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Continuity of the barycentric extension of circle diffeomorphisms with Hölder continuous derivative

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Abstract

The barycentric extension due to Douady and Earle yields a conformally natural extension of a quasisymmetric self-homeomorphism of the unit circle to a quasiconformal self-homeomorphism of the unit disk. We consider such extensions for circle diffeomorphisms with Hölder continuous derivative and show that this operation is continuous with respect to an appropriate topology for the space of the corresponding Beltrami coefficients.

1. *Introduction*

The barycentric extension due to Douady and Earle [**5**] yields a natural extension of a self-homeomorphism of the unit circle $\mathbb{S}$ to a self-homeomorphism of the unit disk $\mathbb{D}$. It plays an important role in the study of complex analytic aspects of Teichmüller spaces. In this paper, we consider barycentric extensions of diffeomorphisms of $\mathbb{S}$ with Hölder continuous derivative. We strengthen results of [**12**], obtaining the quasiconformal extension of such a diffeomorphism and investigating the structure of its Teichmüller space. Our work mainly consists in proving that the extension operator is continuous with respect to a certain smooth topology on the space of these diffeomorphisms and a corresponding topology on the space of the Beltrami coefficients of their quasiconformal extensions.

Our arguments of Teichmüller spaces are modelled on the universal Teichmüller space T, which can be identified with the space $\mathrm{QS}_*(\mathbb{S})$ of all normalised quasisymmetric homeomorphisms of $\mathbb{S}$. In this setting, the Teichmüller projection q is regarded as the boundary extension map on the space $\mathrm{QC}_*(\mathbb{D})$ of all normalised quasiconformal homeomorphisms of $\mathbb{D}$. By the measurable Riemann mapping theorem, the latter space is identified with the space of Beltrami coefficients $\mathrm{Bel}(\mathbb{D}) = L^\infty(\mathbb{D})_1$, which is the open unit ball of measurable functions on $\mathbb{D}$ with the supremum norm. Then, $q : \mathrm{Bel}(\mathbb{D}) \to T$ is continuous with respect to the topology on $\mathrm{QS}_*(\mathbb{S})$ induced by the quasisymmetry constant. The barycentric extension yields a continuous section $e : T \to \mathrm{Bel}(\mathbb{D})$ for q.

The Teichmüller space T_0^α of circle diffeomorphisms with α-Hölder continuous derivative for $\alpha \in (0,1)$ is similarly defined as a subspace of T; the subgroup $\mathrm{Diff}_*^{1+\alpha}(\mathbb{S}) \subset \mathrm{QS}_*(\mathbb{S})$ of all such diffeomorphisms with normalisation can be defined to be T_0^α. The topology on this group is induced by the right translations from the $C^{1+\alpha}$-distance to the identity map. Moreover, the corresponding subspace of Beltrami coefficients is $\mathrm{Bel}_0^\alpha(\mathbb{D}) \subset \mathrm{Bel}(\mathbb{D})$, which consists of all $\mu \in \mathrm{Bel}(\mathbb{D})$ with finite weighted supremum norm

$$\|\mu\|_{\infty,\alpha} = \operatorname*{ess.sup}_{\zeta\in\mathbb{D}} \left(\frac{2}{1-|\zeta|^2}\right)^\alpha |\mu(\zeta)|.$$

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It is proved in [12] that the restriction of the Teichmüller projection to $\mathrm{Bel}_0^\alpha(\mathbb{D})$ yields a continuous map $q:\mathrm{Bel}_0^\alpha(\mathbb{D}) \to T_0^\alpha$. In fact, the topology of T_0^α coincides with the quotient topology induced from $\mathrm{Bel}_0^\alpha(\mathbb{D})$ by q. Moreover, a complex Banach manifold structure has been provided for T_0^α through the Bers embedding. The reader is referred to the survey articles [10] for an introduction to the Teichmüller space T_0^α, and [11] for applications of T_0^α to problems on circle diffeomorphism groups.

The main theorem of this paper asserts the continuity of the section e restricted to T_0^α.

THEOREM 1.1. *The barycentric extension of circle diffeomorphisms with α-Hölder continuous derivative yields a continuous section*

$$e : T_0^\alpha = \mathrm{Diff}_*^{1+\alpha}(\mathbb{S}) \to \mathrm{Bel}_0^\alpha(\mathbb{D})$$

for the Teichmüller projection q.

As is well-known, information about the topological structure of this space can be derived from the existence of a continuous section. We note that $T_0^\alpha = \mathrm{Diff}_*^{1+\alpha}(\mathbb{S})$ is also a topological group [12].

COROLLARY 1.2. *The Teichmüller space T_0^α is contractible.*

In the next section, we will elaborate on these concepts and results.

2. *Preliminaries*

In this section, we summarise several results that will be used as background material for our arguments. These include the definition and properties of the barycentric extension of quasisymmetric self-homeomorphisms of the circle, fundamental results on the universal Teichmüller space, and preliminaries on the space of circle diffeomorphisms with Hölder continuous derivative. For the results mentioned in this section on quasiconformal and quasisymmetric homeomorphisms as well as Teichmüller spaces, the reader is referred to the monograph by Lehto [9].

2.1. *Quasiconformal and quasisymmetric homeomorphisms*

We denote the group of all quasiconformal self-homeomorphisms of the unit disk $\mathbb{D}$ by $\mathrm{QC}(\mathbb{D})$, and the group of all quasisymmetric self-homeomorphism of the unit circle $\mathbb{S}$ by $\mathrm{QS}(\mathbb{S})$. Every $f \in \mathrm{QC}(\mathbb{D})$ extends continuously to a quasisymmetric homeomorphism of $\mathbb{S}$. This boundary extension defines a homomorphism $q : \mathrm{QC}(\mathbb{D}) \to \mathrm{QS}(\mathbb{S})$. Conversely, every $\varphi \in \mathrm{QS}(\mathbb{S})$ extends continuously to a quasiconformal homeomorphism of $\mathbb{D}$, that is, q is surjective. In fact, there are explicit methods for constructing such quasiconformal extensions that define sections $e : \mathrm{QS}(\mathbb{S}) \to \mathrm{QC}(\mathbb{D})$ with $q \circ e = \mathrm{id}|_{\mathrm{QS}}$, such as the Beurling–Ahlfors [3] and Douady–Earle [5] extensions.

2.2. *The barycentric extension*

The barycentric or the Douady–Earle extension $e(\varphi)$ of an orientation-preserving self-homeomorphism $\varphi \in \mathrm{Homeo}(\mathbb{S})$ is given as follows. The *average* of φ taken at $w \in \mathbb{D}$ is defined by

$$\xi_\varphi(w) = \frac{1}{2\pi}\int_{\mathbb{S}} \gamma_w(\varphi(\zeta))|d\zeta| = \frac{1}{2\pi}\int_{\mathbb{S}} \frac{\varphi(\zeta) - w}{1 - \bar{w}\varphi(\zeta)}|d\zeta|,$$

where the Möbius transformation

$$\gamma_w(z) = \frac{z-w}{1-\bar{w}z} \in \text{Möb}(\mathbb{D})$$

maps w to the origin 0. The *barycenter* of φ is the unique point $w_0 \in \mathbb{D}$ such that $\xi_\varphi(w_0) = 0$. The value of the barycentric extension $e(\varphi)$ at the origin 0 is defined to be the barycenter w_0; we set $e(\varphi)(0) = w_0$.

For an arbitrary point $z \in \mathbb{D}$, the barycentric extension $e(\varphi)$ is defined by

$$e(\varphi)(z) = e(\varphi \circ \gamma)(0),$$

where $\gamma \in \text{Möb}(\mathbb{D})$ is any Möbius transformation that maps 0 to z, say, $\gamma = \gamma_z^{-1}$. This is well-defined, since $\xi_{\varphi \circ r}(0) = \xi_\varphi(0)$ for any rotation r, which is a Möbius transformation fixing 0.

An alternative definition was introduced by Lecko and Partyka [8]. For each $w \in \mathbb{D}$, we consider the harmonic extension (the Poisson integral) of $\gamma_w \circ \varphi \in \text{Homeo}(\mathbb{S})$

$$P_w(z) := \frac{1}{2\pi} \int_{\mathbb{S}} \gamma_w \circ \varphi(\zeta) |\gamma_z'(\zeta)| |d\zeta|.$$

Since P_w is a self-homeomorphism of $\mathbb{D}$ by the Radó–Kneser–Choquet theorem, there exists a unique point $z \in \mathbb{D}$ such that $P_w(z) = 0$. We define a map $e_*(\varphi) : \mathbb{D} \to \mathbb{D}$ by $e_*(\varphi)(w) = z$. Then, $e(\varphi) = e_*(\varphi)^{-1}$. Indeed, $e(\varphi)(z) = w$ and $e_*(\varphi)(w) = z$ are equivalent to the conditions

$$\frac{1}{2\pi} \int_{\mathbb{S}} \gamma_w \circ \varphi(\gamma_z^{-1}(\tilde{\zeta})) |d\tilde{\zeta}| = 0 \quad \text{and} \quad \frac{1}{2\pi} \int_{\mathbb{S}} \gamma_w \circ \varphi(\zeta) |\gamma_z'(\zeta)| |d\zeta| = 0,$$

respectively. Using the substitution $\tilde{\zeta} = \gamma_z(\zeta)$, we see that these integrals are equal.

The application of the barycentric extension to a quasisymmetric homeomorphism yields the following fundamental result.

THEOREM [5]. *For every $\varphi \in \text{QS}(\mathbb{S})$, the barycentric extension yields $e(\varphi) \in \text{QC}(\mathbb{D})$.*

In addition to Douady and Earle [5], an exposition on barycentric extensions may be found in Pommerenke [13, Section 5.5], to which we will occasionally refer in the sequel.

2.3. *Conformal naturality*

The barycentric extension $e(\varphi)$ of $\varphi \in \text{Homeo}(\mathbb{S})$ has conformal naturality in the following sense:

$$e(g \circ \varphi \circ \gamma) = g \circ e(\varphi) \circ \gamma$$

for any $g, \gamma \in \text{Möb}(\mathbb{S}) = \text{Möb}(\mathbb{D})$. Indeed, $e(\varphi \circ \gamma) = e(\varphi) \circ \gamma$ follows from the definition of $e(\varphi)$. Moreover, $e(g \circ \varphi) = g \circ e(\varphi)$ follows from the formula

$$\frac{g(z) - g(w)}{1 - \overline{g(w)} g(z)} = e^{i\theta(w)} \frac{z-w}{1-\bar{w}z}$$

for some function $\theta : \mathbb{D} \to \mathbb{R}$ of w independent of z. In fact, if $\xi_\varphi(w_0) = 0$, then

$$\xi_{g \circ \varphi}(g(w_0)) = \frac{1}{2\pi} \int_{\mathbb{S}} \frac{g(\varphi(\zeta)) - g(w_0)}{1 - \overline{g(w_0)} g(\varphi(\zeta))} |d\zeta| = \frac{e^{i\theta(w_0)}}{2\pi} \int_{\mathbb{S}} \frac{\varphi(\zeta) - w_0}{1 - \overline{w_0}\varphi(\zeta)} |d\zeta| = 0.$$

For $f \in \mathrm{QC}(\mathbb{D})$, we denote the complex dilatation of f by $\mu_f(z) = \bar\partial f(z)/\partial f(z)$. The conformal naturality of the barycentric extension for quasisymmetric homeomorphisms in terms of complex dilatations can be described as follows:

$$\mu_{e(g\circ\varphi\circ\gamma)}(z) = \mu_{g\circ e(\varphi)\circ\gamma}(z) = \mu_{e(\varphi)}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)}$$

for any $g, \gamma \in \mathrm{Möb}(\mathbb{S}) = \mathrm{Möb}(\mathbb{D})$ and $\varphi \in \mathrm{QS}(\mathbb{S})$. In particular, this implies

$$|\mu_{e(g\circ\varphi\circ\gamma)}(z)| = |\mu_{e(\varphi)}(\gamma(z))|.$$

2.4. *Continuity of the barycentric extension*

The subgroups consisting of the normalised elements of $\mathrm{QC}(\mathbb{D})$ and $\mathrm{QS}(\mathbb{S})$ fixing three points on $\mathbb{S}$, for instance, $1, i, -1$, are denoted by $\mathrm{QC}_*(\mathbb{D})$ and $\mathrm{QS}_*(\mathbb{S})$, respectively.

By the solution of the Beltrami equation (the measurable Riemann mapping theorem), $\mathrm{QC}_*(\mathbb{D})$ is identified with the space of Beltrami coefficients on $\mathbb{D}$:

$$\mathrm{Bel}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\}.$$

Moreover, $\mathrm{QS}_*(\mathbb{S})$ can be regarded as the *universal Teichmüller space* T, which is equipped with the right uniform topology induced by the quasisymmetry constant $M(\varphi) \geqslant 1$ for $\varphi \in \mathrm{QS}(\mathbb{S})$; a sequence φ_n converges to φ in $\mathrm{QS}(\mathbb{S})$ if $M(\varphi_n \circ \varphi^{-1}) \to 1$ as $n \to \infty$. We note that there are several different ways of defining the quasisymmetry constant M, for instance, using the cross ratio (see the survey [**10**]). Nevertheless, they induce the same topology.

Under the above identification, the restriction of q to $\mathrm{QC}_*(\mathbb{D}) = \mathrm{Bel}(\mathbb{D})$ plays the role of the Teichmüller projection. A basic property of this projection is the following.

Proposition. *The Teichmüller projection*

$$q : \mathrm{Bel}(\mathbb{D}) = \mathrm{QC}_*(\mathbb{D}) \to T = \mathrm{QS}_*(\mathbb{S})$$

is continuous and open.

The section for q given by the barycentric extension is also compatible with the topology.

Theorem [**5**]. *The barycentric extension*

$$e : T = \mathrm{QS}_*(\mathbb{S}) \to \mathrm{Bel}(\mathbb{D}) = \mathrm{QC}_*(\mathbb{D})$$

is continuous. In fact, the composition $e \circ q : \mathrm{Bel}(\mathbb{D}) \to \mathrm{Bel}(\mathbb{D})$ *is real analytic.*

2.5. *Diffeomorphisms with Hölder continuous derivative*

An orientation-preserving diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{S})$ belongs to the class $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ for $\alpha \in (0,1)$ if its derivative is α-Hölder continuous. That is, the lift $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of φ given by $\exp(i\widetilde{\varphi}(x)) = \varphi(e^{ix})$ satisfies

$$|\widetilde{\varphi}'(x) - \widetilde{\varphi}'(y)| \leqslant c|x-y|^\alpha \quad (\forall x, y \in \mathbb{R})$$

for some constant $c \geqslant 0$.

We provide $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ with the right uniform topology defined by the $C^{1+\alpha}$-distance $p_{1+\alpha}(\varphi)$ from id to $\varphi \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$, where

$$p_{1+\alpha}(\varphi) := \sup_{\zeta\in\mathbb{S}} |\varphi(\zeta) - \zeta| + \sup_{x\in\mathbb{R}} |\widetilde{\varphi}'(x) - 1| + \sup_{x\neq y\in\mathbb{R}} \frac{|\widetilde{\varphi}'(x) - \widetilde{\varphi}'(y)|}{|x-y|^\alpha}.$$

A sequence φ_n is defined to converge to φ in $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ if $p_{1+\alpha}(\varphi_n \circ \varphi^{-1}) \to 0$ $(n \to \infty)$. We note that $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ with this topology is a topological group [**12**].

2.6. *Beltrami coefficients corresponding to* $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$

For a Beltrami coefficient $\mu \in \mathrm{Bel}(\mathbb{D})$, we define an α-hyperbolic supremum norm ($\alpha \in (0,1)$) by

$$\|\mu\|_{\infty,\alpha} = \operatorname*{ess.sup}_{z\in\mathbb{D}} \rho_{\mathbb{D}}^{\alpha}(z)\,|\mu(z)|, \quad \rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}.$$

The space of Beltrami coefficients with $\|\mu\|_{\infty,\alpha} < \infty$ is denoted by $\mathrm{Bel}_0^{\alpha}(\mathbb{D})$.
We can characterise $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ by its quasiconformal extension to $\mathbb{D}$.

THEOREM. *A quasisymmetric homeomorphism $\varphi : \mathbb{S} \to \mathbb{S}$ belongs to $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ if and only if it has a quasiconformal extension $f : \mathbb{D} \to \mathbb{D}$ whose complex dilatation μ_f belongs to $\mathrm{Bel}_0^{\alpha}(\mathbb{D})$.*

The 'Only if' part was essentially proved by Carleson [4], using the Beurling–Ahlfors extension of quasisymmetric functions on the real line. The 'If' part was investigated by Anderson and Hinkkanen [2], among others, and settled by Dyn'kin [6] and Anderson, Cantón and Fernández [1]. A different proof for an improved statement that is necessary in the arguments of Teichmüller spaces was given in [12].

2.7. *The Teichmüller space of* $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$

The previous theorem implies that the Teichmüller projection (boundary extension) yields a surjective map

$$q : \mathrm{Bel}_0^{\alpha}(\mathbb{D}) \to \mathrm{Diff}_*^{1+\alpha}(\mathbb{S}),$$

where the group $\mathrm{Diff}_*^{1+\alpha}(\mathbb{S})$ of the normalised elements can be defined to be the Teichmüller space T_0^{α} of circle diffeomorphisms with α-Hölder continuous derivative. Moreover, taking the topology into account, we have proved the following.

THEOREM [12]. *The Teichmüller projection*

$$q : \mathrm{Bel}_0^{\alpha}(\mathbb{D}) \to T_0^{\alpha} = \mathrm{Diff}_*^{1+\alpha}(\mathbb{S})$$

is continuous and open.

Concerning the section given by the barycentric extension, we also obtain that it has the appropriate image.

PROPOSITION [12]. *The image of the barycentric extension*

$$e : T_0^{\alpha} = \mathrm{Diff}_*^{1+\alpha}(\mathbb{S}) \to \mathrm{Bel}(\mathbb{D})$$

is contained in $\mathrm{Bel}_0^{\alpha}(\mathbb{D})$.

3. *An outline of the proof*

This section is devoted to a sketch of the proof of the main theorem (Theorem 1.1). The arguments for the rigorous proof are postponed to the next section. Since the proof is rather technical and complicated, it will be helpful to provide an outline first.
Assuming the results in Section 2.7, we need only prove the continuity of the barycentric extension e, namely,

Theorem 3.1. *We assume that ψ converges to* id *in* $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$. *Then, for every* $\varphi_0 \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$, *the complex dilatation* $\mu_{e(\psi\circ\varphi_0)}$ *converges to* $\mu_{e(\varphi_0)}$ *in* $\mathrm{Bel}_0^\alpha(\mathbb{D})$; *that is,*

$$\sup_{z\in\mathbb{D}}\left(\frac{2}{1-|z|^2}\right)^\alpha |\mu_{e(\psi\circ\varphi_0)}(z)-\mu_{e(\varphi_0)}(z)| \to 0 \quad (\psi \to \mathrm{id}).$$

If $e(\psi \circ \varphi_0) = e(\psi) \circ e(\varphi_0)$, the proof would be easy. However, the barycentric extension e is not a homomorphism; it only has conformal naturality. We reduce the theorem to a simpler form using the following facts.

(1) Composition with a rotation does not change the derivatives of circle diffeomorphisms.
(2) Post-composition with a Möbius transformation does not change the complex dilatations of quasiconformal homeomorphisms.

Then, we can normalise so that φ_0 and ψ fix 1 and the derivative of ψ at 1 is 1. We will estimate the complex dilatations on the real interval $[0, 1) \subset \mathbb{D}$. Moreover, we need only consider the convergence when $|z|$ is sufficiently close to 1, otherwise $2/(1-|z|^2)$ is bounded and the uniform convergence of complex dilatations follows from the convergence $\psi \to \mathrm{id}$ by the arguments for the theorem in Section 2.4. Thus, the above theorem is reduced to the claim below. The precise statement regarding the uniformity under conjugations by rotations will be given in Theorem 6.1 of Section 6.

Hereafter, we will use the following notation. Taking the lift $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of $\varphi \in \mathrm{Diff}(\mathbb{S})$, we define its derivative along $\mathbb{S}$ at $\zeta = e^{ix}$ $(-\pi < x \leqslant \pi)$ by $\varphi'_{\mathbb{S}}(\zeta) := \widetilde{\varphi}'(x)$. The distance $d_{\mathbb{S}}(\zeta, 1)$ between ζ and 1 along $\mathbb{S}$ is then $|x|$. The α-Hölder constant of the derivative of ψ at 1 is given by

$$c_\alpha(\psi)(1) := \sup_{1\neq\zeta\in\mathbb{S}} \frac{|\psi'_{\mathbb{S}}(\zeta)-\psi'_{\mathbb{S}}(1)|}{d_{\mathbb{S}}(\zeta,1)^\alpha}.$$

Claim. We assume that $\psi(1) = \varphi_0(1) = 1$ and $\psi'_{\mathbb{S}}(1) = 1$. If $c_\alpha(\psi)(1)$ converges to 0, then

$$\sup_{t_0\leqslant t<1}\left(\frac{2}{1-t^2}\right)^\alpha |\mu_{e(\psi\circ\varphi_0)}(t)-\mu_{e(\varphi_0)}(t)| \to 0$$

for some $t_0 < 1$ sufficiently close to 1.

The strategy to prove the claim is to use the conjugate by

$$h_t(z) = \frac{z+t}{1+tz} \in \mathrm{M\ddot{o}b}(\mathbb{D}) \quad (-1 < t < 1),$$

which maps the real interval $[-1, 1]$ onto itself with the end points fixed and sends 0 to t. Then, the conformal naturality of the barycentric extension implies that

$$|\mu_{e(\psi\circ\varphi_0)}(t)-\mu_{e(\varphi_0)}(t)| = |\mu_{e(h_t^{-1}\circ\psi\circ\varphi_0\circ h_t)}(0)-\mu_{e(h_t^{-1}\circ\varphi_0\circ h_t)}(0)|.$$

The advantage of this reduction is that we can explicitly represent $\mu_{e(\varphi)}(0)$ for $\varphi \in \mathrm{QS}(\mathbb{S})$ using the Fourier coefficients for φ (including the average of $-\varphi^2$) if $e(\varphi)(0) = 0$, that is, if

$$a_0 := \xi_\varphi(0) = \frac{1}{2\pi}\int_{\mathbb{S}} \varphi(\zeta)|d\zeta| = 0.$$

Under this condition, we have

$$\mu_{e(\varphi)}(0) = \frac{a_{-1}-\overline{a_1}b}{a_1-\overline{a_{-1}}b},$$

$$a_1 := \frac{1}{2\pi}\int_{\mathbb{S}} \bar{\zeta}\varphi(\zeta)\,|d\zeta|, \quad a_{-1} := \frac{1}{2\pi}\int_{\mathbb{S}} \zeta\varphi(\zeta)\,|d\zeta|, \quad b := \frac{-1}{2\pi}\int_{\mathbb{S}} \varphi(\zeta)^2\,|d\zeta|.$$

This follows from [**5**, p.28]. See also [**13**, p.115].

However, there remain the following problems in these arguments.

(1) Handling the weight $(2/(1-t^2))^\alpha$ when $t \to 1$.
(2) Estimating $\mu_{e(\varphi)}(0)$ if $e(\varphi)(0) \neq 0$; the barycenters of $h_t^{-1} \circ \varphi_0 \circ h_t$ and $h_t^{-1} \circ \psi \circ \varphi_0 \circ h_t$ are not necessarily zero.

The solution to problem (1) is provided by a refinement of the following result due to Earle [7]: If $\psi(1) = 1$ and $\psi'_{\mathbb{S}}(1) = 1$ then $h_t^{-1} \circ \psi \circ h_t$ converges to id uniformly on $\mathbb{S}$ as $t \to 1$. This is due to the fact that the conjugation by h_t magnifies the mapping of ψ near 1, and, since the linear approximation of ψ has slope $\psi'_{\mathbb{S}}(1) = 1$, it converges to the identity. Earle provided a more precise statement for it 'with future applications in mind'. We now follow his arguments by utilising the α-Hölder constant $c_\alpha(\psi)(1)$. Integration of the definition of the α-Hölder constant (Proposition 4.1) yields

$$|\psi(\zeta) - \zeta| \leqslant C|\zeta - 1|^{\alpha+1} \quad (\zeta \in \mathbb{S}), \qquad C = \frac{(\pi/2)^{\alpha+1}}{\alpha+1} \cdot c_\alpha(\psi)(1).$$

We can use this to obtain a quantitative version of Earle's result as follows. The proof will be given in Section 4.

Lemma 3.2. *We assume that* $\psi \in \mathrm{Homeo}(\mathbb{S})$ *satisfies*

$$|\psi(\zeta) - \zeta| \leqslant C|\zeta - 1|^{\alpha+1} \quad (\zeta \in \mathbb{S})$$

for some constant $C \leqslant 1/4$. *We set* $\psi_t = h_t^{-1} \circ \psi \circ h_t$ *for* $t \in (0,1)$ *and choose any* $\varepsilon > 0$. *If* $1 - t \leqslant \frac{1}{4}(\varepsilon/(4C))^{1/\alpha}$, *then* $|\psi_t(\zeta) - \zeta| \leqslant \varepsilon$ *for every* $\zeta \in \mathbb{S}$.

This asserts that ψ_t is uniformly close to id in the order of $4^{\alpha+1}C(1-t)^\alpha$ as $t \to 1$. Hence, this order offsets the problematic weight $(2/(1-t^2))^\alpha$. Moreover, the convergence $C \to 0$, which stems from the assumption $c_\alpha(\psi)(1) \to 0$, supports Theorem 3.1.

Towards the solution to problem (2), we consider the barycenter $e(\varphi_t)(0)$ of the conjugate $\varphi_t = h_t^{-1} \circ \varphi_0 \circ h_t$. Even if $e(\varphi_t)(0) \neq 0$, we can estimate the Fourier coefficients for φ_t uniformly if $e(\varphi_t)(0)$ is in a compact subset of $\mathbb{D}$.

For the base point $\varphi_0 \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$, the derivative $(\varphi_0)'_{\mathbb{S}}(1)$ is not necessarily 1. In this case, the close-up of the behaviour of φ_0 in a neighbourhood of 1 by the conjugation of h_t converges to the Möbius transformation h_s satisfying $(h_s)'_{\mathbb{S}}(1) = (\varphi_0)'_{\mathbb{S}}(1)$. More specifically, this is given in the following claim. The corresponding statement regarding the uniformity under normalisation by rotation will be given in Lemma 4.2.

Claim. For $\ell = (\varphi_0)'_{\mathbb{S}}(1) > 0$, we take $h_s \in \mathrm{M\ddot{o}b}(\mathbb{S})$ with $(1-s)/(1+s) = \ell$. Then, φ_t converges uniformly to h_s on $\mathbb{S}$ as $t \to 1$.

We fix t sufficiently close to 1. Then the claim asserts that φ_t is uniformly close to h_s. Under this condition, we can expect that the barycenter $e(\varphi_t)(0)$ should be close to $e(h_s)(0) = s$. This is to be verified in Section 6. Hence, for some $g_1 \in \mathrm{M\ddot{o}b}(\mathbb{D})$ close to h_s^{-1} (written as $g_1 \fallingdotseq h_s^{-1}$), we have

$$e(g_1 \circ \varphi_t)(0) = 0.$$

Similarly, since $\psi_t = h_t^{-1} \circ \psi \circ h_t$ tends to id by assumption,

$$\psi_t \circ \varphi_t = h_t^{-1} \circ \psi \circ \varphi_0 \circ h_t$$

is close to h_s. Hence, for some g_2 $(\fallingdotseq h_s^{-1}) \in \mathrm{M\ddot{o}b}(\mathbb{D})$,

$$e(g_2 \circ \psi_t \circ \varphi_t)(0) = 0.$$

We now represent the complex dilatations as

$$\mu_{e(\varphi_t)}(0) = \mu_{e(g_1 \circ \varphi_t)}(0) = \frac{a_{-1} - \overline{a_1} b}{a_1 - \overline{a_{-1}} b},$$

$$\mu_{e(\psi_t \circ \varphi_t)}(0) = \mu_{e(g_2 \circ \psi_t \circ \varphi_t)}(0) = \frac{a'_{-1} - \overline{a'_1} b}{a'_1 - \overline{a'_{-1}} b'},$$

where a_1, a_{-1}, b are the Fourier coefficients for $g_1 \circ \varphi_t$, and a'_1, a'_{-1}, b' are the Fourier coefficients for $g_2 \circ \psi_t \circ \varphi_t$. Using the fact that $g_1 \fallingdotseq g_2$, we can estimate

$$|\mu_{e(\psi_t \circ \varphi_t)}(0) - \mu_{e(\varphi_t)}(0)|$$

in terms of the approximation of h_s^{-1} by g_1 and g_2. This will be carried out precisely in Section 6.

4. *Convergence of conjugation of circle diffeomorphisms*

In this section, we prove the results on the convergence of conjugation of circle diffeomorphisms by the canonical Möbius transformations. These are inspired by the paper of Earle [7] and are necessary for the proof of the main theorem concerning the solution of the problems mentioned in the previous section.

In what follows, it is convenient to regard $\mathbb{S}$ as being parametrised by arc length. For $\zeta_1, \zeta_2 \in \mathbb{S}$, the length of the shortest circular arc connecting them is denoted by $d_{\mathbb{S}}(\zeta_1, \zeta_2)$. By the universal cover $\zeta = e^{ix} : \mathbb{R} \to \mathbb{S}$, this is given by

$$d_{\mathbb{S}}(\zeta_1, \zeta_2) = \min\{|x_1 - x_2| \mid \zeta_1 = e^{ix_1},\ \zeta_2 = e^{ix_2}\} \leqslant \pi.$$

For $\varphi_1, \varphi_2 \in \mathrm{Homeo}(\mathbb{S})$, we set

$$\|\varphi_1 - \varphi_2\|_{\mathbb{S}} = \sup_{\zeta \in \mathbb{S}} d_{\mathbb{S}}(\varphi_1(\zeta), \varphi_2(\zeta)).$$

We define $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ to be a lift of $\varphi \in \mathrm{Homeo}(\mathbb{S})$ with $\exp(i\widetilde{\varphi}(x)) = \varphi(e^{ix})$. For $\varphi \in \mathrm{Diff}(\mathbb{S})$, its derivative along $\mathbb{S}$ at $\zeta = e^{ix}$ is defined by $\varphi'_{\mathbb{S}}(\zeta) = \widetilde{\varphi}'(x)$. The α-Hölder constant of the derivative of φ at $\eta = e^{iy} \in \mathbb{S}$ is given by

$$c_\alpha(\varphi)(\eta) = \sup_{\eta \neq \zeta \in \mathbb{S}} \frac{|\varphi'_{\mathbb{S}}(\zeta) - \varphi'_{\mathbb{S}}(\eta)|}{d_{\mathbb{S}}(\zeta, \eta)^\alpha} = \sup_{y \neq x \in \mathbb{R}} \frac{|\widetilde{\varphi}'(x) - \widetilde{\varphi}'(y)|}{|x - y|^\alpha}.$$

We first prove an elementary fact on the integration of the α-Hölder continuous derivative at $1 \in \mathbb{S}$.

PROPOSITION 4.1. *We assume that $\psi \in \mathrm{Diff}(\mathbb{S})$ with $\psi(1) = 1$ and $\psi'_{\mathbb{S}}(1) = 1$ satisfies*

$$|\psi'_{\mathbb{S}}(\zeta) - 1| \leqslant c\, d_{\mathbb{S}}(\zeta, 1)^\alpha$$

for some constant $c > 0$. Then,

$$|\psi(\zeta) - \zeta| \leqslant \frac{c(\pi/2)^{\alpha+1}}{\alpha + 1} |\zeta - 1|^{\alpha+1}.$$

Proof. The lift $\widetilde{\psi}$ with $\widetilde{\psi}(0) = 0$ satisfies $|\widetilde{\psi}'(x) - 1| \leqslant c|x|^\alpha$ for $\zeta = e^{ix}$ ($-\pi < x \leqslant \pi$). This can be written as

$$1 - c|x|^\alpha \leqslant \widetilde{\psi}'(x) \leqslant 1 + c|x|^\alpha.$$

Then, integration from 0 to x yields

$$x - \frac{c}{\alpha + 1}|x|^{\alpha+1} \leqslant \widetilde{\psi}(x) \leqslant x + \frac{c}{\alpha + 1}|x|^{\alpha+1}.$$

Hence,

$$|\psi(\zeta)-\zeta| \leqslant |\widetilde{\psi}(x)-x| \leqslant \frac{c}{\alpha+1}|x|^{\alpha+1} \leqslant \frac{c}{\alpha+1}\{(\pi/2)|\zeta-1|\}^{\alpha+1},$$

for $\zeta = e^{ix}$, which is the required inequality. □

For $t \in (-1,1)$, we utilise a particular Möbius transformation of $\mathbb{D}$ given by

$$h_t(z) = \frac{z+t}{1+tz},$$

which maps the real interval $[-1,1]$ onto itself with the end points fixed and sends 0 to t. Lemma 3.2, mentioned earlier, is an application of the arguments in Earle [**7**, Theorem 2] to an orientation-preserving self-homeomorphism $\psi \in \mathrm{Homeo}(\mathbb{S})$ approximating the identity with a prescribed order at the fixed point $1 \in \mathbb{S}$. The conjugate of ψ by h_t expands the local behaviour of ψ near 1 globally to $\mathbb{S}$.

Proof of Lemma 3.2. We set $\omega = h_t(\zeta)$. Then,

$$\begin{aligned}|\psi_t(\zeta)-\zeta| &= |h_t^{-1}(\psi(\omega)) - h_t^{-1}(\omega)| \\ &= \frac{(1-t^2)\,|\psi(\omega)-\omega|}{|1-t\psi(\omega)|\cdot|t\omega-1|} \leqslant \frac{2C(1-t)\,|\omega-1|^{\alpha+1}}{|1-t\psi(\omega)|\cdot|t\omega-1|}.\end{aligned}$$

Using $1-t \leqslant |1-t\psi(\omega)|$ and $|\omega-1| \leqslant 2|t\omega-1|$ for $t \in (0,1)$, we have

$$|\psi_t(\zeta)-\zeta| \leqslant 4C|\omega-1|^{\alpha}.$$

We set $\delta = (\varepsilon/(4C))^{1/\alpha}$. Then, $4C|\omega-1|^{\alpha} \leqslant \varepsilon$ if $|\omega-1| \leqslant \delta$. Hence, we need only consider the case $|\omega-1| \geqslant \delta$.

As before, we have

$$|\psi_t(\zeta)-\zeta| \leqslant \frac{2C(1-t)\,|\omega-1|^{\alpha+1}}{|1-t\psi(\omega)|\cdot|t\omega-1|} \leqslant \frac{4C(1-t)\,|\omega-1|^{\alpha}}{|1-t\psi(\omega)|}.$$

This time, we use $|1-\psi(\omega)| \leqslant 2|1-t\psi(\omega)|$ for $t \in (0,1)$. Moreover, since $C \leqslant 1/4$,

$$\begin{aligned}|1-\psi(\omega)| &\geqslant |\omega-1| - |\psi(\omega)-\omega| \\ &\geqslant |\omega-1|(1-C|\omega-1|^{\alpha}) \geqslant |\omega-1|/2.\end{aligned}$$

Substituting these estimates into the above inequality, we conclude that

$$|\psi_t(\zeta)-\zeta| \leqslant 16C(1-t)|\omega-1|^{\alpha-1} \leqslant 16C(1-t)\delta^{\alpha-1}.$$

If $1-t \leqslant \frac{1}{4}(\varepsilon/(4C))^{1/\alpha}$, then using $\delta = (\varepsilon/(4C))^{1/\alpha}$ we have

$$16C(1-t)\delta^{\alpha-1} \leqslant \varepsilon.$$

This completes the proof of the assertion. □

We will later consider the situation where the constant c in Proposition 4.1, which will be taken as the α-Hölder constant $c_\alpha(\psi)(1)$ of $\psi'_{\mathbb{S}}$ at $1 \in \mathbb{S}$, can be arbitrarily small. Then, we can choose the constant C in Lemma 3.2 as

$$C = \frac{c(\pi/2)^{\alpha+1}}{\alpha+1} \leqslant \frac{1}{4},$$

and apply this lemma to the proof of the main theorem.

We denote the rotation mapping 1 to $\eta \in \mathbb{S}$ by $r_\eta \in \mathrm{Möb}(\mathbb{S})$. The composition with rotations does not change the derivative at any point $\eta \in \mathbb{S}$ of a diffeomorphism $\varphi_0 \in \mathrm{Diff}^1(\mathbb{S})$. Hence, we may assume that φ_0 fixes 1. The previous lemma handled the case that the derivative at 1 is 1. The following lemma treats the general case and asserts the convergence of the conjugate by h_t to an appropriate Möbius transformation.

LEMMA 4.2. *Let $\varphi_0 \in \mathrm{Diff}^1(\mathbb{S})$ and $\eta \in \mathbb{S}$. We consider rotations $r_\eta, r_{\varphi_0(\eta)} \in \mathrm{M\ddot{o}b}(\mathbb{S})$ and set*

$$\varphi_0^\eta = r_{\varphi_0(\eta)}^{-1} \circ \varphi_0 \circ r_\eta,$$

which fixes $1 \in \mathbb{S}$. For $\ell_\eta = (\varphi_0^\eta)'_{\mathbb{S}}(1) > 0$, we take $h_{s_\eta} \in \mathrm{M\ddot{o}b}(\mathbb{S})$ with $s_\eta \in (-1,1)$ satisfying that $(h_{s_\eta})'_{\mathbb{S}}(1) = (1-s_\eta)/(1+s_\eta) = \ell_\eta$. We also set

$$\varphi_t^\eta = h_t^{-1} \circ \varphi_0^\eta \circ h_t \quad (0 < t < 1).$$

Then, for any $\varepsilon_0 \in (0,2]$, there exists $\delta_0 > 0$ depending only on ε_0 and φ_0 but not on $\eta \in \mathbb{S}$ such that if $1 - t \leqslant \delta_0$, then

$$|\varphi_t^\eta(\zeta) - h_{s_\eta}(\zeta)| \leqslant \varepsilon_0$$

for every $\zeta \in \mathbb{S}$ and for every $\eta \in \mathbb{S}$.

Proof. We set $\omega = h_t(\zeta)$. Then,

$$|\varphi_t^\eta(\zeta) - h_{s_\eta}(\zeta)| = |h_t^{-1}(\varphi_0^\eta(\omega)) - h_t^{-1}(h_{s_\eta}(\omega))| = \frac{(1-t^2)\,|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)|}{|1 - t\varphi_0^\eta(\omega)| \cdot |t h_{s_\eta}(\omega) - 1|}.$$

We will estimate the difference between φ_0^η and h_{s_η} near 1.

CLAIM. *For any $\tilde{\varepsilon} > 0$, there exists $\tilde{\delta} > 0$ independent of η such that if $|h_{s_\eta}(\omega) - 1| \leqslant \tilde{\delta}$, then*

$$|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)| \leqslant \tilde{\varepsilon}|h_{s_\eta}(\omega) - 1|.$$

Proof. We consider the lift $\widetilde{\varphi}_0^\eta : \mathbb{R} \to \mathbb{R}$ of φ_0^η with $\widetilde{\varphi}_0^\eta(0) = 0$. Then,

$$\widetilde{\varphi}_0^\eta(x) = \ell_\eta\, x + \{(\widetilde{\varphi}_0^\eta)'(\xi) - (\widetilde{\varphi}_0^\eta)'(0)\}\, x$$

for some $\xi \in \mathbb{R}$ between 0 and x. Since $(\widetilde{\varphi}_0^\eta)'$ is uniformly eqi-continuous independent of η, $|(\widetilde{\varphi}_0^\eta)'(\xi) - (\widetilde{\varphi}_0^\eta)'(0)|$ is bounded by some constant $c(x) > 0$ with $c(x) \to 0$ $(x \to 0)$. Hence,

$$|\widetilde{\varphi}_0^\eta(x) - \ell_\eta x| \leqslant c(x)|x| \quad (\forall \eta \in \mathbb{S}).$$

We consider the same estimate for the lift $\widetilde{h}_{s_\eta} : \mathbb{R} \to \mathbb{R}$ of h_{s_η} with $\widetilde{h}_{s_\eta}(0) = 0$. Since s_η is uniformly bounded away from -1 and 1 (as ℓ_η is uniformly bounded away from 0 and ∞) independent of η, we also have some constant $c_*(x) > 0$ with $c_*(x) \to 0$ $(x \to 0)$ such that

$$|\widetilde{h}_{s_\eta}(x) - \ell_\eta x| \leqslant c_*(x)|x| \quad (\forall \eta \in \mathbb{S}).$$

Moreover, since $\widetilde{h}_{s_\eta}(x) = \widetilde{h}'_{s_\eta}(\xi_*)\, x$ for some $\xi_* \in \mathbb{R}$ and since $\widetilde{h}'_{s_\eta}(\xi_*) \geqslant \min\{\ell_\eta, \ell_\eta^{-1}\}$, we have

$$|x| \leqslant \frac{1}{\min_{\eta \in \mathbb{S}} \ell_\eta^{\pm 1}}\, |\widetilde{h}_{s_\eta}(x)|.$$

Therefore, we obtain that

$$|\widetilde{\varphi}_0^\eta(x) - \widetilde{h}_{s_\eta}(x)| \leqslant (c(x) + c_*(x))|x| \leqslant \frac{c(x) + c_*(x)}{\min_{\eta \in \mathbb{S}} \ell_\eta^{\pm 1}}\, |\widetilde{h}_{s_\eta}(x)|.$$

Here, $\widetilde{h}_{s_\eta}(x) \to 0$ implies $x \to 0$ uniformly. Hence, the coefficient of $|\widetilde{h}_{s_\eta}(x)|$ in the last term tends to 0. Transforming this inequality into that for $\varphi_0^\eta(\omega)$ and $h_{s_\eta}(\omega)$, we can verify the required claim. □

Proof of Lemma 4.2 *continued.* For a given $\varepsilon_0 \in (0, 2]$, we set $\tilde{\varepsilon} = \varepsilon_0/4$ and choose $\tilde{\delta}$ as in the claim. We first consider the case $|h_{s_\eta}(\omega) - 1| \leqslant \tilde{\delta}$. Then, by $1 - t \leqslant |1 - t\varphi_0^\eta(\omega)|$ and $|h_{s_\eta}(\omega) - 1| \leqslant 2|th_{s_\eta}(\omega) - 1|$ for $t \in (0, 1)$, the claim shows that

$$\frac{(1-t^2)\,|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)|}{|1 - t\varphi_0^\eta(\omega)| \cdot |th_{s_\eta}(\omega) - 1|} \leqslant \frac{2(1-t^2)\,|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)|}{(1-t) \cdot |h_{s_\eta}(\omega) - 1|} \leqslant 4\tilde{\varepsilon} = \varepsilon_0.$$

Thus, we obtain $|\varphi_t^\eta(\zeta) - h_{s_\eta}(\zeta)| \leqslant \varepsilon_0$ without imposing any restriction on $t \in (0, 1)$ in this case.

We now consider the case $|h_{s_\eta}(\omega) - 1| \geqslant \tilde{\delta}$. Then, using $|1 - \varphi_0^\eta(\omega)| \leqslant 2|1 - t\varphi_0^\eta(\omega)|$ for $t \in (0, 1)$ in addition, we have

$$\frac{(1-t^2)\,|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)|}{|1 - t\varphi_0^\eta(\omega)| \cdot |th_{s_\eta}(\omega) - 1|} \leqslant \frac{4(1-t^2)\,|\varphi_0^\eta(\omega) - h_{s_\eta}(\omega)|}{|1 - \varphi_0^\eta(\omega)| \cdot |h_{s_\eta}(\omega) - 1|} \leqslant \frac{16(1-t)}{\tilde{\delta}|1 - \varphi_0^\eta(\omega)|}.$$

Here, if $|h_{s_\eta}(\omega) - 1| = \tilde{\delta}$ then

$$\begin{aligned}|1 - \varphi_0^\eta(\omega)| &\geqslant |h_{s_\eta}(\omega) - 1| - |h_{s_\eta}(\omega) - \varphi_0^\eta(\omega)| \\ &\geqslant (1 - \tilde{\varepsilon})|h_{s_\eta}(\omega) - 1| \geqslant \tilde{\delta}/2\end{aligned}$$

by the above claim and $\tilde{\varepsilon} \leqslant 1/2$. However, since φ_0^η is a self-homeomorphism of $\mathbb{S}$, this is also true even for $|h_{s_\eta}(\omega) - 1| > \tilde{\delta}$. Hence,

$$|\varphi_t^\eta(\zeta) - h_{s_\eta}(\zeta)| \leqslant \frac{16(1-t)}{\tilde{\delta}|1 - \varphi_0^\eta(\omega)|} \leqslant \frac{32(1-t)}{\tilde{\delta}^2}.$$

By choosing $\delta_0 = \varepsilon_0\tilde{\delta}^2/32$, we obtain the assertion. □

5. *Average of circle homeomorphisms*

The barycentric extension is defined by considering the average of a circle homeomorphism. In this section, we will establish properties of the average and the vector field given by the average function.

We recall that the Möbius transformation $\gamma_w \in \text{Möb}(\mathbb{D})$ is defined by

$$\gamma_w(z) = \frac{z - w}{1 - \bar{w}z}$$

for each $w \in \mathbb{D}$. We first list properties of γ_w that will be used later. They are verified easily.

PROPOSITION 5.1. *The Möbius transformation $\gamma_w \in \text{Möb}(\mathbb{D})$ for each $w \in \mathbb{D}$ satisfies the following:*

(1) $|\gamma_w(z) - z| \leqslant \dfrac{2|w|}{1 - |w|}$ *for every* $z \in \overline{\mathbb{D}}$;

(2) $|\gamma_w'(\zeta)| = \dfrac{1 - |w|^2}{|\zeta - w|^2}$ *is the Poisson kernel, which satisfies* $\dfrac{1 - |w|}{1 + |w|} \leqslant |\gamma_w'(\zeta)| \leqslant \dfrac{1 + |w|}{1 - |w|}$ *for every* $\zeta \in \mathbb{S}$;

(3) $\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{S}} \gamma_w(\zeta)\,|d\zeta| = -w.$

For $\varphi \in \mathrm{Homeo}(\mathbb{S})$, we define its average taken at $w \in \mathbb{D}$ as

$$\xi_\varphi(w) = \frac{1}{2\pi} \int_{\mathbb{S}} \frac{\varphi(\zeta) - w}{1 - \bar{w}\varphi(\zeta)} \, |d\zeta|.$$

Then, ξ_φ is a complex-valued differentiable function on $\mathbb{D}$ that can be regarded as a vector field on $\mathbb{D}$. If $\varphi \in \mathrm{Homeo}(\mathbb{S})$ is close to id, then the vector field ξ_φ is close to ξ_{id}, as the following claim shows.

PROPOSITION 5.2. *If $\varphi \in \mathrm{Homeo}(\mathbb{S})$ satisfies $\|\varphi - \mathrm{id}\,\|_{\mathbb{S}} \leqslant \varepsilon$, then $|\xi_\varphi(w) - \xi_{\mathrm{id}}(w)| \leqslant 2\varepsilon$ for every $w \in \mathbb{D}$.*

Proof. The definition of ξ implies that

$$|\xi_\varphi(w) - \xi_{\mathrm{id}}(w)| = \left| \frac{1}{2\pi} \int_{\mathbb{S}} \gamma_w(\varphi(\zeta)) \, |d\zeta| - \frac{1}{2\pi} \int_{\mathbb{S}} \gamma_w(\zeta) \, |d\zeta| \right|.$$

Then, this is estimated from above by

$$\frac{1}{2\pi} \int_{\mathbb{S}} |\gamma_w(\varphi(\zeta)) - \gamma_w(\zeta)| \, |d\zeta| \leqslant \frac{1}{2\pi} \int_{\mathbb{S}} \left(\int_{\zeta}^{\varphi(\zeta)} |\gamma'_w(\eta)| \, |d\eta| \right) |d\zeta|,$$

where the inner path integral is along the circular arc from ζ to $\varphi(\zeta)$. Since $d_{\mathbb{S}}(\varphi(\zeta), \zeta) \leqslant \varepsilon$, this integral is strictly bounded by $\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} |\gamma'_w(\eta)| \, |d\eta|$. Hence, we have

$$|\xi_\varphi(w) - \xi_{\mathrm{id}}(w)| \leqslant \frac{1}{2\pi} \int_{\mathbb{S}} \left(\int_{\zeta-\varepsilon}^{\zeta+\varepsilon} |\gamma'_w(\eta)| \, |d\eta| \right) |d\zeta| \leqslant \frac{2\varepsilon}{2\pi} \int_{\mathbb{S}} |\gamma'_w(\eta)| \, |d\eta| = 2\varepsilon,$$

where the last equality is due to the fact that $|\gamma'_w(\eta)|$ is the Poisson kernel by Proposition 5.1 (2). □

REMARK. Since $\xi_{\mathrm{id}}(w) = -w$, by Proposition 5.1 (3), we have $|\xi_\varphi(w) + w| \leqslant 2\varepsilon$ in Proposition 5.2.

The barycenter of $\varphi \in \mathrm{Homeo}(\mathbb{S})$ is defined to be a point $w \in \mathbb{D}$ such that $\xi_\varphi(w) = 0$. It can be shown that it is unique for every $\varphi \in \mathrm{Homeo}(\mathbb{S})$ (see [**5**, Proposition 1; **13**, Lemma 5.20]).

COROLLARY 5.3. *If $\varphi \in \mathrm{Homeo}(\mathbb{S})$ satisfies $\|\varphi - \mathrm{id}\|_{\mathbb{S}} \leqslant \varepsilon$, then the barycenter $w \in \mathbb{D}$ of φ satisfies $|w| \leqslant 2\varepsilon$.*

Proof. The barycenter w of φ satisfies $\xi_\varphi(w) = 0$ by definition. Then, the result follows from Proposition 5.2 and the remark after it. □

We generalise the above proposition to an assertion on the difference between any two average functions and the difference between their derivatives.

PROPOSITION 5.4. *For any $\varphi, \psi \in \mathrm{Homeo}(\mathbb{S})$, the following inequalities are satisfied for every $w \in \mathbb{D}$:*

(1) $|\xi_\varphi(w) - \xi_\psi(w)| \leqslant \dfrac{1 + |w|}{1 - |w|} \|\varphi - \psi\|_{\mathbb{S}}$;

(2) $|\partial\xi_\varphi(w) - \partial\xi_\psi(w)| \leqslant \dfrac{|w|}{(1 - |w|)^2} \|\varphi - \psi\|_{\mathbb{S}}$;

(3) $|\bar{\partial}\xi_\varphi(w) - \bar{\partial}\xi_\psi(w)| \leqslant \dfrac{(2 - |w|)(1 + |w|)^2}{(1 - |w|)^4} \|\varphi - \psi\|_{\mathbb{S}}$.

Proof. (1) Simple computation yields

$$\frac{\varphi(\zeta)-w}{1-\bar{w}\varphi(\zeta)}-\frac{\psi(\zeta)-w}{1-\bar{w}\psi(\zeta)}=\frac{(1-|w|^2)(\varphi(\zeta)-\psi(\zeta))}{(1-\bar{w}\varphi(\zeta))(1-\bar{w}\psi(\zeta))}.$$

Estimating the absolute value of the denominator from below by $(1-|w|)^2$, we have the assertion.

(2) The ∂-derivative of ξ_φ is

$$\partial\xi_\varphi(w)=\frac{1}{2\pi}\int_{\mathbb{S}}\frac{-1}{1-\bar{w}\varphi(\zeta)}\,|d\zeta|,$$

and the same is true for ξ_ψ. Then,

$$\frac{-1}{1-\bar{w}\varphi(\zeta)}-\frac{-1}{1-\bar{w}\psi(\zeta)}=\frac{-\bar{w}(\varphi(\zeta)-\psi(\zeta))}{(1-\bar{w}\varphi(\zeta))(1-\bar{w}\psi(\zeta))}.$$

By the same estimate for the denominator as before, we have the assertion.

(3) The $\bar\partial$-derivative of ξ_φ is

$$\bar\partial\xi_\varphi(w)=\frac{1}{2\pi}\int_{\mathbb{S}}\frac{(\varphi(\zeta)-w)\varphi(\zeta)}{(1-\bar{w}\varphi(\zeta))^2}\,|d\zeta|,$$

and the same is true for ξ_ψ. Then,

$$\begin{aligned}&\frac{(\varphi(\zeta)-w)\varphi(\zeta)}{(1-\bar{w}\varphi(\zeta))^2}-\frac{(\psi(\zeta)-w)\psi(\zeta)}{(1-\bar{w}\psi(\zeta))^2}\\&\quad=\frac{(\varphi(\zeta)-\psi(\zeta))\{\varphi(\zeta)+\psi(\zeta)+\bar{w}(|w|^2-2)\varphi(\zeta)\psi(\zeta)-w\}}{(1-\bar{w}\varphi(\zeta))^2(1-\bar{w}\psi(\zeta))^2}.\end{aligned}$$

We estimate the absolute value of the second factor of the numerator as

$$\begin{aligned}&|\varphi(\zeta)+\psi(\zeta)+\bar{w}(|w|^2-2)\varphi(\zeta)\psi(\zeta)-w|\\&\qquad\leqslant 2+|w|(2-|w|^2)+|w|=(2-|w|)(1+|w|)^2.\end{aligned}$$

By the same estimate for the denominator as before, we have the assertion. □

We will now see that if $\varphi\in\mathrm{Homeo}(\mathbb{S})$ is close to id and normalised so that its barycenter is at the origin $0\in\mathbb{D}$, then $|\xi_\varphi(w)|$ can be estimated from below by $|\xi_{\mathrm{id}}(w)|=|w|$ near the origin.

LEMMA 5.5. *We assume that $\varphi\in\mathrm{Homeo}(\mathbb{S})$ satisfies $\|\varphi-\mathrm{id}\|_{\mathbb{S}}\leqslant\varepsilon$ and $\xi_\varphi(0)=0$. Then*

$$(1-56\varepsilon)|w|\leqslant|\xi_\varphi(w)|$$

for every $w\in\mathbb{D}$ with $|w|\leqslant 1/2$.

Proof. For any such $w\in\mathbb{D}$, we consider the segment connecting to $0\in\mathbb{D}$. We represent this segment by $\gamma(s)$ with the arc length parameter $s\in[0,|w|]$, $\gamma(0)=0$, and $\gamma(|w|)=w$. Then,

$$\xi_\varphi(w)=\int_0^{|w|}\frac{d\xi_\varphi(\gamma(s))}{ds}\,ds=\int_0^{|w|}(\partial\xi_\varphi(\gamma(s))e^{i\theta}+\bar\partial\xi_\varphi(\gamma(s))e^{-i\theta})ds,$$

where $\theta=\arg w$. From this, we have

$$\xi_\varphi(w)+w=e^{i\theta}\int_0^{|w|}(\partial\xi_\varphi(\gamma(s))+1+\bar\partial\xi_\varphi(\gamma(s))e^{-2i\theta})ds.$$

For $|w| \leqslant 1/2$, we apply Proposition 5.4 (2) and (3) with $\psi = \mathrm{id}$ to obtain

$$|\xi_\varphi(w) + w| \leqslant \int_0^{|w|} |\partial \xi_\varphi(\gamma(s)) + 1|\, ds + \int_0^{|w|} |\bar{\partial} \xi_\varphi(\gamma(s))|\, ds$$
$$\leqslant 2\varepsilon|w| + 54\varepsilon|w| = 56\varepsilon|w|.$$

It follows that $(1 - 56\varepsilon)|w| \leqslant |\xi_\varphi(w)|$, which is the required inequality. □

We choose $\varepsilon > 0$ so that $\varepsilon \leqslant 1/112$. Under this condition, if $\|\varphi - \mathrm{id}\|_{\mathbb{S}} \leqslant \varepsilon$ and $\xi_\varphi(0) = 0$, then $|\xi_\varphi(w)| \geqslant |w|/2$ for $|w| \leqslant 1/2$ by Lemma 5.5.

Lemma 5.6. *We assume that $\varphi_0 \in \mathrm{Homeo}(\mathbb{S})$ satisfies $\xi_{\varphi_0}(0) = 0$ and $|\xi_{\varphi_0}(w)| \geqslant |w|/2$ for $|w| \leqslant 1/2$. If $\varphi_1 \in \mathrm{Homeo}(\mathbb{S})$ satisfies $\|\varphi_1 - \varphi_0\|_{\mathbb{S}} < \varepsilon$ with $0 < \varepsilon \leqslant 1/12$, then $\xi_{\varphi_1}(w)$ has a zero, which is the barycenter of φ_1, in $|w| < 6\varepsilon$.*

Proof. Since $\|\varphi_1 - \varphi_0\|_{\mathbb{S}} < \varepsilon$, Proposition 5.4 (1) implies $|\xi_{\varphi_1}(w) - \xi_{\varphi_0}(w)| < 3\varepsilon$ for $|w| \leqslant 1/2$. Moreover, on the circle $|w| = 6\varepsilon \leqslant 1/2$, we have $|\xi_{\varphi_0}(w)| \geqslant |w|/2 = 3\varepsilon$. Then, by the argument principle, the rotation numbers for ξ_{φ_0} and ξ_{φ_1}, regarded as vector fields, are the same along the circle $|w| = 6\varepsilon$. Since $\xi_{\varphi_0}(w)$ has a unique zero in $|w| < 6\varepsilon$, the Poincaré–Hopf theorem implies that $\xi_{\varphi_1}(w)$ also has a zero in $|w| < 6\varepsilon$. □

6. *Proof of the main theorem*

This section is entirely devoted to the proof of the main theorem in the form of Theorem 3.1. In fact, we first show that it can be reduced to Theorem 6.1 below. Then, we prove this theorem by dividing the arguments into several claims.

We fix an arbitrary $\eta \in \mathbb{S}$. Let $r_\eta \in \mathrm{Möb}(\mathbb{S})$ be the rotation that maps 1 to η. By composing with suitable rotations, we have the decomposition

$$r^{-1}_{\psi\circ\varphi_0(\eta)} \circ \psi \circ \varphi_0 \circ r_\eta = \left(r^{-1}_{\psi\circ\varphi_0(\eta)} \circ \psi \circ r_{\varphi_0(\eta)}\right) \circ \left(r^{-1}_{\varphi_0(\eta)} \circ \varphi_0 \circ r_\eta\right)$$

so that both $\varphi_0^\eta := r^{-1}_{\varphi_0(\eta)} \circ \varphi_0 \circ r_\eta$ and $\psi^\eta := r^{-1}_{\psi\circ\varphi_0(\eta)} \circ \psi \circ r_{\varphi_0(\eta)}$ fix 1. Moreover, we can choose $u = u_{\psi,\eta} \in (-1, 1)$ such that $\psi_0^\eta := h_u \circ \psi^\eta$ satisfies $(\psi_0^\eta)'_{\mathbb{S}}(1) = 1$. We note that $\psi_0^\eta(1) = 1$ still holds after this operation. Under these assumptions, we will prove the following.

Theorem 6.1. *We assume that $\psi_0^\eta(1) = \varphi_0^\eta(1) = 1$ and $(\psi_0^\eta)'_{\mathbb{S}}(1) = 1$. Then, there exist constants $t_0 \in [0, 1)$ and $\widetilde{A} > 0$ depending only on φ_0 such that if $t_0 \leqslant t < 1$ then*

$$\left(\frac{2}{1-t^2}\right)^\alpha |\mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t) - \mu_{e(\varphi_0^\eta)}(t)| \leqslant \widetilde{A}\, c_\alpha(\psi_0^\eta)(1)$$

for every $\eta \in \mathbb{S}$ with $c_\alpha(\psi_0^\eta)(1) \leqslant 1/8$.

Theorem 6.1 ⇒ *Theorem* 3.1. If ψ converges to id in $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, as assumed in Theorem 3.1, then the α-Hölder constant $c_\alpha(\psi)(\eta)$, in particular, converges to 0 uniformly with respect to $\eta \in \mathbb{S}$. Since $c_\alpha(\psi)(\varphi_0(\eta)) = c_\alpha(\psi^\eta)(1)$, this also converges to 0 uniformly. Moreover, it follows from the convergence of the derivative of ψ that $\psi'_{\mathbb{S}}(\varphi_0(\eta)) = (\psi^\eta)'_{\mathbb{S}}(1)$ converges to 1 uniformly. This implies that $u_{\psi,\eta}$ converges to 0 uniformly. Therefore, $c_\alpha(\psi_0^\eta)(1)$ also converges to 0 uniformly with respect to $\eta \in \mathbb{S}$.

The conformal naturality implies that

$$\mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t) = \mu_{e(\psi\circ\varphi_0)}(z)\frac{\bar{\eta}}{\eta}, \quad \mu_{e(\varphi_0^\eta)}(t) = \mu_{e(\varphi_0)}(z)\frac{\bar{\eta}}{\eta},$$

for $z = t\eta \in \mathbb{D}$. Then, Theorem 6.1 shows that

$$\sup_{t_0 \leqslant |z| < 1} \left(\frac{2}{1-|z|^2}\right)^{\alpha} |\mu_{e(\psi\circ\varphi_0)}(z) - \mu_{e(\varphi_0)}(z)| \to 0 \quad (\psi \to \mathrm{id}).$$

Moreover, for $z \in \mathbb{D}$ with $|z| < t_0$, $\mu_{e(\psi\circ\varphi_0)}(z)$ converges to $\mu_{e(\varphi_0)}(z)$ uniformly as ψ converge to id uniformly, which was proved in Douady and Earle [**5**, Proposition 2]. This proves Theorem 3.1. □

We consider the conjugate $\varphi_t^{\eta} = h_t^{-1} \circ \varphi_0^{\eta} \circ h_t$ for $t \in (0,1)$. We set $(\varphi_0^{\eta})'_{\mathbb{S}}(1) = \ell_\eta$ and take h_{s_η} with $(1-s_\eta)/(1+s_\eta) = \ell_\eta$. Since $\ell_\eta = (\varphi_0)'_{\mathbb{S}}(\eta)$, there exists some constant $L \geqslant 1$ depending only on φ_0 such that $L^{-1} \leqslant \ell_\eta \leqslant L$ for every $\eta \in \mathbb{S}$. For a certain constant $\varepsilon_0 \in (0,2]$, which will be fixed later, we choose $\delta_0 > 0$ as in Lemma 4.2. *We now consider any $t > 0$ with $0 < 1 - t \leqslant \delta_0$.*

Claim 1. *Under the above assumption, we have*

$$\|h_{s_\eta}^{-1} \circ \varphi_t^{\eta} - \mathrm{id}\|_{\mathbb{S}} \leqslant \pi L \varepsilon_0/2.$$

Moreover, the barycenter $w_{t,\eta}$ of $h_{s_\eta}^{-1} \circ \varphi_t^{\eta}$ satisfies $|w_{t,\eta}| \leqslant \pi L\varepsilon_0$.

Proof. Lemma 4.2 asserts that if $1 - t \leqslant \delta_0$, then $|\varphi_t^{\eta}(\zeta) - h_{s_\eta}(\zeta)| \leqslant \varepsilon_0$ for every $\zeta \in \mathbb{S}$. This condition implies that $d_{\mathbb{S}}(\varphi_t^{\eta}(\zeta), h_{s_\eta}(\zeta)) \leqslant \pi\varepsilon_0/2$. Since $|(h_{s_\eta}^{-1})'(\zeta)| \leqslant L$, by Proposition 5.1 (2) applied to $w = s_\eta$, we have $d_{\mathbb{S}}(h_{s_\eta}^{-1} \circ \varphi_t^{\eta}(\zeta), \zeta) \leqslant \pi L\varepsilon_0/2$ for every $\zeta \in \mathbb{S}$. This proves the first statement. Then, Corollary 5.3 implies that $|w_{t,\eta}| \leqslant \pi L \varepsilon_0$. □

Using this barycenter $w_{t,\eta}$, we set

$$j_{t,\eta}(z) = \frac{z - w_{t,\eta}}{1 - \overline{w_{t,\eta}} z}.$$

Furthermore, we define $g_{t,\eta} = j_{t,\eta} \circ h_{s_\eta}^{-1} \in \mathrm{M\ddot{o}b}(\mathbb{D})$. Then, the constant $\varepsilon_0 \in (0,2]$ is given as follows. We first prove the following inequality:

$$\begin{aligned}
\|g_{t,\eta} \circ \varphi_t^{\eta} - \mathrm{id}\|_{\mathbb{S}} &= \|j_{t,\eta} \circ h_{s_\eta}^{-1} \circ \varphi_t^{\eta} - \mathrm{id}\|_{\mathbb{S}} \\
&\leqslant \|j_{t,\eta} \circ h_{s_\eta}^{-1} \circ \varphi_t^{\eta} - h_{s_\eta}^{-1} \circ \varphi_t^{\eta}\|_{\mathbb{S}} + \|h_{s_\eta}^{-1} \circ \varphi_t^{\eta} - \mathrm{id}\|_{\mathbb{S}} \\
&= \|j_{t,\eta} - \mathrm{id}\,\|_{\mathbb{S}} + \|h_{s_\eta}^{-1} \circ \varphi_t^{\eta} - \mathrm{id}\|_{\mathbb{S}} \\
&\leqslant \frac{\pi}{2} \cdot \frac{2 \cdot \pi L\varepsilon_0}{1 - \pi L \varepsilon_0} + \frac{\pi L\varepsilon_0}{2},
\end{aligned}$$

where the last inequality is due to Proposition 5.1 (1) and Claim 1. We set the last term in the above inequalities as $\tilde{\varepsilon}_0$. *We now choose $\varepsilon_0 \in (0,2]$ so that $0 < \tilde{\varepsilon}_0 \leqslant 1/112$ and $\varepsilon_0 \leqslant (2\pi L)^{-1}$.* This, in particular, implies $|w_{t,\eta}| \leqslant 1/2$ by Claim 1.

Claim 2. *The average function of $g_{t,\eta} \circ \varphi_t^{\eta}$ given by*

$$\xi(w) = \frac{1}{2\pi} \int_{\mathbb{S}} \frac{g_{t,\eta} \circ \varphi_t^{\eta}(\zeta) - w}{1 - \bar{w} g_{t,\eta} \circ \varphi_t^{\eta}(\zeta)} |d\zeta|$$

satisfies $\xi(0) = 0$ and $|\xi(w)| \geqslant |w|/2$ for $|w| \leqslant 1/2$.

Proof. The barycenter of $g_{t,\eta} \circ \varphi_t^{\eta}$ is

$$e(g_{t,\eta} \circ \varphi_t^{\eta})(0) = j_{t,\eta}(e(h_{s_\eta}^{-1} \circ \varphi_t^{\eta})(0)) = j_{t,\eta}(w_{t,\eta}) = 0.$$

This implies that $\xi(0)=0$. Then, Lemma 5.5 with $\|g_{t,\eta}\circ\varphi_t^\eta-\mathrm{id}\|_{\mathbb{S}}\leqslant\tilde\varepsilon_0\leqslant 1/112$ implies that

$$|\xi(w)|\geqslant(1-56\tilde\varepsilon_0)|w|\geqslant|w|/2$$

for $|w|\leqslant 1/2$. □

For the same t with $0<1-t\leqslant\delta_0$ as above, we consider the conjugate $\psi_t^\eta=h_t^{-1}\circ\psi_0^\eta\circ h_t$ and the decomposition

$$g_{t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta=(g_{t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1})\circ(g_{t,\eta}\circ\varphi_t^\eta).$$

Since $|g_{t,\eta}^{-1}(0)|=|h_{s_\eta}(w_{t,\eta})|$ and $|w_{t,\eta}|\leqslant 1/2$, there is $r\in[0,1)$ depending only on L such that $|g_{t,\eta}^{-1}(0)|\leqslant r$. We set $R=(1+r)/(1-r)$. *We take $\varepsilon>0$ arbitrary with $\varepsilon\leqslant 1/(100R)$, and assume hereafter that $\|\psi_t^\eta-\mathrm{id}\|_{\mathbb{S}}\leqslant\varepsilon$.*

CLAIM 3. *The barycenter w_ε of $g_{t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta$ satisfies $|w_\varepsilon|\leqslant 6R\varepsilon$.*

Proof. Since $\|\psi_t^\eta-\mathrm{id}\|_{\mathbb{S}}\leqslant\varepsilon$ and $|g_{t,\eta}^{-1}(0)|\leqslant r$, we see from Proposition 5.1 (2) that

$$\|g_{t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta-g_{t,\eta}\circ\varphi_t^\eta\|_{\mathbb{S}}=\|g_{t,\eta}\circ\psi_t^\eta-g_{t,\eta}\|_{\mathbb{S}}\leqslant R\varepsilon\ (\leqslant 1/100<1/12).$$

Since $g_{t,\eta}\circ\varphi_t^\eta$ is as in Claim 2, Lemma 5.6 asserts that $g_{t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta$ has the barycenter in $|w|\leqslant 6R\varepsilon$. □

Using this barycenter w_ε, we set

$$j_\varepsilon(z)=\frac{z-w_\varepsilon}{1-\overline{w_\varepsilon}z}.$$

Furthermore, we define $g_{\varepsilon,t,\eta}=j_\varepsilon\circ g_{t,\eta}\in\mathrm{M\ddot{o}b}(\mathbb{D})$. Then, the barycenter of $g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta$ is 0. This is due to the fact that

$$e(g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta)(0)=j_\varepsilon(e(g_{t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta)(0))=j_\varepsilon(w_\varepsilon)=0.$$

CLAIM 4. $\|g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta-g_{t,\eta}\circ\varphi_t^\eta\|_{\mathbb{S}}<25R\varepsilon\leqslant 1/4.$

Proof. We have obtained that $\|g_{t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1}-\mathrm{id}\|_{\mathbb{S}}=\|g_{t,\eta}\circ\psi_t^\eta-g_{t,\eta}\|_{\mathbb{S}}\leqslant R\varepsilon$ in the previous proof. Then, Proposition 5.1 (1) and Claim 3 yield

$$\begin{aligned}
&\|g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta-g_{t,\eta}\circ\varphi_t^\eta\|_{\mathbb{S}}\\
&\quad=\|g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1}-\mathrm{id}\|_{\mathbb{S}}\\
&\quad\leqslant\|j_\varepsilon\circ g_{t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1}-g_{t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1}\|_{\mathbb{S}}+\|g_{t,\eta}\circ\psi_t^\eta\circ g_{t,\eta}^{-1}-\mathrm{id}\|_{\mathbb{S}}\\
&\quad\leqslant\frac{\pi}{2}\cdot\frac{2\cdot 6R\varepsilon}{1-6R\varepsilon}+R\varepsilon<25R\varepsilon.
\end{aligned}$$

Since we have chosen $\varepsilon>0$ so that $\varepsilon\leqslant 1/(100R)$, the last term in the above inequality is bounded by $25R\varepsilon\leqslant 1/4$. □

We will compute the complex dilatation of the barycentric extensions of φ_t^η and $\psi_t^\eta\circ\varphi_t^\eta$ at $0\in\mathbb{D}$ and estimate their difference. For this purpose, we replace them with $g_{t,\eta}\circ\varphi_t^\eta$ and $g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta$, respectively. This is possible as post-composition with a Möbius transformation does not affect the complex dilatation. In addition, since the barycenters of both $g_{t,\eta}\circ\varphi_t^\eta$ and $g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta$ are 0, as we have seen above, we can represent the complex dilatations explicitly in terms of the Fourier coefficients for $g_{t,\eta} \circ \varphi_t^\eta$ and $g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta$, as mentioned in Section 3. Namely,

$$\mu_{e(\varphi_t^\eta)}(0) = \mu_{e(g_{t,\eta}\circ\varphi_t^\eta)}(0) = \frac{a_{-1} - \overline{a_1}b}{a_1 - \overline{a_{-1}}b},$$

where

$$a_1 = \frac{1}{2\pi}\int_{\mathbb{S}} \bar{\zeta}(g_{t,\eta} \circ \varphi_t^\eta)(\zeta)\,|d\zeta|, \quad a_{-1} = \frac{1}{2\pi}\int_{\mathbb{S}} \zeta(g_{t,\eta} \circ \varphi_t^\eta)(\zeta)\,|d\zeta|,$$
$$b = \frac{-1}{2\pi}\int_{\mathbb{S}} (g_{t,\eta} \circ \varphi_t^\eta)(\zeta)^2\,|d\zeta|.$$

Similarly,

$$\mu_{e(\psi_t^\eta\circ\varphi_t^\eta)}(0) = \mu_{e(g_{\varepsilon,t,\eta}\circ\psi_t^\eta\circ\varphi_t^\eta)}(0) = \frac{a'_{-1} - \overline{a'_1}b'}{a'_1 - \overline{a'_{-1}}b'},$$

where

$$a'_1 = \frac{1}{2\pi}\int_{\mathbb{S}} \bar{\zeta}(g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta)(\zeta)\,|d\zeta|, \quad a'_{-1} = \frac{1}{2\pi}\int_{\mathbb{S}} \zeta(g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta)(\zeta)\,|d\zeta|,$$
$$b' = \frac{-1}{2\pi}\int_{\mathbb{S}} (g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta)(\zeta)^2\,|d\zeta|.$$

In Claim 4, we obtained the difference between $g_{t,\eta} \circ \varphi_t^\eta$ and $g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta$. Hence,

$$|a_1 - a'_1| \leqslant \frac{1}{2\pi}\int_{\mathbb{S}} |\bar{\zeta}| 25R\varepsilon\,|d\zeta| = 25R\varepsilon,$$
$$|a_{-1} - a'_{-1}| \leqslant \frac{1}{2\pi}\int_{\mathbb{S}} |\zeta| 25R\varepsilon\,|d\zeta| = 25R\varepsilon,$$
$$|b - b'| \leqslant \frac{1}{2\pi}\int_{\mathbb{S}} 2\cdot 25R\varepsilon\,|d\zeta| = 50R\varepsilon.$$

Moreover,

$$|\mu_{e(\varphi_t^\eta)}(0) - \mu_{e(\psi_t^\eta\circ\varphi_t^\eta)}(0)| = \left|\frac{a_{-1} - \overline{a_1}b}{a_1 - \overline{a_{-1}}b} - \frac{a'_{-1} - \overline{a'_1}b'}{a'_1 - \overline{a'_{-1}}b'}\right| =: \frac{N}{|a_1 - \overline{a_{-1}}b|\cdot|a'_1 - \overline{a'_{-1}}b'|}.$$

Simple computation and the above inequalities show that the numerator N is estimated from above by a positive constant multiple of ε, for instance, $300R\varepsilon$.

For the estimate of the denominator from below, we first consider the following:

$$|a_1 - \overline{a_{-1}}b| \geqslant |a_1| - |a_{-1}||b| \geqslant |a_1| - |a_{-1}|,$$
$$|a'_1 - \overline{a'_{-1}}b'| \geqslant |a'_1| - |a'_{-1}||b'| \geqslant |a'_1| - |a'_{-1}|.$$

We set $\delta = |a_1|^2 - |a_{-1}|^2$ and $\delta' = |a'_1|^2 - |a'_{-1}|^2$. Then,

$$|a_1| - |a_{-1}| = \frac{\delta}{|a_1| + |a_{-1}|} \geqslant \frac{\delta}{2}, \quad |a'_1| - |a'_{-1}| = \frac{\delta}{|a'_1| + |a'_{-1}|} \geqslant \frac{\delta'}{2}.$$

Here, we see that $g_{t,\eta} \circ \varphi_t^\eta$ and $g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta$ are uniformly close to id within $\pi/6$. Indeed, the definitions of $\tilde{\varepsilon}_0$ and Claim 4 yield that

$$\|g_{t,\eta} \circ \varphi_t^\eta - \mathrm{id}\,\|_{\mathbb{S}} \leqslant \tilde{\varepsilon}_0 \leqslant 1/112;$$
$$\|g_{\varepsilon,t,\eta} \circ \psi_t^\eta \circ \varphi_t^\eta - g_{t,\eta} \circ \varphi_t^\eta\|_{\mathbb{S}} \leqslant 1/4.$$

Then, by Pommerenke [**13**, Lemma 5.18] interpreting [**5**, Lemma 3], we have that δ and δ' are uniformly bounded away from 0. Thus, we can find some constant $A > 0$ depending only on R such that

$$|\mu_{e(\varphi_t^\eta)}(0) - \mu_{e(\psi_t^\eta \circ \varphi_t^\eta)}(0)| \leqslant A\varepsilon$$

for every $t \in [1-\delta_0, 1)$.

The conformal naturality again yields

$$\mu_{e(\varphi_t^\eta)}(0) = \mu_{e(h_t^{-1}\circ\varphi_0^\eta\circ h_t)}(0) = \mu_{h_t^{-1}\circ e(\varphi_0^\eta)}(h_t(0))\frac{\overline{h_t'(0)}}{h_t'(0)} = \mu_{e(\varphi_0^\eta)}(t),$$

$$\mu_{e(\psi_t^\eta\circ\varphi_t^\eta)}(0) = \mu_{e(h_t^{-1}\circ\psi_0^\eta\circ\varphi_0^\eta\circ h_t)}(0) = \mu_{h_t^{-1}\circ e(\psi_0^\eta\circ\varphi_0^\eta)}(h_t(0))\frac{\overline{h_t'(0)}}{h_t'(0)} = \mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t).$$

Therefore,

$$|\mu_{e(\varphi_0^\eta)}(t) - \mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t)| = |\mu_{e(\varphi_t^\eta)}(0) - \mu_{e(\psi_t^\eta\circ\varphi_t^\eta)}(0)| \leqslant A\varepsilon$$

for every $\eta \in \mathbb{S}$ and every $t \in [1-\delta_0, 1)$.

The assumption for this conclusion was that $\|\psi_t^\eta - \mathrm{id}\|_{\mathbb{S}} \leqslant \varepsilon$ for $\varepsilon \leqslant 1/(100R)$. Proposition 4.1 and Lemma 3.2 imply that if we choose t and $\epsilon := 2\varepsilon/\pi$ in the relation $1 - t = \frac{1}{4}(\epsilon/(4C))^{1/\alpha}$, then we can obtain that condition. Here, $C = C_\eta$ is given by the α-Hölder constant $c_\alpha(\psi_0^\eta)(1)$ as

$$C_\eta = \frac{c_\alpha(\psi_0^\eta)(1)(\pi/2)^{\alpha+1}}{\alpha+1} < 2c_\alpha(\psi_0^\eta)(1),$$

and can be assumed to be bounded by $1/4$. The above relation is alternatively written as

$$\epsilon = 4^{\alpha+1}C_\eta(1-t)^\alpha \leqslant \frac{1}{50\pi R}.$$

Then, we can find a constant t_0 with $1 - \delta_0 \leqslant t_0 < 1$ depending only on R, and hence only on φ_0, such that

$$|\mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t) - \mu_{e(\varphi_0^\eta)}(t)| \leqslant A \cdot \frac{4^{\alpha+1}}{\alpha+1}\left(\frac{\pi}{2}\right)^{\alpha+2} c_\alpha(\psi_0^\eta)(1)(1-t)^\alpha$$

for every $\eta \in \mathbb{S}$ with $c_\alpha(\psi_0^\eta)(1) \leqslant 1/8$ and every $t \in [t_0, 1)$. Therefore, we have

$$\left(\frac{2}{1-t^2}\right)^\alpha |\mu_{e(\psi_0^\eta\circ\varphi_0^\eta)}(t) - \mu_{e(\varphi_0^\eta)}(t)| \leqslant \widetilde{A}\, c_\alpha(\psi_0^\eta)(1)$$

for some constant $\widetilde{A} > 0$ depending only on φ_0. This completes the proof of Theorem 6.1.

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