unit circle $\partial\mathbb{D}$ is conjugate to a Fuchsian group by a quasisymmetric homeomorphism.

Next, we see that most infinite elliptic subgroups $\Gamma$ have an indiscrete orbit in $\mathcal{T}(R)$. Since $\Gamma$ is countable, Theorem 2.14 implies that this is equivalent to the statement that the orbit is not closed.

**Theorem 2.35.** *Assume that an elliptic subgroup $\Gamma \subset \mathrm{Mod}(R)$ has an infinite descending sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of proper subgroups $\Gamma \supsetneqq \Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots$. Then the union $X = \bigcup_{n \geq 1} \mathrm{Fix}(\Gamma_n)$ is not closed in $\mathcal{T}(R)$ and, at each $p \in \overline{X} - X$, $\Gamma$ does not act weakly discontinuously, in other words, the orbit $\Gamma(p)$ is not a discrete set. In* *particular, if an elliptic subgroup $\Gamma$ contains an element of infinite order, the above assumption is always satisfied and the conclusion is valid.*

*Proof.* The strict inclusion relation $\Gamma_n \supsetneqq \Gamma_{n+1}$ gives the strict inclusion relation $\mathrm{Fix}(\Gamma_n) \subsetneqq \mathrm{Fix}(\Gamma_{n+1})$ for every $n$ by passing to a subsequence if necessary. This has been proved in [34]. Then the Baire category theorem implies that $X$ is not closed and $\overline{X} - X$ is dense in $\overline{X}$. Take any point $p \in \overline{X} - X$ and consider a sequence $\{\gamma_n(p)\}_{n=1}^{\infty}$ for $\gamma_n \in \Gamma_n - \Gamma_{n+1}$. Then we see that $\gamma_n(p) \neq p$ and $\lim_{n\to\infty} \gamma_n(p) = p$. This shows that the orbit $\Gamma(p)$ is not a discrete set. □

Note that an arbitrary countable group can be realized as a group of conformal automorphisms of some Riemann surface (cf. [27]). Hence there is an example of an infinite elliptic subgroup $\Gamma$ that does not contain an infinite descending sequence of proper subgroups. For these groups, we do not know whether their orbits are discrete or not.

Finally, we show that each element of the countable group $\mathrm{Mod}_\infty(R)$, which comes from a mapping class trivial near infinity, can be represented by the composition of elliptic modular transformations of infinite order if they exist.

**Theorem 2.36.** *Assume that* $\mathrm{Mod}(R)$ *contains an elliptic modular transformation of infinite order. Then any element of the countable subgroup* $\mathrm{Mod}_\infty(R)$ *can be written as a composition of some elliptic elements of infinite order of* $\mathrm{Mod}(R)$.

*Proof.* For an elliptic modular transformation $\gamma \in \mathrm{Mod}(R)$ of infinite order, we may assume that it fixes the base point $o \in \mathcal{T}(R)$ and hence the corresponding mapping class is realized as a conformal automorphism $g$ of $R$. Take an arbitrary simple closed geodesic $c$ in $R$. Since $\langle g \rangle$ acts on $R$ properly discontinuously, there is an integer $k \neq 0$ such that $g^{kn}(c) \cap c = \emptyset$ for every integer $n \neq 0$. Then, for $\hat{g} = g^k$, a family of simple closed geodesics $\{\hat{g}^n(c)\}_{n\in\mathbb{Z}}$ are mutually disjoint. Moreover in this case, there is a collar neighborhood $A_c$ of $c$ such that $\{\hat{g}^n(A_c)\}_{n\in\mathbb{Z}}$ are mutually disjoint. We rename $\hat{g}$ as $g$ and reset $\gamma = [g]_*$.

For each $n \in \mathbb{Z}$, let $t_{g^n(c)}$ be a quasiconformal automorphism of $R$ that is obtained by a Dehn twist supported on $g^n(A_c) = A_{g^n(c)}$. We set $g_c = g \circ t_c$ and define $\gamma_c = [g_c]_*$. Since $g^{-1} \circ t_c \circ g = t_{g^{-1}(c)}$, we have

$$g_c^n = g^n \circ t_{g^{-(n-1)}(c)} \circ \cdots \circ t_{g^{-1}(c)} \circ t_c$$

for every integer $n \geq 1$. From this expression, we see that the maximal dilatation of $g_c^n$ is equal to that of $t_c$ because the support of the quasiconformal automorphisms $t_{g^{-i}(c)}$ $(0 \leq i \leq n-1)$ are mutually disjoint. This implies that the orbit $\{\gamma_c^n(o)\}_{n\in\mathbb{Z}}$ is bounded. Hence, by Theorem 2.34, $\gamma_c$ is also an elliptic modular transformation. It is easy to see that the order of $\gamma_c$ is infinite.

Since $t_c = g^{-1} \circ g_c$, the corresponding modular transformation $\tau_c = [t_c]_*$ is written as a composition of elliptic elements. Every element of $\mathrm{MCG}_\infty(R)$ can be written as a composition of mapping classes obtained by Dehn twists along simple closed geodesics because this is true for the pure mapping class group of any compact bordered surface. Since every modular transformation $\tau_c$ corresponding to a Dehn twist is the composition of elliptic elements of infinite order, we see that every element of $\mathrm{Mod}_\infty(R)$ is also written as a composition of such elements. □

Note that no non-trivial elliptic modular transformations belong to $\mathrm{Mod}_\infty(R)$ if $R$ is analytically infinite.

## 2.4 Application to the infinite-dimensional Teichmüller theory

We gave some basic concepts in the dynamics of the Teichmüller modular group. They have a general theoretical nature and will be developed by finding interesting applications to Teichmüller theory. Here we review some of such applications.

**Fixed point loci of Teichmüller modular groups**

For a finite-dimensional Teichmüller space $\mathcal{T}(R)$, it is well known that the union of the fixed point loci of all non-trivial elements of the mapping class group $\mathrm{MCG}(R)$ is nowhere dense in $\mathcal{T}(R)$ except in a few cases of low dimensions. For instance, if $R$ is a closed Riemann surface of genus 2, there exists an involution $[g]$ in $\mathrm{MCG}(R)$ that fixes all the points of $\mathcal{T}(R)$. A Riemann surface having such a symmetry is called an exceptional surface. The representation $\iota\colon \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{T}(R))$ is injective for a non-exceptional Riemann surface $R$.

For an infinite-dimensional Teichmüller space, a claim analogous to the above statement says that the union of the fixed point loci of $\mathrm{MCG}(R)$ is contained in a countable union of nowhere dense subsets. This has been proved by Epstein [11]. The complement of this countable union is called a residual set which is dense in $\mathcal{T}(R)$. The existence of a point in the residual set where the isotropy subgroup of $\mathrm{MCG}(R)$ is trivial in particular shows that the representation $\iota\colon \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{T}(R))$ is injective.

On the other hand, from a viewpoint of dynamical systems, we can consider a problem of whether the set of points $p \in \mathcal{T}(R)$ such that $\mathrm{Stab}(p)$ has an element of infinite order is dense in the limit set of $\mathrm{Mod}(R)$, in analogy with the same question in the dynamics of Kleinian groups. However, it is proved in [42] that this is not true for the dynamics of Teichmüller modular groups. This means that the fixed point loci of $\mathrm{Mod}(R)$ is a thinner set even in the limit set and similar arguments for proving this fact give the following extension of the aforementioned result.

**Theorem 2.37** ([42]). *The interior of the set of points $p \in \mathcal{T}(R)$ for which* $\mathrm{Stab}(p)$ *is trivial is dense in $\mathcal{T}(R)$.*

**Biholomorphic automorphisms of Teichmüller spaces**

Now it is known that every biholomorphic automorphism of the Teichmüller space of dimension greater than one is a Teichmüller modular transformation. For finite-dimensional Teichmüller spaces, this result was first proved by Royden [44]. In the general case, the proof is carried out by the combination of two theorems. Earle and Gardiner [3] proved the automorphism theorem, which asserts that the above claim is true if a Riemann surface satisfies a so-called isometry property. Then Markovic [31] finally proved that every non-exceptional Riemann surface satisfies the isometry property. See the exposition in Volume II of this Handbook [12]. See also [9] for an adaptation of this idea to the proof in the finite-dimensional case.

If we assume the isometry property against the chronological order, then we can state an essential part of the automorphism theorem as follows.

**Theorem 2.38.** *Assume that the Teichmüller space $\mathcal{T}(R)$ has dimension greater than one. Then, for every biholomorphic automorphism $\phi$ of $\mathcal{T}(R)$ and for every point $p \in \mathcal{T}(R)$, there exists an element $\gamma_p \in \mathrm{Mod}(R)$ such that $\phi(p) = \gamma_p(p)$.*

For an analytically finite Riemann surface, once this theorem is proved (actually Royden's arguments imply this statement), then it is easy to obtain the result that any biholomorphic automorphism is a Teichmüller modular transformation. This is due to the fact that the Teichmüller modular group acts properly discontinuously in this case. However, in the general case, we still need an extra argument to reach the desired result. This step was included in the proof of the automorphism theorem in [3]. Fujikawa [15] found that there is a certain case where the original argument of Royden can be applied without change. The assumption for this case is described by using the region of discontinuity of the Teichmüller modular group.

**Theorem 2.39.** *For a biholomorphic automorphism $\phi$ of $\mathcal{T}(R)$, assume that there exists a subgroup $\Gamma$ of $\mathrm{Mod}(R)$ with $\Omega(\Gamma) \neq \emptyset$ such that, for every point $p \in \Omega(\Gamma)$, there is an element $\gamma_p \in \Gamma$ satisfying $\phi(p) = \gamma_p(p)$. Then $\phi$ coincides with an element of $\Gamma$.*

*Proof.* Choose a point $p \in \Omega(\Gamma)$. The stabilizer of $p$ is a finite group in general and this does not make a trouble in the proof as is shown in [15], but here we assume that the stabilizer is trivial for the sake of simplicity. Actually, the choice of such a $p$ is always possible because the set of all such $p$ is dense in $\mathcal{T}(R)$ as is seen before.

Take a disk $U_p(r)$ with center at $p$ and radius $r > 0$ such that $\gamma(U_p(r)) \cap U_p(r) = \emptyset$ for every non-trivial $\gamma \in \Gamma$. Then consider a smaller disk $U_p(r/2)$ of radius $r/2$ and choose an arbitrary point $q \in U_p(r/2)$. It is clear that $d_T(\gamma(q), q) > r$ for every non-trivial $\gamma \in \Gamma$. We consider $\gamma_p^{-1} \circ \gamma_q \in \Gamma$ and, from the fact that the biholomorphic automorphism $\phi$ preserves the Kobayashi distance on $\mathcal{T}(R)$, we have

$$\begin{aligned} d_T(\gamma_p^{-1} \circ \gamma_q(q), q) &= d_T(\gamma_q(q), \gamma_p(q)) \\ &\leq d_T(\gamma_q(q), \gamma_p(p)) + d_T(\gamma_p(p), \gamma_p(q)) \\ &= d_T(\phi(q), \phi(p)) + d_T(p, q) < r. \end{aligned}$$

This estimate implies that $\gamma_p^{-1} \circ \gamma_q$ should be trivial and hence $\gamma_q = \gamma_p$ for every $q \in U_p(r/2)$. Therefore $\phi = \gamma_p$ restricted to $U_p(r/2)$. Then, by the rigidity of holomorphic functions, we conclude that $\phi$ coincides with $\gamma_p$ on $\mathcal{T}(R)$. □

In order to apply this theorem, it is necessary to find geometric or algebraic conditions under which $\Omega(\Gamma)$ is not empty. Theorem 2.31 says that this is true if $R$ satisfies the bounded geometry condition. Here we give the following condition on $\Gamma$ to guarantee $\Omega(\Gamma) \neq \emptyset$. The assumption $\Omega(\Gamma) = \Phi(\Gamma)$ is satisfied, for example, when $\Gamma$ is countable by Theorem 2.14.

**Lemma 2.40.** *Assume that $R$ is non-elementary. If $\Gamma \subset \mathrm{Mod}(R)$ is a closed subgroup such that $\Omega(\Gamma) = \Phi(\Gamma)$, then $\Omega(\Gamma) \neq \emptyset$.*

*Proof.* As in the proof of Theorem 2.31, we can choose a neighborhood $U$ of some $p \in \mathcal{T}(R)$ and a simple closed geodesic $c$ on $R$ so that the smallest subgroup $\Gamma_0$ that contains $\{\gamma \in \Gamma \mid \gamma(p) \in U\}$ is contained in $\mathrm{Mod}_c(R)$. Note that this is possible even if $R$ does not satisfy the bounded geometry condition. See [42].

Take the closure $\overline{\Gamma}_0$ of $\Gamma_0$. Since $\mathrm{Mod}_c(R)$ and $\Gamma$ are closed, $\overline{\Gamma}_0$ is contained in both of them. Since $\mathrm{Mod}_c(R)$ is stationary, so is $\overline{\Gamma}_0$. Hence $\overline{\Gamma}_0$ is stationary and closed, and by Theorem 2.23, $\overline{\Gamma}_0$ acts stably on $\mathcal{T}(R)$. This implies that $\Gamma$ acts stably at $p$ and thus $\Phi(\Gamma) \neq \emptyset$. □

# 3 The action on the asymptotic Teichmüller space

We regard Teichmüller space as a fiber space over a certain base space called the asymptotic Teichmüller space. An asymptotically conformal homeomorphism of a Riemann surface is a quasiconformal map that is close to a conformal map as we go to the infinity of the surface. The asymptotic Teichmüller space is defined by replacing the roles of conformal homeomorphisms with asymptotically conformal ones in the definition of the Teichmüller space. Since the quasiconformal mapping class group acts on the Teichmüller space preserving the fibers, its action can be divided into that on each fiber and that on the asymptotic Teichmüller space. In this section, we are mainly concerned with the former action, which is given by a group of asymptotically conformal mapping classes. It acts on the Teichmüller space as an asymptotically elliptic subgroup of the Teichmüller modular group, having certain similarity to Teichmüller modular groups of analytically finite Riemann surfaces.

## 3.1 Asymptotic Teichmüller spaces and modular groups

We introduce the asymptotic Teichmüller space and define the action of the quasiconformal mapping class group on this space.

### Asymptotic Teichmüller spaces

The asymptotic Teichmüller space has been introduced by Gardiner and Sullivan [26] for the unit disk and by Earle, Gardiner and Lakic [4], [5], [6] for an arbitrary Riemann surface.

**Definition 3.1.** We say that a quasiconformal homeomorphism $f$ of a Riemann surface $R$ is *asymptotically conformal* if, for every $\epsilon > 0$, there exists a compact bordered subsurface $V$ of $R$ such that the maximal dilatation $K(f|_{R-V})$ of the restriction of $f$ to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms $f_1$ and $f_2$ of $R$ are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h\colon f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity (relative to $\partial_\infty R$ if $\partial_\infty R \neq \emptyset$). The *asymptotic Teichmüller space* $\mathcal{AT}(R)$ of $R$ is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms $f$ of $R$.

Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\alpha\colon \mathcal{T}(R) \to \mathcal{AT}(R)$ that maps each Teichmüller equivalence class $[f] \in \mathcal{T}(R)$ to the asymptotic equivalence class $[[f]] \in \mathcal{AT}(R)$. The asymptotic Teichmüller space $\mathcal{AT}(R)$ has a complex manifold structure such that $\alpha$ is holomorphic. Each fiber of the projection $\alpha$ is a separable closed subspace of $\mathcal{T}(R)$. Moreover $\alpha$ induces a quotient distance $d_{AT}$ on $\mathcal{AT}(R)$ from the Teichmüller distance, which is called the *asymptotic Teichmüller distance*. We do not know yet whether this distance coincides with the Kobayashi distance on $\mathcal{AT}(R)$ or not. See [5], [6] and [8].

The asymptotic Teichmüller space $\mathcal{AT}(R)$ is of interest only when $R$ is analytically infinite. Otherwise $\mathcal{AT}(R)$ is trivial, that is, it consists of just one point. Conversely, if $R$ is analytically infinite, then $\mathcal{AT}(R)$ is not trivial. In fact, it is infinite-dimensional and non-separable.

### Asymptotic Teichmüller modular groups

Like in the case of Teichmüller space, every mapping class $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $\mathcal{AT}(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to $d_{AT}$. Note that since the projection $\alpha\colon \mathcal{T}(R) \to \mathcal{AT}(R)$ is not known to be a holomorphic split submersion, the fact that $[g]_{**}$ is holomorphic is not so trivial. See [6] and [7].

**Definition 3.2.** Let $\mathrm{Aut}(\mathcal{AT}(R))$ be the group of all biholomorphic isometric automorphisms of $\mathcal{AT}(R)$. For a homomorphism

$$\iota_{AT}\colon\ \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{AT}(R))$$

given by $[g] \mapsto [g]_{**}$, we define the *asymptotic Teichmüller modular group* $\mathrm{Mod}_{AT}(R)$ of $R$ to be the image $\iota_{AT}(\mathrm{MCG}(R))$.

Unlike the representation $\iota\colon\ \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{T}(R))$, the homomorphism $\iota_{AT}$ is not injective, namely, $\mathrm{Ker}\ \iota_{AT} \neq \{[\mathrm{id}]\}$ unless $R$ is either the unit disc or the once-punctured disc. See [4].

**Definition 3.3.** We call an element of $\mathrm{Ker}\ \iota_{AT}$ *asymptotically trivial* and call $\mathrm{Ker}\ \iota_{AT}$ the *asymptotically trivial mapping class group*. We also call an element of the corresponding subgroup $\iota(\mathrm{Ker}\ \iota_{AT})$ of $\mathrm{Mod}(R)$ asymptotically trivial.

The action of $\mathrm{Mod}_{AT}(R)$ on $\mathcal{AT}(R)$ has been studied by Fujikawa [17]. In particular, the limit set of $\mathrm{Mod}_{AT}(R)$ in $\mathcal{AT}(R)$ is investigated.

### 3.2 Asymptotically elliptic subgroups

In order to investigate the action of the quasiconformal mapping class group on a fiber over the asymptotic Teichmüller space, we consider the stabilizer subgroup of the fiber in the Teichmüller modular group. The projection of this subgroup to the asymptotic Teichmüller modular group fixes the base point of the fiber on the asymptotic Teichmüller space.

#### Asymptotically elliptic modular transformations

Now we define asymptotic conformality for quasiconformal mapping classes and asymptotic ellipticity for Teichmüller modular transformations.

**Definition 3.4.** A mapping class $[g] \in \mathrm{MCG}(R)$ is called *asymptotically conformal* if there is a quasiconformal homeomorphism $f$ of $R$ such that $fgf^{-1}$ is homotopic to an asymptotically conformal automorphism of $f(R)$ (relative to the boundary at infinity if it is not empty). A Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is called *asymptotically elliptic* if $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ has a fixed point $[[f]]$ on $\mathcal{AT}(R)$.

It is clear that a mapping class $[g] \in \mathrm{MCG}(R)$ is asymptotically conformal if and only if the corresponding Teichmüller modular transformation $[g]_* \in \mathrm{Mod}(R)$ is asymptotically elliptic. An elliptic modular transformation is of course asymptotically elliptic. However the converse is not true. A trivial example is a mapping class caused by a single Dehn twist. This is not a conformal mapping class, but it acts trivially on $\mathcal{AT}(R)$. In particular, it has a fixed point on $\mathcal{AT}(R)$. Petrovic [45] first dealt with an asymptotically conformal mapping class that acts on $\mathcal{AT}(R)$ non-trivially (in fact non-periodically) and that has no fixed point on $\mathcal{T}(R)$. See also [40] for another example.

When $R$ is analytically finite, every Teichmüller modular transformation is asymptotically elliptic since $\mathcal{AT}(R)$ consists of a single point. Asymptotically elliptic modular transformations are generalization of the Teichmüller modular transformations of analytically finite Riemann surfaces in this sense.

Similarly, we define asymptotic ellipticity for subgroups of $\mathrm{Mod}(R)$.

**Definition 3.5.** A subgroup $\Gamma = \iota(G)$ of $\mathrm{Mod}(R)$ is called asymptotically elliptic if $\iota_{AT}(G) \subset \mathrm{Mod}_{AT}(R)$ has a common fixed point on $\mathcal{AT}(R)$.

It is clear from the definition that a subgroup consisting of asymptotically trivial modular transformations is asymptotically elliptic.

Elliptic subgroups are always countable. For asymptotically elliptic subgroups, this is not valid in general, but if we impose the bounded geometry condition on $R$, this is true as is shown in [38].

**Theorem 3.6.** *Assume that $R$ satisfies the bounded geometry condition. Then every asymptotically elliptic subgroup $\Gamma$ of* $\mathrm{Mod}(R)$ *is countable.*

*Proof.* By Lemma 2.27, we can take a stationary subgroup $\Gamma'$ of countable index in $\Gamma$. If $\Gamma$ is uncountable, then so is $\Gamma'$. On the other hand, $\Gamma'$ acts discontinuously on $\mathcal{T}(R)$ by Theorem 2.29. In particular, the uncountable group $\Gamma'$ acts discontinuously on the fiber over the fixed point on $\mathcal{AT}(R)$, which is separable. However, this is impossible. □

Like in the case where $\mathrm{Mod}(R)$ is countable, if the entire $\mathrm{Mod}(R)$ is asymptotically elliptic, this restrictive condition gives us a stronger consequence.

**Theorem 3.7** ([38]). *If* $\mathrm{Mod}(R)$ *itself is asymptotically elliptic, then* $\mathrm{Mod}(R)$ *is countable and acts discontinuously on $\mathcal{T}(R)$.*

There is an example of $R$ such that $\mathrm{Mod}(R)$ is asymptotically elliptic. Furthermore, the entire $\mathrm{Mod}(R)$ can be asymptotically trivial. See [35] and [37] for these examples.

### The action on the fiber

We consider the action of an asymptotically elliptic subgroup $\Gamma \subset \mathrm{Mod}(R)$ restricted to the fiber over the fixed point on $\mathcal{AT}(R)$. For any point $p \in \mathcal{T}(R)$, we denote the fiber of the projection $\alpha\colon \mathcal{T}(R) \to \mathcal{AT}(R)$ containing $p$ by $T_p$, that is, $T_p = \alpha^{-1}(\alpha(p))$.

If $\Gamma \subset \mathrm{Mod}(R)$ is asymptotically elliptic having a common fixed point $\alpha(p) \in \mathcal{AT}(R)$, then $\Gamma$ preserves the fiber $T_p$. We investigate an abelian action of such a subgroup and obtain the following.

**Theorem 3.8.** *Assume that $\partial_\infty R = \emptyset$. Let $\Gamma$ be an asymptotically elliptic subgroup of* $\mathrm{Mod}(R)$ *that is finitely generated infinite abelian. Then, for every point $p \in \mathcal{T}(R)$ over the fixed point of $\Gamma$ on $\mathcal{AT}(R)$, one of the following alternative conditions is satisfied:*

(1) *$\Gamma$ fixes $p$;*

(2) *$\Gamma$ acts discontinuously at $p$ and the orbit $\Gamma(p)$ is bounded;*

(3) *$\Gamma(p)$ is divergent, that is, $\Gamma$ is of divergent type.*

*In any case, $\Gamma(p)$ is a discrete set.*

Before the proof of Theorem 3.8, we extend the definition of the stationary property for a subgroup of $\mathrm{MCG}(R)$ to any sequence of mapping classes. A sequence $\{[g_i]\}_{i=1}^{\infty}$ in $\mathrm{MCG}(R)$ is called *stationary* if there exists a compact bordered subsurface $V$ of $R$ such that every representative $g_i$ of each mapping class $[g_i]$ satisfies $g_i(V) \cap V \neq \emptyset$. On the contrary, a sequence $\{[g_i]\}_{i=1}^{\infty}$ is called *escaping* if, for every compact bordered subsurface $V$ of $R$, there exists some representative $g_i$ of each mapping class $[g_i]$

such that $\{g_i(V)\}$ diverges to the infinity of $R$ as $i \to \infty$. Note that a sequence $\{[g_i]\} \subset \mathrm{MCG}(R)$ can be neither stationary nor escaping, but we can always choose a subsequence that is either stationary or escaping. A sequence $\{\gamma_i\}$ in $\mathrm{Mod}(R)$ is also called stationary or escaping if so is $\{[g_i]\} \subset \mathrm{MCG}(R)$ for $\gamma_i = [g_i]_*$.

*Proof.* If $\Gamma$ is stationary, then Corollary 2.25 says that $\Gamma$ is of divergent type. This is also true for any stationary subsequence $\{\gamma_i\}$ in $\Gamma$ and we see that $\{\gamma_i(p)\}$ diverges to the infinity of $\mathcal{T}(R)$ for such a sequence.

Suppose that there is a subsequence $\{\gamma_i\}$ in $\Gamma$ such that $\{\gamma_i(p)\}$ has an accumulation point in $T_p$. By replacing the subsequence if necessary, we may assume that $\{\gamma_i(p)\}$ converges to $p$. Moreover, by the previous paragraph, we see that this subsequence $\{\gamma_i\}$ is escaping. Then we use Lemma 3.9 below to show that the whole group $\Gamma$ fixes the point $p$. This is the situation of Condition (1).

Next, suppose that there is a subsequence $\{\gamma_i\}$ in $\Gamma$ such that $\{\gamma_i(p)\}$ is bounded in $T_p$. Then $\{\gamma_i\}$ should be an escaping subsequence as before, and in this case, we see by Lemma 3.9 that the whole orbit $\Gamma(p)$ is bounded. This is the situation of either Conditions (1) or (2). By excluding the case discussed in the previous paragraph, we have Condition (2).

Finally, if there is no subsequence $\{\gamma_i\}$ in $\Gamma$ such that $\{\gamma_i(p)\}$ is bounded, then this means that the orbit $\Gamma(p)$ is divergent. This is the situation of Condition (3). □

**Lemma 3.9** ([36], [40]). *Assume that $\partial_\infty R = \emptyset$. Let $\Gamma$ be an asymptotically elliptic abelian subgroup of* $\mathrm{Mod}(R)$. *Let $\{\gamma_i\}$ be an escaping sequence of $\Gamma$. Then the following are satisfied for any point $p \in \mathcal{T}(R)$ over the fixed point of $\Gamma$ on $\mathcal{AT}(R)$.*

- *If $\{\gamma_i(p)\}$ converges to $p$, then $\Gamma$ fixes $p$.*
- *If $\{\gamma_i(p)\}$ is bounded, then $\Gamma$ is of bounded type.*

*In both cases, $\Gamma$ is elliptic.*

By Theorem 2.34, we see that Conditions (1) or (2) of Theorem 3.8 occur if and only if $\Gamma$ is elliptic. Note that there is a case where $\Gamma$ satisfies (2) but has no fixed point in $T_p$, which is shown in [39]. Condition (3) occurs if and only if $\Gamma$ is asymptotically elliptic but not elliptic. In this case, $\Gamma$ acts discontinuously on $\mathcal{T}(R)$. This gives the following corollary.

**Corollary 3.10.** *Assume that $\partial_\infty R = \emptyset$. Let $\Gamma$ be an asymptotically elliptic subgroup of* $\mathrm{Mod}(R)$ *that is finitely generated infinite abelian. Then either $\Gamma$ is elliptic or $\Gamma$ acts discontinuously on $\mathcal{T}(R)$.*

Note that if $R$ satisfies the bounded geometry condition in Theorem 3.8 and Corollary 3.10, then we can weaken the assumption on $\Gamma$ for the same claim. Namely, we have only to assume that $\Gamma$ is an infinite abelian group. This is based on Theorem 2.29.

As an application of the previous facts, we have the following result, which has been obtained in [39] and [19]. We believe that this should be proved without any assumption on $R$.

**Proposition 3.11.** *Assume that $R$ satisfies the bounded geometry condition. Then no non-trivial elliptic modular transformation of* $\mathrm{Mod}(R)$ *is asymptotically trivial.*

*Proof.* Let $\gamma$ be an elliptic modular transformation of $\mathrm{Mod}(R)$. If $\gamma$ is of infinite order, then by Theorem 2.35, there is an orbit of $p \in \mathcal{T}(R)$ under $\langle\gamma\rangle$ that is not a discrete set. On the other hand, if $\gamma$ is asymptotically trivial, then in particular $\langle\gamma\rangle$ preserves the fiber $T_p$, and the orbit should be a discrete set by Theorem 3.8. This is a contradiction. In the case where $\gamma$ is of finite order, we see that $\gamma$ cannot be asymptotically trivial by a certain geometric argument. □

## 3.3 The asymptotically trivial mapping class group

The asymptotically trivial mapping class group contains the stable mapping class group. They do not necessarily coincide, but when $R$ satisfies the bounded geometry condition, they coincide. We explain the relationship between these groups and then discuss certain results obtained from their coincidence.

### Relation to the stable mapping class group

It is evident from the definition that the stable mapping class group is contained in the asymptotically trivial mapping class group and the pure mapping class group. Moreover, there is an inclusion relation between the latter two groups.

**Theorem 3.12** ([16], [20]). *The following inclusion relations are satisfied in general:*

$$\mathrm{MCG}_\infty(R) \subset \mathrm{Ker}\ \iota_{AT} \subset \mathrm{MCG}_\partial(R).$$

We expect that the closure $\overline{\mathrm{MCG}_\infty(R)}$ of the stable mapping class group in the compact-open topology should contain $\mathrm{Ker}\ \iota_{AT}$. Since $\mathrm{MCG}_\partial(R)$ is closed, the inclusion $\overline{\mathrm{MCG}_\infty(R)} \subset \mathrm{MCG}_\partial(R)$ is clear.

If $R$ has a sequence of mutually disjoint simple closed geodesics whose lengths tend to zero, then a mapping class given by the simultaneous Dehn twists along all these curves belongs to $\mathrm{Ker}\ \iota_{AT}$ but not to $\mathrm{MCG}_\infty(R)$. However, if $R$ satisfies the bounded geometry condition, then there is no such sequence of curves, and in fact there is no such mapping class.

**Theorem 3.13** ([19], [20]). *Assume that $R$ satisfies the bounded geometry condition. Then* $\mathrm{MCG}_\infty(R) = \mathrm{Ker}\ \iota_{AT}$ *is satisfied.*

An application of this theorem will be given in the next section.

### Finite subgroups in the asymptotic Teichmüller modular group

We deal with periodic elements, and more generally, finite subgroups of the asymptotic Teichmüller modular group. Recall that every finite subgroup of the Teichmüller modular group has a fixed point on the Teichmüller space, which is a special case of Theorem 2.34. We consider a similar property on the asymptotic Teichmüller space.

We assume that $R$ satisfies the bounded geometry condition. Let $[g] \in \mathrm{MCG}(R)$ be a mapping class such that $[g]_{**} \in \mathrm{Mod}_{AT}(R)$ is periodic of order $n$. This means that $[g^n] \in \mathrm{Ker}\ \iota_{AT}$, and since $\mathrm{Ker}\ \iota_{AT} = \mathrm{MCG}_\infty(R)$ by Theorem 3.13, we have $[g^n] \in \mathrm{MCG}_\infty(R)$. Then we see that, outside some topologically finite bordered subsurface, $[g]$ is a periodic mapping class. By standard arguments, we can find a complex structure such that $[g]$ can be realized as a conformal automorphism off the subsurface, that is, $[g]$ is asymptotically conformal. This is equivalent to saying that this complex structure gives a fixed point of $[g]_{**}$ on $\mathcal{AT}(R)$. Therefore, every periodic element of $\mathrm{Mod}_{AT}(R)$ has a fixed point on $\mathcal{AT}(R)$. This has been proved in [19].

The Nielsen realization theorem for the mapping class group $\mathrm{MCG}(R)$ is solved by finding a fixed point on $\mathcal{T}(R)$. Analogously, we formulate the following fixed point theorem for $\mathrm{Mod}_{AT}(R)$, the asymptotic version of the realization theorem. The proof is also carried out by a similar argument as above relying on the fact that $\mathrm{Ker}\ \iota_{AT} = \mathrm{MCG}_\infty(R)$.

**Theorem 3.14** ([20]). *Assume that $R$ satisfies the bounded geometry condition. Then every finite subgroup of* $\mathrm{Mod}_{AT}(R)$ *has a common fixed point on* $\mathcal{AT}(R)$.

In the light of Theorem 2.34, we further propose the following.

**Problem 3.15.** Find a common fixed point on $\mathcal{AT}(R)$ when the orbit of a subgroup of $\mathrm{Mod}_{AT}(R)$ is bounded.

### Realization in asymptotic Teichmüller modular groups

Every countable group can be realized as a group of conformal automorphisms of some Riemann surface. Actually, a stronger result has been periodically proved since the first proof was given by Greenberg [27], which asserts that we can find a Riemann surface $R$ whose conformal automorphism group is precisely isomorphic to the given countable group.

This fact says that every countable group can be obtained as the stabilizer of some point in some Teichmüller modular group. Then we may ask the same question for the asymptotic Teichmüller modular group. If we see that the kernel of the representation $\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(\mathcal{AT}(R))$ contains no conformal mapping classes besides the trivial one, then every countable group can also be realized as the stabilizer subgroup of $\mathrm{Mod}_{AT}(R)$. In fact, Proposition 3.11 gives the following theorem. Note that it is easy to make a hyperbolic Riemann surface $R$ to satisfy the bounded geometry condition as well as to avoid extra asymptotically conformal automorphisms of $R$ other than the conformal ones.

**Theorem 3.16.** *For any countable group $H$, there exists a hyperbolic Riemann surface $R$ satisfying the bounded geometry condition such that the stabilizer subgroup for some point of $\mathcal{AT}(R)$ in* $\mathrm{Mod}_{AT}(R)$ *is isomorphic to $H$.*

Here we will give a concrete construction of a Riemann surface $R$ such that the Thompson group is realized in some stabilizer subgroup of $\mathrm{Mod}_{AT}(R)$, according to de Faria, Gardiner and Harvey [2].

Let $E$ be the middle-third Cantor set in the unit interval and set $R = \mathbb{C} - E$, which has one puncture at $\infty$. Given a hyperbolic metric, $R$ satisfies the bounded geometry condition. Indeed, each step for the construction of the Cantor set by removing the middle-third interval defines a pair of pants, and this procedure induces a canonical pants decomposition of $R$ such that all the lengths of boundary geodesics of the pairs of pants are uniformly bounded from above and from below. Then there is a quasiconformal homeomorphism $f$ of $R = \mathbb{C} - E$ preserving the upper and lower half-planes respectively such that for any non-cuspidal topological ends $e$ and $e'$ of $f(R)$, there are neighborhoods $U$ and $U'$ of $e$ and $e'$ respectively that are conformally equivalent. Set $p = [f] \in \mathcal{T}(R)$.

Let $G$ be the subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes that have representatives preserving the upper and lower half-planes. Then, by the choice of $p$, we see that each mapping class of $G$ is realized as an asymptotically conformal automorphism of the Riemann surface $f(R)$ corresponding to $p$. This means that $\Gamma = \iota(G)$ is an asymptotically elliptic subgroup of $\mathrm{Mod}(R)$. Since $R$ satisfies the bounded geometry condition, Theorem 3.6 tells us that $\Gamma$ is a countable group. Also note that $\mathrm{MCG}(R)$ itself is stationary because every representative of each mapping class maps any neighborhood of the puncture in such a way that it has non-empty intersection with its image. Hence by Theorem 2.29, $\Gamma$ acts discontinuously on $\mathcal{T}(R)$.

The *Thompson group* $F$ is the group of all piecewise-linear automorphisms of the unit interval $[0, 1]$ fixing 0 and 1 having the following property. For some integer $n \geq 0$, the domain and the range are divided into $n+1$ subintervals. These subintervals are obtained by $n$ time half-division of intervals such that at each step we choose one of the intervals made by the previous steps and divide it into two half intervals. (The subdivision in the domain and in the range is not the same.) Then such a division of the domain and the range intervals gives a unique piecewise-linear homeomorphism by the correspondence of the subintervals in order. The Thompson group $F$ is an infinitely generated group without torsion. It has been proved in [2] that $\iota_{AT}(G)$ is isomorphic to $F$. This means that $F$ can be realized as a subgroup of the stabilizer for $\alpha(p) \in \mathcal{AT}(R)$ in $\mathrm{Mod}_{AT}(R)$.

## 4 Quotient spaces by Teichmüller modular groups

If a Riemann surface $R$ is analytically finite, the moduli space $M(R)$ of all complex structures on $R$ is obtained as a quotient space of the Teichmüller space $\mathcal{T}(R)$ by the Teichmüller modular group $\mathrm{Mod}(R)$. In this case, $\mathrm{Mod}(R)$ acts properly discontinuously on $\mathcal{T}(R)$ and hence $M(R)$ inherits complex and geometric structures from $\mathcal{T}(R)$. However, this is not always the case where $R$ is a general Riemann surface. We have to consider other quotients which inherit certain structures of $\mathcal{T}(R)$. Especially, we introduce the stable moduli space and the enlarged moduli space. The former is obtained as the completion of the quotient of the region of stability by $\mathrm{Mod}(R)$ whereas the latter is the quotient of $\mathcal{T}(R)$ by the stable mapping class group.

In this section, we assume that a Riemann surface $R$ is non-elementary.

## 4.1 Geometric moduli spaces

We introduce a new moduli space, which has a complete distance induced from the Teichmüller distance. We give two different ways for its construction and show that the resulting spaces coincide.

### The moduli space of stable points

No matter how the action of $\mathrm{Mod}(R)$ is far from discontinuity, we can define the moduli space $M(R) = \mathcal{T}(R)/\mathrm{Mod}(R)$ which is a topological space for the quotient topology. We call this $M(R)$ the *topological* moduli space. Moreover a pseudo-distance $d_M$ on $M(R)$ is induced from the Teichmüller distance $d_T$ on $\mathcal{T}(R)$. Namely, letting $\pi : \mathcal{T}(R) \to M(R)$ the projection, we define the pseudo-distance by

$$d_M(\sigma, \tau) = \inf\{d_T(p, q) \mid \pi(p) = \sigma,\ \pi(q) = \tau\}$$

for any $\sigma$ and $\tau$ in $M(R)$. However, this is not always a distance because the infimum is not necessarily attained. Hence we want to consider the following smaller subset in $M(R)$.

**Definition 4.1.** The *moduli space of stable points* is defined by

$$M_\Phi(R) = \Phi(\mathrm{Mod}(R))/\mathrm{Mod}(R),$$

where $\Phi(\mathrm{Mod}(R))$ is the region of stability for $\mathrm{Mod}(R)$.

For the region of discontinuity $\Omega(\mathrm{Mod}(R))$, the quotient space

$$M_\Omega(R) = \Omega(\mathrm{Mod}(R))/\mathrm{Mod}(R)$$

inherits complex and geometric structures from $\mathcal{T}(R)$. In particular, $M_\Omega(R)$ is a complex Banach orbifold. On the other hand, $M_\Phi(R)$ is an open subset of $M(R)$ including $M_\Omega(R)$ where the restriction of the pseudo-distance $d_M$ becomes a distance. If $R$ satisfies the bounded geometry condition, then $M_\Phi(R) = M_\Omega(R)$ by Theorem 2.30.

The distance $d_M$ on $M_\Phi(R)$ defines the length of a path in $M_\Phi(R)$. For any two points in $M_\Phi(R)$, consider all paths in $M_\Phi(R)$ connecting these points and take the infimum over their lengths. This defines an intrinsic distance $d_M^i$ on $M_\Phi(R)$, which is called the inner distance with respect to $d_M$. Clearly $d_M^i \geq d_M$.

**Definition 4.2.** The metric completion of $M_\Phi(R)$ with respect to the inner distance $d_M^i$ is denoted by $\overline{M_\Phi(R)}^i$ and called the *stable moduli space*.

### Closure equivalence

We use a stronger equivalence relation than the usual orbit equivalence under $\mathrm{Mod}(R)$. This makes the quotient space a metric space.

**Definition 4.3.** For a subgroup $\Gamma$ of $\mathrm{Mod}(R)$, we define two points $p$ and $q$ in $\mathcal{T}(R)$ to be equivalent if $q \in \overline{\Gamma(p)}$. This gives an equivalence relation and the equivalence class containing $p$ is $\overline{\Gamma(p)}$. This is called *closure equivalence*. The quotient space by the closure equivalence is denoted by $\mathcal{T}(R)/\!/\,\Gamma$.

Let $\bar{\pi}\colon \mathcal{T}(R)/\Gamma \to \mathcal{T}(R)/\!/\,\Gamma$ be the canonical projection. The inverse image $\bar{\pi}^{-1}(s)$ for $s \in \mathcal{T}(R)/\!/\,\Gamma$ coincides with the closure $\overline{\{\sigma\}}$ of a single point set $\{\sigma\}$ in $\mathcal{T}(R)/\Gamma$, where $\sigma$ is an arbitrary point in $\bar{\pi}^{-1}(s)$. This corresponds to the fact that the equivalence classes containing $p \in \mathcal{T}(R)$ are $\Gamma(p)$ and $\overline{\Gamma(p)}$ for the orbit equivalence and for the closure equivalence, respectively. Clearly, $\overline{\{\sigma\}} = \{\sigma\}$ if and only if the corresponding orbit $\Gamma(p)$ is closed, namely, $\Gamma$ acts at $p$ weakly stably. The Teichmüller distance $d_T$ induces a quotient distance $d_*$ on $\mathcal{T}(R)/\!/\,\Gamma$; it satisfies a property that $d_*(s,s') = 0$ implies $s = s'$. This is because the equivalence classes are closed in $\mathcal{T}(R)$. Hence $\mathcal{T}(R)/\!/\,\Gamma$ is a complete metric space.

Now, by setting $\Gamma = \mathrm{Mod}(R)$, we have our definition of the moduli space.

**Definition 4.4.** The complete metric space $\mathcal{T}(R)/\!/\,\mathrm{Mod}(R)$ with the distance $d_*$ is called the *geometric* moduli space and denoted by $M_*(R)$.

If $\mathrm{Mod}(R)$ acts on $\mathcal{T}(R)$ weakly stably, then the geometric moduli space $M_*(R)$ is nothing but the topological moduli space $M(R)$ and the pseudo-distance $d_M$ coincides with the distance $d_*$. However, if it does not act weakly stably, the projection $\bar{\pi}\colon M(R) \to M_*(R)$ is not injective and $d_M$ is not a distance on $M(R)$. In fact, $M(R)$ does not satisfy the first separability axiom ($T_1$-axiom) in this case.

**Proposition 4.5** ([42]). *The following conditions are equivalent:*

(1) *the Teichmüller modular group* $\mathrm{Mod}(R)$ *acts weakly stably on* $\mathcal{T}(R)$*;*

(2) *the projection* $\bar{\pi}\colon M(R) \to M_*(R)$ *is injective;*

(3) *the topological moduli space* $M(R)$ *is a* $T_1$*-space, in other words, every single point constitutes a closed set;*

(4) *the pseudo-distance* $d_M$ *on* $M(R)$ *is a distance.*

A sufficient condition for $R$ and $\mathrm{Mod}(R)$ not to satisfy the conditions in Proposition 4.5 is also given in [42] as follows.

**Theorem 4.6.** *Assume that $R$ satisfies the bounded geometry condition and* $\mathrm{Mod}(R)$ *contains an elliptic element of infinite order. Then the topological moduli space $M(R)$ is not a $T_1$-space. In particular, for an infinite cyclic cover $R$ of an analytically finite Riemann surface, $M(R)$ is not a $T_1$-space.*

*Proof.* Since $\mathrm{Mod}(R)$ contains an elliptic element of infinite order, it does not act weakly discontinuously by Theorem 2.35. Since $R$ satisfies the bounded geometry condition, this implies that $\mathrm{Mod}(R)$ does not act weakly stably by Theorem 2.30. Then Proposition 4.5 asserts that $M(R)$ is not a $T_1$-space. □

**Genericity of stable points**

We give several properties of the stable points which show that they are generic in $\mathcal{T}(R)$ in the following sense. We apply these properties to the investigation of the structure of moduli spaces.

**Theorem 4.7** ([42]). *Assume that $R$ is non-elementary. The region of stability $\Phi(\mathrm{Mod}(R))$ is open, connected and dense in $\mathcal{T}(R)$.*

Note that we have seen that $\Phi(\mathrm{Mod}(R))$ is open by Theorem 2.28. The following corollary is an easy consequence of the density of $\Phi(\mathrm{Mod}(R))$.

**Corollary 4.8.** *The geometric moduli space $M_*(R)$ is isometric to the completion $\overline{M_\Phi(R)}$ of the moduli space of the stable points with respect to the distance $d_M$.*

Concerning the connectivity, $\Phi(\mathrm{Mod}(R))$ has a stronger property than just a topological one. Namely, the distance between two points in $\Phi(\mathrm{Mod}(R))$ measured by $d_M$ is comparable in some sense, with the length of a path connecting them in $\Phi(\mathrm{Mod}(R))$, which approximates the distance measured by $d_M^i$. This in particular gives the following.

**Theorem 4.9** ([42]). *The geometric moduli space $M_*(R)$ is locally bi-Lipschitz equivalent to the stable moduli space $\overline{M_\Phi(R)}^i$.*

If $R$ satisfies the bounded geometry condition, $\Phi(\mathrm{Mod}(R)) = \Omega(\mathrm{Mod}(R))$. This implies that our moduli space $M_*(R)$ has an open dense connected subregion $\Omega(\mathrm{Mod}(R))/\mathrm{Mod}(R)$ which has the complex Banach orbifold structure induced from $\mathcal{T}(R)$.

One of the problems we are interested in is to give a characterization of each point in $M_*(R)$ explaining this equivalence class geometrically.

## 4.2 Several Teichmüller spaces

In general, we can define the quotient $\mathcal{T}(R)/\Gamma$ by a subgroup $\Gamma$ of $\mathrm{Mod}(R)$ as a certain reduction of the Teichmüller space or a certain extension of the moduli space in some appropriate sense.

**An example: the reduced Teichmüller space**

As an example, we present a familiar Teichmüller space, which can be defined as the quotient of the following subgroup of the Teichmüller modular group. Let $\mathrm{MCG}_\#(R)$ be the subgroup of $\mathrm{MCG}(R)$ consisting of all elements $[g]$ such that $g$ is freely homotopic to the identity of $R$, where $R$ is assumed to have the boundary at infinity $\partial_\infty R$ but the homotopy is not assumed to be relative to $\partial_\infty R$. It is clear that $\mathrm{MCG}_\#(R)$ is normal in $\mathrm{MCG}(R)$. As usual, we set $\mathrm{Mod}_\#(R) = \iota(\mathrm{MCG}_\#(R))$.

**Proposition 4.10.** *Assume that $R$ is non-elementary. The subgroup $\mathrm{Mod}_\#(R)$ is the intersection of the subgroups $\mathrm{Mod}_c(R)$ taken over all non-trivial simple closed curves $c$ on $R$. Hence $\mathrm{Mod}_\#(R)$ acts stably on $\mathcal{T}(R)$.*

*Proof.* The first statement is well-known. See for instance [11]. The second statement is a consequence of Proposition 2.8. □

The space $\mathcal{T}(R)/\mathrm{Mod}_\#(R) = \mathcal{T}(R)/\!/\mathrm{Mod}_\#(R)$ is called the *reduced Teichmüller space* $T^\#(R)$ with the quotient distance $d^\#$, and $\mathrm{Mod}(R)/\mathrm{Mod}_\#(R)$ is the reduced Teichmüller modular group $\mathrm{Mod}^\#(R)$. It acts on $(T^\#(R), d^\#)$ isometrically.

**Relative Teichmüller spaces**

We have already seen the important roles of the subgroup $\mathrm{Mod}_c(R)$. Here we consider the quotient space of $\mathcal{T}(R)$ by this group. In Proposition 2.13, we have seen that $\mathrm{Mod}_c(R)$ is of countable index in $\mathrm{Mod}(R)$. And, since $\mathrm{Mod}_c(R)$ is stationary and closed if $R$ is non-elementary, it acts stably on $\mathcal{T}(R)$ by Theorem 2.23. Moreover, if $R$ satisfies the bounded geometry condition, then it acts discontinuously on $\mathcal{T}(R)$.

**Definition 4.11.** The quotient space $\mathcal{T}^c(R) = \mathcal{T}(R)/\mathrm{Mod}_c(R)$ is called the *relative Teichmüller space* with respect to $c$.

Since $\mathrm{Mod}_c(R)$ acts stably on $\mathcal{T}(R)$, the relative Teichmüller space $\mathcal{T}^c(R)$ is a complete metric space with the quotient distance $\hat{d}$. This divides the action of $\mathrm{Mod}(R)$ on $\mathcal{T}(R)$ into the stable part under $\mathrm{Mod}_c(R)$ on $\mathcal{T}(R)$ and the countable part under $\mathrm{Mod}(R)/\mathrm{Mod}_c(R)$ on $\mathcal{T}^c(R)$. More precisely, let $P \subset \mathcal{T}(R)$ be the orbit of $p$ under $\mathrm{Mod}(R)$ and $\widehat{P} \subset \mathcal{T}^c(R)$ the image of $P$ under the projection $\mathcal{T}(R) \to \mathcal{T}^c(R)$. Assume that $\mathrm{Mod}_c(R)$ acts discontinuously on $\mathcal{T}(R)$. In this case, if $\widehat{P}$ is closed in $\mathcal{T}^c(R)$, then $\widehat{P}$ is discrete and, as a consequence, we see that $\mathrm{Mod}(R)$ acts discontinuously on $\mathcal{T}(R)$. This yields a result similar to Lemma 2.6.

Another feature of $\mathcal{T}^c(R)$ is the fact that $\mathcal{T}^c(R)$ is not separable if $R$ is of infinite topological type, which is obtained in [42]. If we impose the extra assumption that $R$ satisfies the bounded geometry condition, this fact can be easily seen as is shown below. From the non-separability of $\mathcal{T}^c(R)$, we can prove that the topological moduli space $M(R)$ is not separable either. In fact, every countable subset is nowhere dense in $M(R)$, and this is also true for the geometric moduli space $M_*(R)$.

**Theorem 4.12.** *Assume that $R$ satisfies the bounded geometry condition. Then the geometric moduli space $M_*(R)$ is not separable if $R$ is of infinite topological type.*

*Proof.* If $R$ satisfies the bounded geometry condition, then $\mathrm{Mod}_c(R)$ acts discontinuously on $\mathcal{T}(R)$ by Theorem 2.29. On the other hand, $\mathcal{T}(R)$ is not separable when $R$ is of infinite topological type. Then $\mathcal{T}^c(R) = \mathcal{T}(R)/\mathrm{Mod}_c(R)$ is not separable. Since $\mathrm{Mod}_c(R)$ is of countable index in $\mathrm{Mod}(R)$, $M(R) = \mathcal{T}(R)/\mathrm{Mod}(R)$ is not separable either. By considering the moduli space of stable points $M_\Phi(R)$, which is open and dense in $M(R)$, we also see that $M_*(R)$ is not separable by Corollary 4.8. □

**The intermediate Teichmüller space**

We consider quotient spaces of $\mathcal{T}(R)$ by the stable mapping class group and the asymptotically trivial mapping class group. When $R$ satisfies the bounded geometry condition, by Theorem 3.13, they coincide.

**Definition 4.13.** For the subgroup $\mathrm{Mod}_\infty(R)$ of $\mathrm{Mod}(R)$ corresponding to the stable mapping class group, the quotient space $\widetilde{M}(R) = \mathcal{T}(R)/\mathrm{Mod}_\infty(R)$ is called the *enlarged moduli space*.

If $R$ is of infinite topological type and satisfies the bounded geometry condition, then $\mathrm{Mod}_\infty(R)$ acts on $\mathcal{T}(R)$ discontinuously and freely by Theorem 2.16. Then the enlarged moduli space $\widetilde{M}(R)$ is a complex Banach manifold which has complex and metric structures induced from $\mathcal{T}(R)$. Since $\mathrm{Mod}_\infty(R)$ is a normal subgroup of $\mathrm{Mod}(R)$, the quotient group $\mathrm{Mod}^\infty(R) = \mathrm{Mod}(R)/\mathrm{Mod}_\infty(R)$ acts on $\widetilde{M}(R)$ as a biholomorphic and isometric automorphism group that induces a quotient map onto the topological moduli space $M(R)$. This will be a way of considering a geometric structure on $M(R)$.

**Definition 4.14.** For the subgroup $\iota(\mathrm{Ker}\ \iota_{AT})$ of $\mathrm{Mod}(R)$ corresponding to the asymptotically trivial mapping class group, the quotient space $\mathcal{IT}(R) = \mathcal{T}(R)/\iota(\mathrm{Ker}\ \iota_{AT})$ is called the *intermediate Teichmüller space* .

Since $\mathrm{Ker}\,\iota_{AT}$ acts on $\mathcal{AT}(R)$ trivially, the definition of $\mathcal{IT}(R)$ immediately gives the following.

**Proposition 4.15** ([19]). *The projection $\mathcal{T}(R) \to \mathcal{IT}(R)$ factorizes the projection $\alpha\colon \mathcal{T}(R) \to \mathcal{AT}(R)$. Hence there are natural projections from $\mathcal{IT}(R)$ onto both $\mathcal{AT}(R)$ and $M(R)$. In fact, $\mathcal{IT}(R)$ is the smallest quotient space of $\mathcal{T}(R)$ by a subgroup of* $\mathrm{Mod}(R)$ *having this property.*

Since $\mathrm{Mod}_\infty(R) \subset \mathrm{Ker}\,\iota_{AT}$ by Theorem 3.12, the enlarged moduli space $\widetilde{M}(R)$ lies always between $\mathcal{T}(R)$ and $\mathcal{IT}(R)$. If $R$ is analytically finite, then $\widetilde{M}(R) = \mathcal{IT}(R) = M(R)$, and the asymptotic Teichmüller space $\mathcal{AT}(R)$ is just one point.

On the other hand, if $R$ is the unit disk $\mathbb{D}$, then $\mathcal{T}(\mathbb{D}) = \widetilde{M}(\mathbb{D}) = \mathcal{IT}(\mathbb{D})$. Indeed, Ker $\iota_{AT}$ is trivial for the unit disk $\mathbb{D}$, and thus $\mathcal{IT}(\mathbb{D}) = \mathcal{T}(\mathbb{D})/\iota(\text{Ker } \iota_{AT}) = \mathcal{T}(\mathbb{D})$.

Now we assume that $R$ satisfies the bounded geometry condition. Then we have $\mathrm{MCG}_\infty(R) = \text{Ker } \iota_{AT}$ by Theorem 3.13 and hence $\widetilde{M}(R) = \mathcal{IT}(R)$. In this case, we have the relationship between $\mathrm{Mod}^\infty(R)$ and $\mathrm{Mod}_{AT}(R)$.

**Theorem 4.16** ([19]). *Assume that $R$ is of infinite topological type and satisfies the bounded geometry condition. Then the asymptotic Teichmüller modular group* $\mathrm{Mod}_{AT}(R)$ *is geometrically isomorphic to the automorphism group* $\mathrm{Mod}^\infty(R)$ *of* $\widetilde{M}(R) = \mathcal{IT}(R)$.

For the representation of the quasiconformal mapping class group $\mathrm{MCG}(R)$ in the automorphism group $\mathrm{Aut}(\mathcal{T}(R))$ of Teichmüller space, it has been proved that the kernel is trivial and the image is the entire group in almost all cases. In contrast to these facts, for the representation of $\mathrm{MCG}(R)$ in the automorphism group $\mathrm{Aut}(\mathcal{AT}(R))$ of the asymptotic Teichmüller space, we obtain that the kernel is characterized topologically as the stable mapping class group $\mathrm{MCG}_\infty(R)$ and the image can be represented as the automorphism group of the intermediate Teichmüller space $\mathcal{IT}(R)$ in the case where $R$ satisfies the bounded geometry condition.

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