PLANAR RIEMANN SURFACES WITH UNIFORMLY DISTRIBUTED CUSPS: PARABOLICITY AND HYPERBOLICITY

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Abstract. We consider a planar Riemann surface R made of a non-compact simply connected plane domain from which an infinite discrete set of points is removed. We give several conditions for the collars of the cusps in R caused by these points to be uniformly distributed in R in terms of Euclidean geometry. Then we associate a graph G with R by taking the Voronoi diagram for the uniformly distributed cusps and show that G represents certain geometric and analytic properties of R.

1. Introduction

In this paper, we investigate the relationship between certain geometric and analytic properties of a Riemann surface $\hat{S}$ and a Riemann surface S obtained from $\hat{S}$ by removing an infinite discrete set of points $\{p_n\}_n$. We provide the Poincaré metric for S; each point of $\{p_n\}_n$ is a cusp with respect to this metric. On the other hand, we look at $\{p_n\}_n$ by considering the original metric for $\hat{S}$. As a typical case, we focus on the following situation: for a non-compact simply connected planar Riemann surface $\hat{R}$ and for an infinite discrete set $\{p_n\}_n$ in $\hat{R}$, we have $R := \hat{R} \setminus \{p_n\}_n$. As far as we are concerned with conformal structure, we can assume that $\hat{R} = \mathbb{D}$ or $\hat{R} = \mathbb{C}$.

We formulate the following question as a type problem: determine the type $\hat{R} = \mathbb{D}$ or $\hat{R} = \mathbb{C}$ by geometric properties of R. Note that the type of $\hat{R}$ is invariant under quasiconformal equivalence of R. Indeed, suppose that there is a quasiconformal homeomorphism $f : R \longrightarrow R'$. Since punctures are removable singularities for quasiconformality, f extends to a quasiconformal homeomorphism of $\hat{R}$ onto $\hat{R}'$. Then they are $\mathbb{D}$ or $\mathbb{C}$ simultaneously. As a basic answer to the type problem, we have that $\hat{R} = \mathbb{D}$ if and only if R possesses Green's function (Theorem 6.1). We will seek a geometric interpretation of this condition on R.

To understand the geometry of R closely, we have to put an assumption on the cusps $\{p_n\}_n$ that they are uniformly distributed. This means that from every point $z \in R$ the distance to the 1-collars $\{C_n\}_n$ of $\{p_n\}_n$ is uniformly bounded with respect to the Poincaré metric on R. Under this condition, we consider the Voronoi diagram for $\{C_n\}_n$ and construct a graph G from the tessellation induced by this diagram. We can regard G as a discrete model for R by providing the path metric with edge length 1 for it. Then the major results obtained in this paper can be summarized as follows.

Theorem 1.1. *The Riemann surface R satisfies any of the following properties if and only if the graph G satisfies the corresponding one: Gromov hyperbolicity; linear isoperimetric inequality; parabolicity for the Laplacian.*

These results will be given in Sections 4, 5 and 6 (Theorems 4.6, 5.2 and 6.6). In particular, concerning the type problem as above, we can give a characterization by using the graph G, namely, $\hat{R} = \mathbb{D}$ if and only if G is not parabolic.

As a secondary topic in this paper, we examine the uniform distribution of cusps $\{p_n\}_n$. In Section 3, we consider necessary and sufficient conditions for cusps $\{p_n\}_n$ to be uniformly distributed in $S = \hat{S} \setminus \{p_n\}_n$

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in general. In Section 7, examples are given where the uniform separation of points $\{p_n\}_n$ in $\hat{R}$ are not necessarily satisfied by the uniform distribution of the cusps $\{p_n\}_n$. In Section 8, we deal with a special Denjoy domain called a train and consider its condition for the uniform distribution of its cusps in connection with its Gromov hyperbolicity.

2. Background

2.1. **Quasi-isometry and Gromov hyperbolicity.** We say that the curve γ in a metric space (X, d) is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). Hereafter L stands for the length of a path with respect to the given metric. The metric space X is said to be *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$. Along this paper we assume that every edge of any graph has length 1.

For a geodesic metric space (X, d) and $x_1, x_2, x_3 \in X$, a geodesic triangle $\Delta = \{x_1, x_2, x_3\}$ is the union of three geodesics $J_1 = [x_1x_2], J_2 = [x_2x_3], J_3 = [x_3x_1]$. We say that Δ is *δ-thin* for a constant $\delta \geq 0$ if for every $x \in J_i$ we have that $d(x, \bigcup_{j\neq i} J_j) \leq \delta$. The space (X, d) is *Gromov δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. In order to simplify the notation, we say that X is Gromov hyperbolic or just hyperbolic instead of saying that (X, d) is Gromov δ-hyperbolic.

A function between two metric spaces $f : (X, d_X) \longrightarrow (Y, d_Y)$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b$$

for every $x_1, x_2 \in X$. Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \geq 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is easy to check that to be quasi-isometric is an equivalence relation on the set of metric spaces. The basic theorem concerning hyperbolicity is as follows (see [12, p. 88]).

Theorem 2.1. *Let us consider an (a, b)-quasi-isometric embedding between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is hyperbolic, then X is hyperbolic. Besides, if f is c-full for some $c \geq 0$, then X is hyperbolic if and only if Y is hyperbolic.*

A *geodesic* in X is a $(1, 0)$-quasigeodesic. The word *geodesic* will always be used with this meaning except for the case of simple closed geodesics (which are just local geodesics).

2.2. **The Poincaré metric on a Riemann surface and a collar of a cusp.** If a Riemann surface S has a universal cover $\Pi : \mathbb{D} \longrightarrow S$, we can define the Poincaré metric in R, i.e., the metric obtained by projecting the metric $ds = 2|dz|/(1 - |z|^2)$ of the unit disk $\mathbb{D}$ by Π. Recall that the universal cover of any planar domain $\Omega \subset \mathbb{C}$ with at least two finite boundary points is the unit disk $\mathbb{D}$. Alternatively, we may use the upper half-plane $\mathbb{H}$ with the metric $ds = |dz|/y$ as the universal cover. With this metric, S is a complete Riemannian manifold with constant curvature -1 and, in particular, S is a geodesic metric space.

If S' is a closed connected subset of S with smooth boundary, we consider in S' the *inner distance*

$$d_{S'}(z, w) := \inf \big\{ L_S(\gamma) \,|\, \gamma \text{ is a curve in } S' \text{ joining } z \text{ and } w \big\} \geq d_S(z, w).$$

One can check that S' with this inner distance is also a geodesic metric space.

The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1 (that is, non-expanding), when we consider the respective Poincaré metrics. A Riemann surface that admits the Poincaré metric is usually called a hyperbolic Riemann surface, but to distinguished it with the Gromov hyperbolicity, we do not use the term "hyperbolic" in this sense.

A *collar* in a Riemann surface S with the Poincaré metric about a simple closed geodesic σ is a doubly connected domain in S "bounded" by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from σ; such collar is equal to $\{p \in S : d_S(p,\sigma) < d\}$, for some positive constant d. The constant d is called the *width* of the collar. Collar Lemma (see [23]) says that always there exists the collar of σ of width $w = \operatorname{Arccosh}\coth(L_S(\sigma)/2)$.

Let S be a Riemann surface with the Poincaré metric having a cusp q. Note that if $S \subset \mathbb{C}$ then every isolated point in the boundary ∂S of S taken in $\mathbb{C}$ is a cusp. A *collar* in S about q is a doubly connected domain in S "bounded" both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q. It is well known that the length of the boundary curve is equal to the area of the collar (see, e.g., [6]). A collar of area β is called a β-collar. For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint. Besides, the collar of the simple closed geodesic σ does not intersect the 2-collar of a cusp (see [23], [25] and [7, Chapter 4]).

2.3. **Linear isoperimetric inequality for Riemann surfaces.** A Riemann surface S satisfies the *linear isoperimetric inequality* (LII) if there exists a constant c such that $A_S(\Omega) \le c\,L_S(\partial\Omega)$ for every relatively compact domain $\Omega \subset S$. Throughout, A_S, L_S and d_S refer to Poincaré area, length and distance of S; if $S = \mathbb{C}$, then these symbols refer to Euclidean metric. We denote by $c(S)$ the sharp linear isoperimetric constant of S, i.e.,

$$c(S) := \sup_{\Omega} \frac{A_S(\Omega)}{L_S(\partial\Omega)}.$$

This is also called the *Cheeger constant* if we take the inverse of this value.

A reduction is that it suffices to prove LII for geodesic domains. A domain $\Omega \subset S$ is said to be a *geodesic domain* if $\partial\Omega$ is a finite number of simple closed geodesics and if $A_S(\Omega)$ is finite. Note that Ω does not need to be relatively compact for it could contain a finite number of cusps. From this point of view, the boundary of a cusp will be considered as an improper geodesic of zero length. Let us denote by $c_g(S)$ the sharp linear isoperimetric constant of S for geodesic domains.

Lemma 2.2. *Let S be a Riemann surface with the Poincaré metric. Then $c_g(S) \le c(S) \le c_g(S) + 1$.*

This result was proved in [11, Lemma 1.2] with additive constant 2 and improved in [19, Theorem 7].

2.4. **Stability of LII under quasi-isometry.** Let G be a graph. Recall that the degree of a vertex v in G is the number of its neighbors, and it is denoted by $\deg v$. We say that a graph G has *bounded degree* if there exists a constant D such that $\deg v \le D$ for every vertex v. For a finite subset S of $V(G)$ we define its boundary ∂S by $\partial S = \{p \in V(G) \mid d_G(p,S) = 1\}$. Then the linear isoperimetric constant of G is defined by $c(G) := \sup_S \#S/\#\partial S$, and we say that G satisfies LII if $c(G) < \infty$. There are several equivalent conditions for LII. See [9] and [18] for example. Among them, *non-amenability* of G is equivalent to satisfying LII.

Let M be a complete Riemannian manifold. The *injectivity radius* of $p \in M$ is defined as the supremum of those $r > 0$ such that the metric open ball $B_M(p,r)$ of center p and radius r is simply connected; we denote it by $\iota(p,M)$ or $\iota(p)$. The injectivity radius $\iota(M)$ of M is the infimum over $p \in M$ of $\iota(p)$. We say that M has *bounded geometry* if it has a lower bound for its Ricci curvature and positive injectivity radius.

Kanai proved in [15, 17] the stability of isoperimetric inequalities under quasi-isometries between complete Riemannian manifolds with bounded geometry and graphs with bounded degree.

Theorem 2.3. *Let $f : X \to Y$ be a quasi-isometry. If X and Y are complete Riemannian manifolds with bounded geometry or graphs with bounded degree, then X and Y satisfy LII or not simultaneously.*

Actually, [15, Theorem 4.1] gives the stability of LII between complete Riemannian manifolds; [15, Lemma 4.2] gives the stability of LII between graphs; [15, Lemmas 4.2 and 4.5] give the stability of LII between a complete Riemannian manifold and a graph.

3. Uniformly distributed cusps

Let $\hat{S}$ be a Riemann surface and $\{p_n\}_n$ an infinite discrete set in $\hat{S}$. Consider $S = \hat{S} \setminus \{p_n\}_n$ equipped with the Poincaré metric d_S. Denote by C_n the 1-collar $C_1(p_n)$ of a cusp p_n in S. We say that the cusps $\{p_n\}_n$ are *uniformly distributed* if there exists a constant M such that $d_S\big(z, \{C_n\}_n\big) \le M$ for every $z \in S$.

It is easy to obtain the following necessary condition for the cusps to be uniformly distributed. Denote by $\mathcal{G}(S)$ the set of simple closed geodesics in S.

Proposition 3.1. *If* $\{p_n\}_n$ *are uniformly distributed, then* $\inf_{\gamma\in\mathcal{G}(S)} L_S(\gamma) > 0$.

Proof. Assume that this infimum is 0. Then there exist simple closed geodesics in S with collars of width as large as we wish. Since collars of cusps and of simple closed geodesic are disjoint, $\{p_n\}_n$ are not uniformly distributed. □

Suppose that $\hat{S} \subset \mathbb{C}$ is a planar Riemann surface. In this case, we can also describe a necessary condition in terms of the Euclidean metric $d_{\mathbb{C}}$. Denote by $\mathcal{A}(S)$ the set of annuli

$$A := \{z \in \mathbb{C} : r_1(A) < |z - z_0| < r_2(A)\}$$

contained in S such that both $\{|z - z_0| \le r_1(A)\}$ and $\{|z - z_0| \ge r_2(A)\}$ contain at least two points in the boundary of S taken in the Riemann sphere $\hat{\mathbb{C}}$. We see that if $\{p_n\}_n$ are uniformly distributed then

$$\sup_{A\in\mathcal{A}(R)} \frac{r_2(A)}{r_1(A)} < \infty.$$

Indeed, if the supremum is infinite, then we can find an annulus $A \in \mathcal{A}(S)$ such that $r_2(A)/r_1(A)$ is arbitrarily large. Also, there exists a simple closed geodesic γ in S freely homotopic to the core curve of A. Since the modulus of A can be arbitrarily large, the length $L_A(\gamma_A)$ of the simple closed geodesic with respect to the Poincaré metric on A tends to zero. By $L_S(\gamma) \le L_A(\gamma_A)$, we have $\inf_{\gamma\in\mathcal{G}(S)} L_S(\gamma) = 0$ and Proposition 3.1 gives that $\{p_n\}_n$ are not uniformly distributed.

Next, we will find a sufficient condition for $\{p_n\}_n$ to be uniformly distributed. We use the following natural result concerning collars of a cusp. Perhaps this is already known but, since we cannot found it in the literature, we include here a short proof. Let $N_r(Y) := \{x \in X : d(x, Y) \le r\}$ denote the r-neighborhood of a subset Y in a metric space X for $r > 0$.

Lemma 3.2 (Another Collar Lemma)**.** *Let* S *be a Riemann surface with the Poincaré metric and* $S' \subseteq S$ *a connected subsurface. Assume that there is a cusp* p *both in* S *and in* S'*. Then the* β*-collar* C'_β *of* p *in* S' *is contained in the* β*-collar* C_β *of* p *in* S *for* $0 < \beta \le 2$.

Proof. Since $S' \subseteq S$, the Poincaré metric satisfies $d_S(z, w) \le d_{S'}(z, w)$ for every $z, w \in S'$. Fix now $0 < \varepsilon < \beta \le 2$. Recall that ∂C_ε and $\partial C'_\varepsilon$ are simple closed curves in S and S', respectively. Furthermore, $\varepsilon = L_{S'}(\partial C'_\varepsilon) \ge L_S(\partial C'_\varepsilon)$. We will show that

$$C'_\varepsilon \subseteq N_{\log \frac{\sinh(\varepsilon/2)}{\varepsilon/2}}\big(C_\varepsilon\big), \tag{3.3}$$

where the neighborhood is defined with respect the metric in S.

If $C'_\varepsilon \subseteq C_\varepsilon$, then (3.3) holds. Otherwise, let $u \in \partial C'_\varepsilon \setminus C_\varepsilon$ be the farthest point from ∂C_ε in $\partial C'_\varepsilon \setminus C_\varepsilon$ with respect to d_S. We consider a universal covering map Π from the upper half-plane $\mathbb{H}$ onto S such that a lift of C_β is given by

$$\{z \in \mathbb{H} \,|\, 0 \le \Re z \le 1,\ \Im z > 1/\beta\}$$

and such that $\Pi(ia) = u$ for some $a > 0$. Then $L_S(\partial C'_\varepsilon) \ge d_{\mathbb{H}}(ia, 1 + ia)$ and hence

$$\sinh^2\frac{\varepsilon}{2} \ge \sinh^2\frac{L_S(\partial C'_\varepsilon)}{2} \ge \sinh^2\frac{d_{\mathbb{H}}(ia, 1 + ia)}{2} = \frac{|1 + ia - ia|^2}{4\Im(1 + ia)\Im(ia)} = \frac{1}{4a^2}.$$

This implies that $1/a \le 2\sinh(\varepsilon/2)$, and thus

$$d_S(u, \partial C_\varepsilon) = \int_a^{1/\varepsilon} \frac{dt}{t} = \log\frac{1}{a\,\varepsilon} \le \log\frac{\sinh(\varepsilon/2)}{\varepsilon/2},$$

which shows the required inclusion (3.3).

Since this inclusion holds for every $0 < \varepsilon < \beta$ and since $d_S(z, w) \le d_{S'}(z, w)$ for every $z, w \in S'$, we have

$$C'_\beta \subseteq N_{\log \frac{\sinh(\varepsilon/2)}{\varepsilon/2}}\big(C_\beta\big)$$

for every $0 < \varepsilon < \beta \le 2$. Letting $\varepsilon \to 0$, we conclude that $C'_\beta \subseteq C_\beta$. □

We apply this lemma to the previous setting: for a planar Riemann surface $\hat{S} \subseteq \mathbb{C}$ and for an infinite discrete set $\{p_n\}_n$ in $\hat{S}$, consider a Riemann surface $S = \hat{S} \setminus \{p_n\}_n$ with the Poincaré metric. Here the boundary ∂S of S is taken in $\mathbb{C}$.

Lemma 3.4. *Assume that p is a cusp in S and set $r = d_{\mathbb{C}}(p, \partial S \setminus \{p\})$. Then for each $0 < \beta \leq 2$ there exists a constant c_β depending only on β such that*

$$B(p, e^{-2\pi/\beta} r) \setminus \{p\} \subseteq C_\beta \subseteq B(p, c_\beta r) \setminus \{p\},$$

where C_β is the β-collar of p in S and $B(p, \rho) = \{z \in \mathbb{C} : |z - p| < \rho\}$ denotes the Euclidean open ball.

Proof. Denote by C'_β the β-collar of p in the punctured disk $B(p, r) \setminus \{p\}$ and by C''_β the β-collar of p in $\mathbb{C} \setminus \{p, q\}$, where $q \in \partial S$ is a point satisfying $|p - q| = r$. Since

$$B(p, r) \setminus \{p\} \subseteq S \subseteq \mathbb{C} \setminus \{p, q\},$$

Lemma 3.2 gives $C'_\beta \subseteq C_\beta \subseteq C''_\beta$. Since $C'_\beta = B(p, e^{-2\pi/\beta} r) \setminus \{p\}$, we have the first inclusion. Denote by C^*_β the β-collar of 0 in $\mathbb{C} \setminus \{0, 1\}$. Define now $c_\beta := \sup\{|z| : z \in C^*_\beta\}$. Then $C^*_\beta \subseteq B(0, c_\beta) \setminus \{0, 1\}$, and applying a translation, a dilation and a rotation, we have $C''_\beta \subseteq B(p, c_\beta r) \setminus \{p, q\}$. □

By using this lemma, we consider a sufficient condition for $\{p_n\}_n$ to be uniformly distributed. Recall that we denote $C_1(p_n) = C_n$.

Proposition 3.5. *For $S = \hat{S} \setminus \{p_n\}_n$, assume that there exists a constant c verifying the following: for each $z \in S \setminus \{C_n\}_n$, there exists some cusp $p \in \{p_n\}_n$ such that*

$$|z - p| = d_{\mathbb{C}}(z, \partial S) \leq c \, d_{\mathbb{C}}(p, \partial S \setminus \{p\}).$$

Then $\{p_n\}_n$ are uniformly distributed.

Proof. Set $r = d_{\mathbb{C}}(p, \partial S \setminus \{p\})$. By Lemma 3.4, the 1-collar $C_1(p)$ of p in S contains $B(p, e^{-2\pi} r) \setminus \{p\}$. We consider the disk $S' := B(z, |z - p|) \subseteq S$. If $z \in C_1(p)$, then $d_S(z, \{C_n\}_n) = 0$. Otherwise, there is a point $w \in \partial C_1(p)$ belonging to the Euclidean segment joining z and p. By assumption $|z - p| \leq cr$ and by the fact above we have $|w - p| \geq e^{-2\pi} r$. Then the Poincaré metric in S' holds $d_{S'}(z, w) \leq \log(2ce^{2\pi} - 1)$. Since $d_S(z, w) \leq d_{S'}(z, w)$, this shows that the distance from z to $C_1(p)$, and hence to $\{C_n\}_n$ is bounded by a uniform constant. □

This proposition can be generalized as follows.

Theorem 3.6. *For $S = \hat{S} \setminus \{p_n\}_n$, assume that there exists a positive constant $\varepsilon < e^{-2\pi}$ verifying the following: for each $z \in S \setminus \{C_n\}_n$ there exist a cusp $p \in \{p_n\}_n$ and a curve g joining z and p such that*

$$L_{\mathbb{C}}\big(g \setminus B(p, e^{-2\pi} r)\big) \leq \varepsilon^{-1} r \ ; \quad N_{\varepsilon r}(g) \setminus B(p, e^{-2\pi} r) \subset S,$$

where $r = d_{\mathbb{C}}(p, \partial S \setminus \{p\})$. Then $\{p_n\}_n$ are uniformly distributed.

Proof. By Lemma 3.4, the punctured disk $B(p, e^{-2\pi} r) \setminus \{p\}$ is contained in the 1-collar $C_1(p)$ of p. Hence, the curve g' defined as $g' := g \setminus B(p, e^{-2\pi} r)$ joins z and $C_1(p)$, and

$$d_S\big(z, \{C_n\}_n\big) \leq d_S\big(z, C_1(p)\big) \leq L_S(g') = \int_{g'} \lambda_S(z) \, |dz| \leq \int_{g'} \frac{2 \, |dz|}{d_{\mathbb{C}}(z, \partial S)} \leq \int_{g'} \frac{2 \, |dz|}{\varepsilon r} \leq \frac{2}{\varepsilon^2} \, .$$

Therefore, $\{p_n\}_n$ are uniformly distributed. □

Note that Proposition 3.5 also follows from Theorem 3.6. Indeed, for each $z \in S \setminus \{C_n\}_n$, take the cusp $p \in \{p_n\}_n$ as in Proposition 3.5. Set $\varepsilon := \min\{e^{-2\pi}/2, 1/(c - e^{-2\pi})\}$ and choose the Euclidean segment joining z and p as g. Then the assumption of Theorem 3.6 is verified for these ε and g. We expect that this theorem might provide a necessary and sufficient condition for $\{p_n\}_n$ to be uniformly distributed.

4. Graphs for the Voronoi diagram and quasi-isometry

We consider uniformly distributed cusps for the following special Riemann surfaces and associate them with certain graphs. Let $\hat{R}$ be a non-compact simply connected planar Riemann surface. Then we can assume that $\hat{R} = \mathbb{D}$ or $\hat{R} = \mathbb{C}$. Define $R := \hat{R} \setminus \{p_n\}_n$ where $\{p_n\}_n$ is an infinite discrete set in $\hat{R}$ and provide R with the Poincaré metric. Denote by C_n the 1-collar $C_1(p_n)$ of the cusp p_n in R.

Consider the tessellation of R given by the *Voronoi diagram* of $\{C_n\}_n$, i.e., the tessellation with tiles $\{T_n\}_n$ defined as $T_n = \{z \in R \,|\, d_R\big(z, C_n\big) = d_R\big(z, \{C_m\}_m\big)\}$. Denote by G^* the graph obtained as the 1-skeleton of this tessellation, with edges of length 1. Let G be the dual graph of this tessellation, i.e., the graph with vertices $V(G) = \{v_n\}_n$ such that $[v_n, v_m] \in E(G)$ if and only if $T_n \cap T_m$ has positive length, and with every edge of length 1.

Since 2-collars of different cusps are disjoint, the 2-collar of p_n is contained in T_n and we obtain the following inequalities.

Lemma 4.1. *If $R := \hat{R} \setminus \{p_n\}_n$, then $A_R(T_n \setminus C_n) \geq 1$ for every n and $d_R(\partial C_m, \partial C_n) \geq 2 \log 2$ for every $m \neq n$.*

Since $L_R(\partial C_n) = 1$ and the distance from every point in T_n to C_n is at most M if $\{p_n\}_n$ is uniformly distributed, we have the following result.

Lemma 4.2. *Let $R = \hat{R} \setminus \{p_n\}_n$ and $\{p_n\}_n$ uniformly distributed with constant M. Then* $\operatorname{diam}_R(T_n \setminus C_n) \leq 2M + 1/2$ *for every n.*

Based on these lemmas about the tessellation of R, we obtain the following two claims concerning the graphs G and G^*.

Lemma 4.3. *Let $R = \hat{R} \setminus \{p_n\}_n$ and $\{p_n\}_n$ uniformly distributed with constant M. Then there exists a constant $D = D(M)$ such that* $\deg v \leq D$ *for every $v \in V(G)$ and* $\deg v^* \leq D$ *for every $v^* \in V(G^*)$*

Proof. Fix any vertex $v^* \in V(G^*)$ and consider any neighbor w^* of v^* in G^*. The edge $[v^*, w^*]$ is contained in the boundary of some T_m, and by Lemma 4.2 the set $T_m \setminus C_m$ is contained in the closed ball $\overline{B_R(v^*, 2M + 1/2)}$. Denote by $I(v^*)$ the set of indices m such that T_m contains an edge starting from v^*. It is clear that the cardinality of the set $I(v^*)$ is $\deg v^*$. By Lemma 4.1,

$$\begin{aligned}\deg v^* = \sum_{m \in I(v^*)} 1 &\leq \sum_{m \in I(v^*)} A_R\big(T_m \setminus C_m\big) \leq A_R\big(B_R(v^*, 2M + 1/2)\big) \\ &\leq A_{\mathbb{D}}\big(B_{\mathbb{D}}(0, 2M + 1/2)\big) = 4\pi \sinh^2 \frac{2M + 1/2}{2}\,.\end{aligned}$$

Consider now any fixed $v_m \in V(G)$ and choose a point $z_m \in \partial C_m$. Then $T_m \setminus C_m$ is contained in the closed ball $\overline{B_R(z_m, M + 1/2)}$. If $[v_m, v_n] \in E(G)$, then every point $T_n \setminus C_n$ is at distance at most $2M + 1/2$ from $T_m \setminus C_m$ by Lemma 4.2. Hence, $T_n \setminus C_n$ is contained in the closed ball $\overline{B_R(z_m, 3M + 1)}$. By Lemma 4.1,

$$\begin{aligned}\deg v_m &\leq \sum_{\{n \,|\, [v_m, v_n] \in E(G)\}} A_R\big(T_n \setminus C_n\big) \leq A_R\big(B_R(z_m, 3M + 1)\big) \\ &\leq A_{\mathbb{D}}\big(B_{\mathbb{D}}(0, 3M + 1)\big) = 4\pi \sinh^2 \frac{3M + 1}{2}.\end{aligned}$$

Hence, it suffices to choose

$$D(M) := 4\pi \sinh^2 \frac{3M + 1}{2}$$

for the statement. □

Lemma 4.4. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. For every $r > 0$ there exists a constant $K(r)$ such that if $z \in T_m \setminus C_m$, $w \in T_n \setminus C_n$ and $d_R(z, w) \leq r$ then $d_G(v_m, v_n) \leq K(r)$.*

Proof. Choose a geodesic γ in R joining z and w. Denote by M the constant of uniformly distributiveness of $\{p_n\}_n$. Lemma 4.2 gives that if γ intersects a tile T_j then $T_j \setminus C_j$ is contained in the closed ball $\overline{B_R(z, r+2M+1/2)}$. If some vertex of the tessellation belongs to γ, we can modify slightly γ in a neighborhood of that vertex in order to obtain a curve g in R joining z and w with the following properties:

(a) there is no vertex of the tessellation in g (except perhaps z or w);
(b) if g intersects a tile T_j, then $T_j \setminus C_j$ is contained in the closed ball $\overline{B_R(z, r+2M+1/2)}$.

Denote by $\mathcal{N}$ the set of indices j such that $T_j \setminus C_j$ is contained in the closed ball $\overline{B_R(z, r+2M+1/2)}$. If N denotes the cardinality of the set $\mathcal{N}$, then g induces a path σ in G joining v_m and v_n with $d_G(v_m, v_n) \le L(\sigma) \le N-1$. By Lemma 4.1,

$$d_G(v_m, v_n) \le N-1 \le \sum_{j\in\mathcal{N}} A_R\big(T_j \setminus C_j\big) - 1 \le A_R\big(B_R(z, r+2M+1/2)\big) - 1$$
$$\le A_{\mathbb{D}}\big(B_{\mathbb{D}}(0, r+2M+1/2)\big) - 1 = 4\pi \sinh^2 \frac{r+2M+1/2}{2} - 1 =: K(r),$$

which gives the required constant. □

Now we are ready to show a claim which guarantees that the graphs defined by tessellation of R and R itself except for the collars of the cusps have similar properties. We provide the inner distance for $R_1 := R \setminus \{C_n\}_n$.

Theorem 4.5. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. Then $R_1 = R \setminus \{C_n\}_n$, G and G^* are quasi-isometric.*

Proof. Define a map $f : R_1 \to G$ in the following way: if z belongs to the interior of T_n for some n, then define $f(z) := v_n$; if $z \in \{\partial T_n\}_n$ then choose any m such that $z \in \partial T_m$ and define $f(z) := v_m$. We have $f(R_1) = V(G)$ and f is (1/2)-full.

Consider $z, w \in R_1$ with $f(z) = v_m$ and $f(w) = v_n$. Denote by M the constant of uniform distribution of $\{p_n\}_n$. Choose a geodesic $\sigma = \{v_{n_0} = v_m, v_{n_1}, v_{n_2}, \dots, v_{n_r} = v_n\}$ in G joining v_m and v_n; then $r = d_G\big(f(z), f(w)\big)$. Let h_0 be a geodesic joining z with ∂C_m in R_1, h_{r+1} a geodesic joining w with ∂C_n in R_1 and h_j a geodesic joining $\partial C_{n_{j-1}}$ with ∂C_{n_j} in R_1 for $1 \le j \le r$. Since $\{p_n\}_n$ is uniformly distributed, $L_{R_1}(h_j) \le 2M$ for $1 \le j \le r$. Let g_j be a curve contained in ∂C_{n_j} with length at most 1/2 joining h_j with h_{j+1} for $0 \le j \le r$. Then $h := h_0 \cup h_1 \cup \cdots \cup h_{r+1} \cup g_0 \cup g_1 \cup \cdots \cup g_r$ is a curve joining z with w in R_1. By Lemma 4.2,

$$d_{R_1}(z, w) \le L_{R_1}(h) \le M + 2Mr + M + \frac{1}{2}(r+1) = \Big(2M + \frac{1}{2}\Big) d_G\big(f(z), f(w)\big) + 2M + \frac{1}{2}\,.$$

Choose now a geodesic γ joining z and w in R_1 and denote by k the positive integer satisfying $k-1 \le L_{R_1}(\gamma) < k$. Let $\{z_0 = z, z_1, z_2, \dots, z_k = w\}$ be the points in γ with

$$1 - \frac{1}{k} \le d_{R_1}(z_{j-1}, z_j) = L_{R_1}(\gamma)/k < 1$$

for $1 \le j \le k$. By Lemma 4.4, there exists a constant K such that $d_G(f(z_{j-1}), f(z_j)) \le K$ for $1 \le j \le k$. Then

$$d_G\big(f(z), f(w)\big) \le \sum_{j=1}^{k} d_G\big(f(z_{j-1}), f(z_j)\big) \le \sum_{j=1}^{k} K \frac{L_{R_1}(\gamma) + 1}{k} = K d_{R_1}(z, w) + K.$$

For $a := \max\{2M + 1/2, K\}$ and $b := \max\{1, K\}$, we see that f is an (1/2)-full (a, b)-quasi-isometry.

Since the edges of G and G^* have length 1, and since there exists a constant D such that the degree of any vertex in G or G^* is at most D by Lemma 4.3, [21, Theorem 4.1] gives that G and G^* are quasi-isometric. □

As a natural consequence from the quasi-isometric equivalence, we can consider the Gromov hyperbolicity for these spaces.

Theorem 4.6. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. Then $R_1 = R \setminus \{C_n\}_n$, R, G and G^* are hyperbolic or not simultaneously.*

Proof. By Theorem 4.5 and Theorem 2.1, we have that R_1, G and G^* are hyperbolic or not simultaneously. Hence, it suffices to prove that R is hyperbolic if and only if R_1 is hyperbolic.

We have the following facts:

(a) ∂C_n is a compact set with $R \setminus \partial C_n$ non-connected for every n;
(b) $\operatorname{diam}_R \partial C_n \leq L_R(\partial C_n) = 1$ for every n;
(c) $d_R(\partial C_m, \partial C_n) \geq 2\log 2$ for every $m \neq n$ by Lemma 4.1;
(d) C_n is δ-hyperbolic for some constant δ and for every n, since any two 1-collars are isometric.

These properties allow us to use [24, Theorem 2.4.], which gives that R is hyperbolic if and only if R_1 is hyperbolic. □

5. Isoperimetric inequalities

We consider the linear isoperimetric inequality (LII) for the Riemann surface $R = \hat{R} \setminus \{p_n\}_n$ defined by uniformly distributed cusps. To this end, we modify R to another Riemannian surface R_0 by replacing the collar of each cusp p_n with a disk with a suitable conformal metric. Then we compare R with R_0 as well as the graphs G and G^* from a viewpoint of LII.

Denote by $\mathbb{D}^*$ the punctured unit disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. It is well known that the density $\lambda_{\mathbb{D}^*}$ of the Poincaré metric in $\mathbb{D}^*$ is

$$\lambda_{\mathbb{D}^*}(z) = \frac{1}{|z| \log \frac{1}{|z|}} \,,$$

and that $\{0 < |z| < e^{-2\pi/\beta}\}$ is the β-collar of the cusp at 0. Let us consider a fixed smooth function $f : [0,1) \to (0,\infty)$ with the following properties: $f \equiv 1 + e^{2\pi}/(2\pi)$ in a neighborhood of 0, f is a concave function on $[0, e^{-2\pi}]$, and $f(x) = 1/(x\log(1/x))$ if $x \in [e^{-2\pi}, 1)$. Then $\rho = f(|z|)|dz|$ is a complete conformal metric on $\mathbb{D}$ which coincides with the Poincaré metric of $\mathbb{D}^*$ in the complement of the 1-collar of the cusp at 0, which "fills" this cusp. Note that $\rho \leq d_{\mathbb{D}^*}$ in $\mathbb{D}^*$. Denote by a_0 the area of $\{|z| < e^{-2\pi}\}$ with respect to the metric ρ; then $a_0 < A_{\mathbb{D}^*}\big(\{0 < |z| < e^{-2\pi}\}\big) = 1$. Since $\{|z| \leq e^{-2\pi}\}$ is a compact set, there exists a constant c_0 such that the curvature of ρ satisfies $K_\rho \geq c_0$ for some constant $c_0 \leq -1$.

Given any $R = \hat{R} \setminus \{p_n\}_n$, we define a new surface $R_0 := R \cup \{p_n\}_n$ with the same metric as R in the complement of the 1-collars of the cusps $\{p_n\}_n$, and such that the 1-collar of each cusp p_n is replaced by a disk B_n with a metric isometric to the restriction of ρ to $\{|z| < e^{-2\pi}\}$. It is clear that R_0 and $\hat{R}$, considered just as topological spaces, are the same.

Since $\rho \leq d_{\mathbb{D}^*}$ in $\mathbb{D}^*$, the area with respect to the conformal metric in R_0 satisfies $A_{R_0}(\Omega) \leq A_R(\Omega)$ for every $\Omega \subset R$. We also have that the curvature of R_0 satisfies $K_{R_0} \geq c_0$.

Lemma 5.1. *There is a constant c_1 such that $A_\rho(\Omega) \leq c_1 L_\rho(\partial\Omega)$ for every domain $\Omega \subset \{|z| \leq e^{-2\pi}\}$ with respect to the conformal metric ρ.*

Proof. Since f is a continuous function, there exist universal constants k_1 and k_2 such that $A_\rho(\Omega) \leq k_1 A_{\text{euc}}(\Omega)$ and $L_{\text{euc}}(\partial\Omega) \leq k_2 L_\rho(\partial\Omega)$ for every domain $\Omega \subset \{|z| \leq e^{-2\pi}\}$. Here A_{euc} and L_{euc} stand for the Euclidean area and length respectively. Using the Euclidean isoperimetric inequality, we deduce

$$A_\rho(\Omega) \leq k_1 A_{\text{euc}}(\Omega) \leq \frac{k_1}{4\pi} L_{\text{euc}}(\partial\Omega)^2 \leq \frac{k_1 k_2^2}{4\pi} L_\rho(\partial\Omega)^2 = k_3 L_\rho(\partial\Omega)^2$$

for every domain $\Omega \subset \{|z| \leq e^{-2\pi}\}$, where we set $k_3 = k_1 k_2^2/(4\pi)$.

Assume that Ω satisfies $A_\rho(\Omega) \leq k_3$. If $L_\rho(\partial\Omega) \leq 1$, then $A_\rho(\Omega) \leq k_3 L_\rho(\partial\Omega)^2 \leq k_3 L_\rho(\partial\Omega)$. If $L_\rho(\partial\Omega) \geq 1$, then $A_\rho(\Omega) \leq k_3 \leq k_3 L_\rho(\partial\Omega)$. On the contrary, assume that Ω satisfies $A_\rho(\Omega) > k_3$. Then $k_3 < A_\rho(\Omega) \leq k_3 L_\rho(\partial\Omega)^2$ and we have $L_\rho(\partial\Omega) > 1$. Hence, $A_\rho(\Omega) \leq a_0 < a_0 L_\rho(\partial\Omega)$. Therefore, if we define $c_1 := \max\{k_3, a_0\}$, then $A_\rho(\Omega) \leq c_1 L_\rho(\partial\Omega)$ for every domain $\Omega \subset \{|z| \leq e^{-2\pi}\}$. □

The main result in this section is as follows.

Theorem 5.2. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. Then R_0 and R satisfy LII or not simultaneously.*

Proof. (a) Assume that R_0 satisfies LII. Consider a geodesic domain Ω in R. Let $C_{n_1},\dots,C_{n_m}$ be the 1-collars of cusps contained in Ω. Denote by Ω_0 the domain in R_0 obtained from Ω by filling the cusps $p_{n_1},\dots,p_{n_m}$. Recall that we denote by B_n the ball in R_0 obtained from C_n by filling the cusp p_n. Since 1-collars of different cusps are disjoint and the collar of the simple closed geodesic σ does not intersect the 1-collar of a cusp, we have $L_R(\partial\Omega) = L_{R_0}(\partial\Omega_0)$ and

$$\begin{aligned} A_R(\Omega) &= A_R\big(\Omega \setminus \{C_{n_1},\dots,C_{n_m}\}\big) + \sum_{j=1}^{m} A_R(C_{n_j}) \\ &= A_{R_0}\big(\Omega \setminus \{B_{n_1},\dots,B_{n_m}\}\big) + \frac{1}{a_0}\sum_{j=1}^{m} A_{R_0}(B_{n_j}) \\ &\le \frac{1}{a_0}\, A_{R_0}(\Omega_0) \le \frac{c(R_0)}{a_0}\, L_{R_0}(\partial\Omega_0) = \frac{c(R_0)}{a_0}\, L_R(\partial\Omega). \end{aligned}$$

Hence, $c_g(R) \le c(R_0)/a_0$ and Lemma 2.2 gives $c(R) \le c(R_0)/a_0 + 1$.

(b) Assume that R satisfies LII. Consider a domain Ω_0 in R_0. Without loss of generality we may assume that Ω_0 is a simply connected domain, for otherwise we can "fill the holes" of Ω_0 to obtain a simply connected domain with more area and shorter boundary. Let $B_{n_1},\dots,B_{n_r}$ be the balls intersecting Ω_0 but not contained in Ω_0. Let $\Omega_0^1,\dots,\Omega_0^k$ be the connected components of $\Omega_0 \setminus \big(\overline{B_{n_1}} \cup\cdots\cup \overline{B_{n_r}}\big)$.

Consider any curve g contained in $\partial\Omega_0 \setminus \{B_{n_i}\}_i$ joining two points of $\{\partial B_{n_i}\}_i$. Assume that g joins two points of the same circle ∂B_{n_i}. It is well known that if g' is the arc in ∂B_{n_i} with the same endpoints as g and homotopic to g, then $L_{R_0}(g') = L_R(g') \le L_R(g) = L_{R_0}(g)$. Next assume that g joins two points of the different circles ∂B_{n_i} and $\partial B_{n_{i'}}$. Lemma 4.1 gives that $L_{R_0}(g) = L_R(g) \ge 2\log 2 > 1 = L_{R_0}(\partial B_n)$ for every n. Since the numbers of the connected components of $\partial\Omega_0^j \cap (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})$ and $\partial\Omega_0^j \setminus (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})$ are the same for every $1 \le j \le k$, we have

$$L_{R_0}\big(\partial\Omega_0^j \cap (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big) \le L_{R_0}\big(\partial\Omega_0^j \setminus (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big).$$

Summing up for all j yields that

$$\begin{aligned} \sum_{j=1}^{k} L_{R_0}\big(\partial\Omega_0^j \cap (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big) &\le \sum_{j=1}^{k} L_{R_0}\big(\partial\Omega_0^j \setminus (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big) \\ &= L_{R_0}\big(\partial\Omega_0 \setminus \big(\overline{B_{n_1}}\cup\cdots\cup\overline{B_{n_r}}\big)\big) \le L_{R_0}\big(\partial\Omega_0\big). \end{aligned}$$

We also have

$$L_{R_0}\big(\partial(\Omega_0 \cap (B_{n_1}\cup\cdots\cup B_{n_r})) \setminus \partial\Omega_0\big) \le L_{R_0}\big(\partial\Omega_0\big).$$

These inequalities imply that

$$\begin{aligned} &\sum_{j=1}^{k} L_{R_0}\big(\partial\Omega_0^j\big) \\ &= \sum_{j=1}^{k} L_{R_0}\big(\partial\Omega_0^j \cap (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big) + \sum_{j=1}^{k} L_{R_0}\big(\partial\Omega_0^j \setminus (\partial B_{n_1}\cup\cdots\cup\partial B_{n_r})\big) \le 2L_{R_0}\big(\partial\Omega_0\big); \\ &L_{R_0}\big(\partial(\Omega_0 \cap (B_{n_1}\cup\cdots\cup B_{n_r}))\big) \\ &= L_{R_0}\big(\partial(\Omega_0 \cap (B_{n_1}\cup\cdots\cup B_{n_r})) \cap \partial\Omega_0\big) + L_{R_0}\big(\partial(\Omega_0 \cap (B_{n_1}\cup\cdots\cup B_{n_r})) \setminus \partial\Omega_0\big) \le 2L_{R_0}\big(\partial\Omega_0\big). \end{aligned}$$

Using these inequalities together with $A_{R_0}\big(\Omega_0^j\big) \le A_R\big(\Omega_0^j\big)$ (possibly some ball B_n is contained in Ω_0^j) and Lemma 5.1, we obtain that

$$\begin{aligned} A_{R_0}(\Omega_0) &= \sum_{j=1}^{k} A_{R_0}\big(\Omega_0^j\big) + A_{R_0}\big(\Omega_0 \cap (B_{n_1} \cup \cdots \cup B_{n_r})\big) \\ &\le \sum_{j=1}^{k} A_R\big(\Omega_0^j\big) + A_{R_0}\big(\Omega_0 \cap (B_{n_1} \cup \cdots \cup B_{n_r})\big) \\ &\le c(R) \sum_{j=1}^{k} L_R\big(\partial\Omega_0^j\big) + c_1 L_{R_0}\big(\partial(\Omega_0 \cap (B_{n_1} \cup \cdots \cup B_{n_r}))\big) \le \big(2\, c(R) + 2\, c_1\big) L_{R_0}\big(\partial\Omega_0\big). \end{aligned}$$

This shows that $c(R_0) \le 2(c(R) + c_1)$. □

By using Theorem 2.3, we can extend Theorem 5.2 to a claim which is also true for the Riemannian surface R_0 and the graphs G and G^*. To do this, we have only to prepare the following lemma.

Lemma 5.3. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. Then R_0 and $R_1 = R \setminus \{C_n\}_n$ are quasi-isometric and R_0 has bounded geometry.*

Proof. Since $\operatorname{diam}_{R_0}(B_n) = \operatorname{diam}_{R_0}(\partial B_n) \le 1/2$ for every n and $d_{R_0}(B_m, B_n) \ge 2\log 2$ for any $m \ne n$, applying [24, Theorem 2.1] twice, we have that R_0 and $R_1 = R_0 \setminus \{B_n\}_n$ are quasi-isometric.

Recall that the curvature of R_0 satisfies $K_{R_0} \ge c_0$. Proposition 3.1 gives that there exists a positive constant k_0 with $\iota(z, R) \ge k_0$ for every $z \in R_1$; since the balls $\{B_n\}_n$ are isometric, one can check that the injectivity radius of R_0 is positive. Hence, R_0 has bounded geometry. □

Corollary 5.4. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. Then R, R_0, G and G^* satisfy LII or not simultaneously.*

Proof. We have that R_0, G and G^* are quasi-isometric by Theorem 4.5 and Lemma 5.3. We also obtain that R_0 has bounded geometry by Lemma 5.3 and that G and G^* are of bounded degree by Lemma 4.3. Then Theorem 2.3 and Theorem 5.2 give the assertion. □

6. The type problem

We continue to consider a Riemann surface R given by $R = \hat{R} \setminus \{p_n\}_n$ for a discrete set $\{p_n\}_n$ in a non-compact simply connected Riemann surface $\hat{R}$, which is either $\mathbb{D}$ or $\mathbb{C}$. Note that the uniform distribution of the cusps $\{p_n\}_n$ are not assumed in the first part of this section. We formulate the following question on R as the type problem: determine $\hat{R}$ is $\mathbb{D}$ or $\mathbb{C}$ in terms of the geometry of R.

We see that the existence of Green's function on R gives a complete answer to the type problem. We recall that a *Green's function* in a complete Riemannian manifold M is a positive fundamental solution of the Laplace-Beltrami operator on M. If M satisfies LII then M has Green's function. It is well known that a Riemann surface has Green's function if and only if it possesses non-constant positive superharmonic functions (see [1, p. 204] or [26, p. 434]). It is stated in [1, p. 249] that a domain in $\mathbb{C}$ has Green's function if and only if its Euclidean boundary has positive logarithmic capacity.

Since any discrete set has zero logarithmic capacity, we have the following characterization for $\hat{R} = \mathbb{D}$.

Theorem 6.1. *For $R = \hat{R} \setminus \{p_n\}_n$ as above, $\hat{R} = \mathbb{D}$ if and only if R has Green's function.*

We also have a geometric characterization for $\hat{R} = \mathbb{D}$. We say that a sequence of points $\{p_n\}_n$ in $\hat{R}$ is *uniformly separated* if $d_{\hat{R}}(p_n, p_m) \ge c$ for every $n \ne m$ and some positive constant c. Uniformly separated sequences play a main role in the study of LII and hyperbolicity (see, e.g., [2], [11] and [22]).

Theorem 6.2. *For $R = \hat{R} \setminus \{p_n\}_n$ as above, $\hat{R} = \mathbb{D}$ if and only if there exists a subsequence $\{p_{n_k}\}_k \subseteq \{p_n\}_n$ such that $\hat{R} \setminus \{p_{n_k}\}_k$ satisfies LII.*

Proof. Assume that $\hat{R} \setminus \{p_{n_k}\}_k$ satisfies LII for some subsequence $\{p_{n_k}\}_k \subseteq \{p_n\}_n$. Seeking for a contradiction, assume that $\hat{R} = \mathbb{C}$. By [11, Theorem 4], we see that $\partial\big(\hat{R} \setminus \{p_{n_k}\}_k\big) = \{p_{n_k}\}_k$ has positive logarithmic capacity, which is a contradiction. Hence $\hat{R} = \mathbb{D}$.

Assume now that $\hat{R} = \mathbb{D}$. Define n_k inductively as follows. Choose $n_1 := 1$. If we have chosen $n_1 < n_2 < \cdots < n_{k-1}$, we can define

$$n_k := \min\big\{n > n_{k-1} \,|\, d_{\mathbb{D}}(p_n, p_{n_1}), d_{\mathbb{D}}(p_n, p_{n_2}), \dots, d_{\mathbb{D}}(p_n, p_{n_{k-1}}) \geq 1\big\},$$

since $\{p_n\}_n$ is a discrete set in $\mathbb{D}$. Then $\{p_{n_k}\}_k$ is uniformly separated in $\mathbb{D}$, since $d_{\mathbb{D}}(p_{n_j}, p_{n_k}) \geq 1$ for every $j \neq k$, and we have that $\hat{R} \setminus \{p_{n_k}\}_k$ satisfies LII by [11, Theorem 3]. □

Corollary 6.3. *If $R = \hat{R} \setminus \{p_n\}_n$ satisfies LII, then $\hat{R} = \mathbb{D}$.*

Hereafter, we assume that the cusps $\{p_n\}_n$ are uniformly distributed in order to associate the graph G (or G^* but we omit this hereafter) with the type problem. Theorem 5.2 and Corollary 6.3 yield the following consequence.

Corollary 6.4. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. If the graph G satisfies LII, then $\hat{R} = \mathbb{D}$.*

On a graph G with the path metric, we can define a discrete Laplacian and thus think of Green's function and parabolicity. It is well known that the parabolicity is equivalent to the condition that the simple random walk on G is recurrent. Also, this condition implies that G does not satisfy LII. Moreover, by results in Kanai [16, Theorems 1 and 2], R_0 and G have Green's functions or not simultaneously since they are quasi-isometric. To convert this claim to that for R, we need the following.

Lemma 6.5. *The Riemann surface $\hat{R}$ and the Riemannian surface R_0 have non-constant positive superharmonic functions or not simultaneously.*

Proof. The Riemannian surface R_0 coincides with $\hat{R} \subseteq \mathbb{C}$ as a Riemann surface. Then the Laplace-Beltrami operator Δ_0 on R_0 and the ordinary Laplacian Δ are the same up to a multiple of a positive function. Hence the statement follows. □

Hence the generalization of Corollary 6.4 to an equivalent condition is obtained as a consequence from Theorem 6.1 and Kanai's.

Theorem 6.6. *Let $R = \hat{R} \setminus \{p_n\}_n$ with $\{p_n\}_n$ uniformly distributed. The graph G has Green's function if and only if $\hat{R} = \mathbb{D}$.*

Proof. By Lemma 6.5, Corollary 5.4 and the above argument, we see that $\hat{R}$ and G have Green's functions or not simultaneously. The existence of Green's function on $\hat{R}$ is equivalent to the condition $\hat{R} = \mathbb{D}$. □

Now we will show that a quasi-isometry of R preserves the (non-)parabolicity as in the above theorem. We need some preliminary results in order to prove our next theorem.

We recall here the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [4, p.107]): for any $0 < \varepsilon < \operatorname{Arcsinh} 1$, any Riemann surface S equipped with the Poincaré metric can be partitioned into a thick part

$$S(\varepsilon) := \{z \in S : \iota(z) \geq \varepsilon\},$$

and a thin part $S \setminus S(\varepsilon)$ whose components are either collars of cusps or collars of closed geodesics of length less than or equal to 2ε.

The following result is an straightforward computation.

Lemma 6.7. *Let S be a Riemann surface with the Poincaré metric having a puncture p. If C denotes the 1-collar of p and $z \in C$, then*

$$d_S(z, \partial C) = \log \frac{1}{2 \sinh \iota(z)} .$$

Furthermore, the set $\{z \in C |\, \iota(z) < \varepsilon\}$ is equal to the $(2 \sinh \varepsilon)$-collar of p for any $0 < \varepsilon \leq \operatorname{Arcsinh}(1/2)$.

In [8, Lemma 6.1] appears the following result.

Lemma 6.8. *Let S and S' be planar Riemann surfaces with the Poincaré metric, and let $f : S \longrightarrow S'$ be a c-full (a,b)-quasi-isometry. Then, given $0 < \varepsilon, \varepsilon_1 < \operatorname{Arcsinh} 1$, there exist $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon_1$, which just depend on $\varepsilon, \varepsilon_1, a, b, c$, so that*

$$f(S(\varepsilon)) \subset S'(\varepsilon') \subset N_c(f(S(\tilde{\varepsilon}))).$$

It was proved by Kanai [16, Theorem 1] that the absence of Green's function (parabolicity) is invariant under quasi-isometries between Riemannian manifolds with bounded geometry. We have the following version of Kanai's result without bounded geometry.

Theorem 6.9. *Let $R = \hat{R} \setminus \{p_n\}_n$ and $R' = \hat{R}' \setminus \{p'_n\}_n$ with $\{p_n\}_n$ and $\{p'_n\}_n$ uniformly distributed. If R and R' are quasi-isometric, then $\hat{R}$ and $\hat{R}'$ are $\mathbb{D}$ or $\mathbb{C}$ simultaneously.*

Proof. Since $\{p_n\}_n$ and $\{p'_n\}_n$ are uniformly distributed, injectivity radius can be close to zero only in collars of cusps by Proposition 3.1. Then there exists $0 < \varepsilon_1 < \operatorname{Arcsinh}(1/2)$ such that for every $0 < \varepsilon < \varepsilon_1$ we have that $R \setminus R(\varepsilon)$ and $R' \setminus R'(\varepsilon)$ are the union of the $(2\sinh\varepsilon)$-collars of the punctures in R and R', respectively, by Lemma 6.7.

Let $f : R \to R'$ be a c-full (a,b)-quasi-isometry. Fix $0 < \varepsilon < \varepsilon_1$. By Lemma 6.8, there exist $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon_1$ so that

$$f(R(\varepsilon)) \subset R'(\varepsilon') \subset N_c(f(R(\tilde{\varepsilon}))).$$

Lemma 6.7 gives

$$R(\tilde{\varepsilon}) = N_{-\log(2\sinh\tilde{\varepsilon})}(R_1), \qquad f(R_1) \subset R'(\varepsilon') \subset N_{c-a\log(2\sinh\tilde{\varepsilon})+b}(f(R_1)),$$

and the restriction $f|_{R_1} : R_1 \to R'(\varepsilon')$ is also a quasi-isometry. We remark here that it is easy to see that the restriction of the distance in R to R_1 and the inner distance of R_1 are quasi-isometric. Applying [24, Theorem 2.1] twice, we have that R'_1 and $R'(\varepsilon')$ are quasi-isometric, and then R_1 and R'_1 are quasi-isometric. Lemma 5.3 gives that R_0 and R_1 are quasi-isometric and R_0 has bounded geometry, and that R'_0 and R'_1 are quasi-isometric and R'_0 has bounded geometry.

Therefore, R_0 and R'_0 are quasi-isometric and Kanai's Theorem in [16] gives that R_0 has Green's function if and only if R'_0 has Green's function. Hence, $\hat{R}$ and $\hat{R}'$ are $\mathbb{D}$ or $\mathbb{C}$ simultaneously by Lemma 6.5. □

One can also consider the Gromov hyperbolicity both for a Riemann surface $R = \hat{R} \setminus \{p_n\}_n$ with uniformly distributed $\{p_n\}_n$ and for the associated graph G. By Theorem 4.6, they are hyperbolic or not simultaneously. We might ask a question about the relationship between the condition $\hat{R} = \mathbb{D}$ and the condition that R (or G) is hyperbolic. However, Corollary 8.6 below gives hyperbolic Denjoy domains with $\hat{R} = \mathbb{C}$.

7. Examples regarding uniformly separated points

It is clear that uniformly separated points $\{p_n\}_n$ in $\hat{R}$ are not necessarily uniformly distributed cusps in $R = \hat{R} \setminus \{p_n\}_n$. Conversely, one might think that uniformly distributed cusps are uniformly separated. However, the following examples show that this is not the case for $\hat{R} = \mathbb{C}$ and in $\hat{R} = \mathbb{D}$, respectively.

Example 7.1. For each integer $m \geq 1$, we consider the $16m$ points $p_{m,k} = \sqrt{m}\, e^{2\pi i k/(16m)}$ with $k = 0, 1, \dots, 16m-1$. It is clear that $\{p_{m,k}\}_{m,k}$ is not uniformly separated in $\mathbb{C}$. However, one can check that $\{p_{m,k}\}_{m,k}$ is uniformly distributed in $R = \mathbb{C} \setminus \{p_{m,k}\}_{m,k}$ by using Proposition 3.5.

Indeed, take any $z \in \mathbb{C}$ with $\sqrt{n-1} \leq |z| < \sqrt{n}$ for some $n \geq 1$. Then the nearest point $p = p_{m_0,k_0}$ $(m_0 = n-1, n)$ from z to the discrete set $\{p_{m,k}\}_{m,k}$ is within at most Euclidean distance

$$\sqrt{n} - \sqrt{n-1} + \frac{2\pi\sqrt{n}}{16n} < \frac{2}{\sqrt{n}}.$$

On the other hand, any two distinct points in $\{p_{n-1,k}\}_k \cup \{p_{n,k'}\}_{k'}$ are at least Euclidean distance $1/(4\sqrt{n})$ away from each other. Since the ratio of the first distance to the second is bounded independently of n, we can take the constant c as in Proposition 3.5. Hence $\{p_{m,k}\}_{m,k}$ are uniformly distributed cusps in R.

Example 7.2. Take a finite Riemann surface S of genus 2 with two punctures equipped with the Poincaré metric. Take the simple closed geodesic γ surrounding the two punctures. This gives a geodesic subdomain Ω_2 with two cusps and with the geodesic boundary γ. We consider a regular covering surface $\tilde{S}$ of S with respect to Ω_2. This means that $\pi_1(\tilde{S}) < \pi_1(S)$ is defined by the normal closure of $\pi_1(\Omega_2)$ in $\pi_1(S)$. Geometrically, $\tilde{S}$ is constructed as follows. We fill the punctures of S to make a closed surface Σ of genus 2. Then take the universal covering map $\Pi : \mathbb{D} \longrightarrow \Sigma$ and remove the preimage of the two punctures under Π from $\mathbb{D}$ to make $\tilde{S}$. The restriction of Π to $\tilde{S} \subset \mathbb{D}$ gives the regular covering map $\Pi : \tilde{S} \longrightarrow S$ with the covering transformation group isomorphic to $\pi_1(\Sigma)$.

Consider the preimage of Ω_2 under $\Pi : \tilde{S} \longrightarrow S$, which consists of infinitely many copies of Ω_2. We replace them with geodesic domains $\{\Omega_k\}_{k\geq 3}$, where Ω_k has one geodesic boundary isometric to γ, no genus and k cusps that are uniformly distributed in Ω_k with a constant M independent of k. We denote the resulting Riemann surface by R, which can be represented as $\mathbb{D} \setminus \{p_n\}_n$. Moreover, the cusps $\{p_n\}_n$ are uniformly distributed in R by its construction. However, R does not satisfies LII because $A_R(\Omega_k)/L_R(\gamma) \to \infty$ as $k \to \infty$.

If $\{p_n\}_n$ are uniformly separated in $\mathbb{D}$, then $R = \mathbb{D} \setminus \{p_n\}_n$ satisfies LII by [11, Theorem 3]. Hence, we see that $\{p_n\}_n$ are not uniformly separated.

8. Denjoy domains

A *Denjoy domain* Ω is a domain in the complex plane $\mathbb{C}$ whose boundary is contained in the real axis. Since $\Omega \cap \mathbb{R}$ is an open set in $\mathbb{R}$, it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega \cap \mathbb{R} = \bigcup_{n\in\Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n\in\Lambda}$ are pairwise disjoint.

Along this section we just consider Denjoy domains which can be written as $R = \mathbb{C} \setminus \{p_n\}_n$ with $p_0 = 0$ and $\{p_n\}_n$ a non-bounded increasing sequence. These domains are called tight trains (see [20] and [3]) and are important since they are the simplest examples of infinite ends; furthermore, in a tight train it is possible to give a fairly precise description of the ending geometry. See, e.g. [5], [13], [14], where they call a similar but more general surface (allowing twists) a flute space.

We say that a curve in R is a *fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in $(-\infty, 0)$ and (p_n, p_{n+1}) for some $n > 0$; we denote by γ_n the fundamental geodesic corresponding to n and its length by $2l_n := L_R(\gamma_n)$. We will need the following result given in [3, Theorem 5.1].

Theorem 8.1. *Let R be a Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$. Then, R is hyperbolic if and only if there exists a constant c such that $d_R(z, \mathbb{R}) \leq c$ for every $z \in \{\gamma_n\}_n$.*

Denote by $\mathbb{R}^+$ the positive real half-axis. We have the following consequence of Theorem 8.1.

Corollary 8.2. *Let R be a Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$. Then, R is hyperbolic if and only if there exists a constant c such that $d_R(z, \mathbb{R}^+) \leq c$ for every $z \in \{\gamma_n\}_n$.*

Proof. Assume that R is hyperbolic. Theorem 8.1 gives that there exists a constant c such that $d_R(z, \mathbb{R}) \leq c$ for every $z \in \{\gamma_n\}_n$. Fix $n > 0$ and $z \in \gamma_n$. By symmetry we can assume that $\Im z \geq 0$. Let us define γ_n^+ as $\gamma_n^+ := \gamma_n \cap \{z \in \mathbb{C} \,|\, \Im z \geq 0\}$; then γ_n^+ is a geodesic minimizing the distance from $(-\infty, 0)$ to (p_n, p_{n+1}). Denote by z_1 and z_2 the endpoints of γ_n^+ in $(-\infty, 0)$ and (p_n, p_{n+1}), respectively. Let z_c be the point in γ_n^+ at distance c from z_1; we also have $d_R\big(z_c, (-\infty, 0)\big) = c$. If $z \in [z_c z_2]$, then $d_R(z, \mathbb{R}^+) \leq c$. If $z \in [z_1 z_c]$, then $d_R(z, \mathbb{R}^+) \leq d_R(z, z_c) + d_R(z_c, \mathbb{R}^+) \leq 2c$.

The other implication is a direct consequence of Theorem 8.1, since $d_R(z, \mathbb{R}) \leq d_R(z, \mathbb{R}^+)$. □

A *Y-piece* is a compact bordered Riemann surface with the Poincaré metric which is topologically a sphere without three open disks and whose boundary curves are simple closed geodesics. They are standard tools for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [10, Chapter X.3] and [7, Chapter 1].

A *generalized Y-piece* is a bordered or non-bordered Riemann surface with the Poincaré metric which is topologically a sphere without n open disks and m points, with integers $n, m \geq 0$ and $n + m = 3$, so that the n boundary curves are simple closed geodesics and the m deleted points are cusps. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \le m \le 3$.

Let Y_n be the generalized Y-piece in R bounded by γ_n and γ_{n+1} for $n > 0$. Let Y_0 be the generalized Y-piece in R bounded by γ_1. The hexagon H_n is the intersection $H_n := Y_n \cap \{z \in \mathbb{C} | \Im z \ge 0\}$ for some $n \ge 0$.

Theorem 8.3. *Let R be a Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$. Then the following hold:*

(1) *If $\{p_n\}_n$ is uniformly distributed, then R is hyperbolic and $\inf_n l_n > 0$;*
(2) *If R is hyperbolic, $\inf_n l_n > 0$ and $\sup_n |l_n - l_{n+1}| < \infty$, then $\{p_n\}_n$ is uniformly distributed.*

Proof. (1) Assume that $\{p_n\}_n$ is uniformly distributed with constant M. Proposition 3.1 gives $\inf_n 2l_n \ge \inf_{\gamma\in\mathcal{G}(R)} L_R(\gamma) > 0$. Fix $n > 0$ and $z \in \gamma_n$. There exists m with $d_R(z, C_m) \le M$. Since $L_R(\partial C_m) = 1$, $d_R(w, \mathbb{R}) \le 1/4$ for every $w \in \partial C_m$ and $d_R(z, \mathbb{R}) \le M + 1/4$. Then Theorem 8.1 gives that R is hyperbolic.

(2) Assume now that R is hyperbolic and that there exist positive constants c_1, c_2, with $l_n \ge c_1$ and $|l_n - l_{n+1}| \le c_2$ for every $n > 0$. Corollary 8.2 gives that there exists a constant c such that $d_R(z, \mathbb{R}^+) \le c$ for every $z \in \{\gamma_n\}_n$. Define γ_n^+ as $\gamma_n^+ := \gamma_n \cap \{z \in \mathbb{C} | \Im z \ge 0\}$ and denote by η_n the geodesic in R from γ_n to γ_{n+1} (note that $\eta_n \subset (-\infty, 0)$).

Since $l_n \ge c_1$ for every $n > 0$, there exists a constant c_3 with $L_R(\eta_n) \le c_3$ for every $n > 0$. Since $l_n \ge c_1$ and $|l_n - l_{n+1}| \le c_2$ for every $n > 0$, there exists a constant c_4 with $d_R(x, C_n \cup C_{n+1}) \le c_4$ for every $x \in (p_n, p_{n+1})$ and $n > 0$. Let

$$c_5 := \max\{d_R(x, C_0 \cup C_1) | \, x \in (0, p_1)\} \ ; \qquad c_6 := \max\{c_4, c_5\}.$$

Then $d_R(x, C_n \cup C_{n+1}) \le c_6$ for every $x \in (p_n, p_{n+1})$ and $n \ge 0$. Fix $n > 0$, $z \in \gamma_n$ and $x \in \mathbb{R}^+$ with $d_R(z, x) = d_R(z, \mathbb{R}^+) \le c$. By symmetry we can assume that $\Im z \ge 0$. Let m with $x \in (p_m, p_{m+1})$. Then

$$d_R(z, C_m \cup C_{m+1}) \le d_R(z, x) + d_R(x, C_m \cup C_{m+1}) \le c + c_6.$$

Fix $n > 0$ and $z \in \eta_n$. Since $L_R(\eta_n) \le c_3$,

$$d_R(z, \{C_m\}_m) \le d_R(z, \eta_n \cup (\gamma_n \cup \gamma_{n+1})) + c + c_6 \le c_3/2 + c + c_6 =: c_7.$$

Hence, $d_R(z, \{C_m\}_m) \le c_7$ for every $z \in \partial H_n$ and $n > 0$.

Consider $z \in Y_n$ for some $n \ge 0$. By symmetry we can assume that $\Im z \ge 0$, and then $z \in H_n$. If $n = 0$, then

$$d_R(z, C_0 \cup C_1) \le c_8 := \max\{d_R(w, C_0 \cup C_1) | \, w \in H_0\}.$$

Assume now that $n > 0$. Since $A_R(H_m) = \pi$ for every $m > 0$, there exists a constant c_9 such that $d_R(w, \partial H_m) \le c_9$ for every $w \in H_m$ and $m > 0$.

Let $z_0 \in \partial H_n$ with $d_R(z, z_0) = d_R(z, \partial H_n) \le c_9$. Then

$$d_R(z, \{C_m\}_m) \le d_R(z, z_0) + d_R(z_0, \{C_m\}_m) \le c_9 + c_7.$$

Hence,

$$d_R(z, \{C_n\}_n) \le \max\{c_8, c_7 + c_9\}$$

for every $z \in R$. □

We have an example showing that the second statement in Theorem 8.3 does not hold without the hypothesis $\sup_n |l_n - l_{n+1}| < \infty$. We will need the following technical result proved in [3, Corollary 5.28].

Lemma 8.4. *Let R be a Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$. If there exist a subsequence $\{n_k\}_k$ and a constant c such that $l_{n_k} \le c$ and $n_{k+1} - n_k \le c$ for every k, then R is hyperbolic.*

Example 8.5. Let R be the Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$ with $l_{2k-1} = k$ and $l_{2k} = 1$ for every $k > 0$. Then $\inf_n l_n = 1 > 0$ and $\sup_n |l_n - l_{n+1}| = \infty$. Lemma 8.4, with $n_k = 2k$ and $c = 2$, gives that R is hyperbolic. Since $l_{2k} = 1$ for every $k > 0$ and $\lim_{k\to\infty} l_{2k-1} = \infty$, $\{p_n\}_n$ is not uniformly distributed.

The following consequence of Theorem 8.3 shows that it is possible to have R hyperbolic when $\hat{R} = \mathbb{C}$.

Corollary 8.6. *Let R be a Denjoy domain $R = \mathbb{C} \setminus \{p_n\}_n$. Assume that there exist positive constants c_1 and c_2 such that $c_1 \le l_n \le c_2$ for every $n > 0$. Then $\{p_n\}_n$ is uniformly distributed and R is hyperbolic.*

Proof. The following facts hold:

(a) γ_n is a compact set with $R \setminus \gamma_n$ non-connected for every n;
(b) $\operatorname{diam}_R \gamma_n \le L_R(\gamma_n)/2 = l_n \le c_2$ for every $n > 0$;
(c) Since $l_n \le c_2$ for every $n > 0$, there exists a constant c_3, which just depends on c_2, such that γ_n has a collar of width c_3 for every $n > 0$. Therefore, $d_R(\gamma_m, \gamma_n) \ge 2c_3$ for every $m \ne n$;
(d) Since $l_n \le c_2$ for every $n > 0$, there exists a constant c_4, which just depends on c_2, such that Y_n is c_4-hyperbolic for every $n > 0$ (see, e.g., [22, Proposition 3.2]).

These properties allows to use [24, Theorem 2.4.], which gives that R is hyperbolic.

Since $\inf_n l_n \ge c_1 > 0$ and $\sup_n |l_n - l_{n+1}| \le c_2 - c_1 < \infty$, Theorem 8.3 gives that $\{p_n\}_n$ is uniformly distributed. □

References

[1] Ahlfors, L. V., Sario, L., *Riemann surfaces.* Princeton Univ. Press, 1960.
[2] Alvarez, V., Pestana, D., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces of infinite type, *Rev. Mat. Iberoam.* **15** (1999), 353-427.
[3] Alvarez, V., Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Denjoy domains, *Geom. Dedicata* **121** (2006), 221-245.
[4] Ballman, W., Gromov, M., Schroeder, V., *Manifolds of Non-positive Curvature.* Birkhauser, Boston, 1985.
[5] Basmajian, A., Hyperbolic structures for surfaces of infinite type, *Trans. Amer. Math. Soc.* **336** (1993), 421-444.
[6] Bers, L., An Inequality for Riemann Surfaces. Differential Geometry and Complex Analysis. H. E. Rauch Memorial Volume. Springer-Verlag, 1985.
[7] Buser, P., *Geometry and Spectra of Compact Riemann Surfaces.* Birkhäuser, Boston, 1992.
[8] Cant , A., Granados, A., Portilla, A., Rodr uez, J. M., Quasi-isometries and isoperimetric inequalities in planar domains. To appear in *J. Math. Soc. Japan.*
[9] Ceccherini-Silberstein, T., Grigorchuk, R., de la Harpe, P., Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces, *Proc. Steklov Inst. Math.* **224** (1999), 57-97.
[10] Chavel, I., *Eigenvalues in Riemannian Geometry.* Academic Press, New York, 1984.
[11] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Annales Acad. Sci. Fenn. A. I.* **15** (1990), 165-183.
[12] Ghys, E., de la Harpe, P., *Sur les Groupes Hyperboliques d'après Mikhael Gromov.* Progress in Mathematics, Volume 83, Birkhäuser. 1990.
[13] Haas, A., Dirichlet points, Garnett points and infinite ends of hyperbolic surfaces I, *Ann. Acad. Sci. Fenn. Series AI* **21** (1996), 3-29.
[14] Haas, A., Susskind, P., The geometry at infinity of a hyperbolic Riemann surface of infinite type, *Geom. Dedicata* **130** (2007), 1-24.
[15] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.
[16] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227-238.
[17] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122-137.
[18] Kapovich, I. The non-amenability of Schreier graphs for infinite index quasiconvex subgroups of hyperbolic groups, *Ensign. Math.* **48** (2002), 359-375.
[19] Matsuzaki, K., Isoperimetric constants for conservative Fuchsian groups, *Kodai Math. J.* **28** (2005), 292-300.
[20] Matsuzaki, K., Rodríguez, J. M., Twists and Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica, English Series* **27** (2011), 29-44.
[21] Portilla, A., Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M., Gromov hyperbolic tessellation graphs, to appear in *Utilitas Math.* Preprint in http://gama.uc3m.es/index.php/jomaro.html
[22] Portilla, A., Rodríguez, J. M., Tourís, E., The topology of balls and Gromov hyperbolicity of Riemann surfaces, *Diff. Geom. Appl.* **21** (2004), 317-335.
[23] Randol, B., Cylinders in Riemann surfaces, *Comment. Math. Helv.* **54** (1979), 1-5.
[24] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.
[25] Shimizu, H., On discontinuous groups operating on the product of upper half-planes, *Ann. of Math.* **77** (1963), 33-71.
[26] Tsuji, M., *Potential Theory in Modern Function Theory.* Chelsea, 1959.