Symmetric groups that are not the symmetric conjugates of Fuchsian groups

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ABSTRACT. A symmetric automorphism of the unit circle is the boundary extension of an asymptotically conformal automorphism of the unit disk. A symmetric group is a quasisymmetric group whose elements are symmetric automorphisms. In this paper, we consider a problem whether a symmetric group is conjugate to a Fuchsian group by a symmetric homeomorphism or not. Our answer is negative.

1. Introduction

A *quasiconformal group* is a discrete group of quasiconformal automorphisms of the unit disk Δ whose maximal dilatations are uniformly bounded. A quasisymmetric automorphism of the unit circle $\partial\Delta$ is the boundary extension of a quasiconformal automorphism of Δ. A *quasisymmetric group* is a discrete group of quasisymmetric automorphisms of $\partial\Delta$ whose quasisymmetric constants are uniformly bounded. The boundary extension of a quasiconformal group to $\partial\Delta$ is a quasisymmetric group. Due to Sullivan [13] and Tukia [14], every quasiconformal group is conjugate to a conformal group (Fuchsian group) by a quasiconformal homeomorphism $\Delta \to \Delta$.

On the other hand, since there is no canonical extension of the automorphisms of $\partial\Delta$ to Δ preserving the group structure (cf. [2] and [5]), it was difficult to see that every quasisymmetric group is conjugate to a Fuchsian group by a quasisymmetric homeomorphism $\partial\Delta \to \partial\Delta$, or equivalently, every quasisymmetric group is the boundary extension of a quasiconformal group. Recently, this is shown to be true by Markovic [10], based on a famous result by Tukia [15], Gabai [6] and Casson-Jungreis [1] that a quasisymmetric group, which is a convergence group in the sense of Gehring-Martin [9], is conjugate to a Fuchsian group by a topological homeomorphism $\partial\Delta \to \partial\Delta$.

An *asymptotically conformal group* is a quasiconformal group whose elements are asymptotically conformal automorphisms of Δ. A symmetric automorphism of

$\partial\Delta$ is the boundary extension of an asymptotically conformal automorphism of Δ, which was originally introduced by Gardiner-Sullivan [8]. A *symmetric group* is a quasisymmetric group whose elements are symmetric automorphisms of $\partial\Delta$. It is clear that the boundary extension of an asymptotically conformal group to $\partial\Delta$ is a symmetric group. However, we do not know whether the converse is true or not.

In this note, we consider an analogous problem to the above context; whether a symmetric group is conjugate to a Fuchsian group by a symmetric homeomorphism $\partial\Delta \to \partial\Delta$ or not. Our answer is negative, and in fact, we will prove that, to every infinite non-rigid Fuchsian group, there exists a corresponding symmetric group that is not conjugate to any Fuchsian group by a symmetric homeomorphism. Here, a Fuchsian group G is said to be *rigid* if the Teichmüller space of the orbifold Δ/G consists of a single point. We state our main result precisely as follows.

THEOREM 1.1. *Let G be an infinite non-rigid Fuchsian group possibly with torsion and possibly infinitely generated. Then there exists a quasisymmetric homeomorphism $\bar{f} : \partial\Delta \to \partial\Delta$ such that $G_* = \bar{f}G\bar{f}^{-1}$ is a symmetric group but there exists no symmetric homeomorphism $h : \partial\Delta \to \partial\Delta$ such that hG_*h^{-1} is a Fuchsian group.*

Actually, we obtain the symmetric group G_* in this theorem as the boundary extension of an asymptotically conformal group.

This theorem can be paraphrased as a statement on the existence of a fixed point of the isometric action of G on a closed subspace of the universal Teichmüller space, which is a fiber over the asymptotic universal Teichmüller space. The development of this observation will be discussed elsewhere.

2. Fuchsian groups and quasi-homomorphisms

In this section, we prove that there exists a non-trivial homogenous quasi-homomorphism for every infinite Fuchsian group G. Here, a map $\varphi : G \to \mathbb{R}$ is said to be a *quasi-homomorphism* if there exists a constant $D \geq 0$ such that

$$|\varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2)| \leq D$$

for any g_1 and g_2 in G. Moreover, it is *homogenous* if $\varphi(g^n) = n\varphi(g)$ for any $g \in G$ and for any $n \in \mathbb{Z}$. It is clear that a homomorphism is a homogenous quasi-homomorphism.

Note that, the orbifold $R = \Delta/G$ often has an infinite cyclic cover and hence there exists a surjective homomorphism $\varphi : G \to \mathbb{Z}$. However, this is not always the case. If the first homology group $H_1(G, \mathbb{Z}) = G/[G, G]$ contains no element of infinite order, then there is no surjective homomorphism $\varphi : G \to \mathbb{Z}$, and vice versa. We need to be concerned about such a case. First, we remark the following elementary claims.

PROPOSITION 2.1. *For every free group G possibly infinitely generated, there exists a surjective homomorphism $\varphi : G \to \mathbb{Z}$.*

PROPOSITION 2.2. *For a surjective homomorphism $\theta : G \to G'$ and a non-trivial homogenous quasi-homomorphism $\varphi' : G' \to \mathbb{R}$, the composition $\varphi = \varphi' \circ \theta : G \to \mathbb{R}$ is also a non-trivial homogenous quasi-homomorphism.*

A *hyperbolic group* is a finitely generated group that is a Gromov hyperbolic space with respect to the word metric. It is known that every cofinite area Fuchsian

group is hyperbolic. The following fact is crucial in our arguments, which can be found in [4] and [12].

LEMMA 2.3. *For every infinite hyperbolic group G, there exists a non-trivial homogenous quasi-homomorphism $\varphi : G \to \mathbb{R}$.*

Now we are ready to prove the following.

THEOREM 2.4. *For every infinite Fuchsian group G possibly with torsion and possibly infinitely generated, there is a non-trivial homogenous quasi-homomorphism $\varphi : G \to \mathbb{R}$.*

PROOF. We divide our arguments into the following cases. Case 0: G is elementary. In this case, G is virtually abelian and hence there exists a surjective homomorphism $\varphi : G \to \mathbb{Z}$.

Case 1: G is non-elementary and finitely generated. In this case, G is isomorphic to a cofinite area Fuchsian group G' which is hyperbolic. Then, by Lemma 2.3, we see that there is a non-trivial homogenous quasi-homomorphism for G.

Case 2: G is infinitely generated and has at most finitely many conjugacy classes of elliptic elements. Let $R = \Delta/G$ be the orbifold for G and R' the underlying Riemann surface without cone singularity obtained by forgetting the cone singularities of R. Then there exists a surjective homomorphism $\theta : G \to \pi_1(R')$ onto the fundamental group of R'. Since R' is topologically infinite, $\pi_1(R')$ is a free group. By Proposition 2.1, there exists a surjective homomorphism $\varphi' : \pi_1(R') \to \mathbb{Z}$. Hence the composition $\varphi = \varphi' \circ \theta$ yields a required map.

Case 3: G is infinitely generated and has infinitely many conjugacy classes of elliptic elements. In the orbifold $R = \Delta/G$, we choose finitely many cone singularities x_i with branch order ν_i $(1 \le i \le n)$ such that

$$-2 + \sum_{i=1}^{n} \left(1 - \frac{1}{\nu_i} \right) > 0.$$

Let S be a sphere with n cone singularities of order $(\nu_1, \ldots, \nu_n)$ and let G' be its orbifold fundamental group, which is isomorphic to a cocompact Fuchsian group with torsion uniformizing the orbifold. Then there exists a surjective homomorphism $\theta : G \to G'$. Since G' is a hyperbolic group, Lemma 2.3 and Proposition 2.2 show that there is a non-trivial homogenous quasi-homomorphism for G. $\qquad\square$

3. A discrete model

For a countable group G in general, we define a Banach space $L(G)$ of all real-valued functions $\xi : G \to \mathbb{R}$ endowed with the supremum norm $\|\xi\|_\infty = \sup_{g \in G} |\xi(g)|$. Also we consider a subspace $L_0(G)$ that consists of all elements $\xi \in L(G)$ vanishing at infinity. Namely, $\xi \in L(G)$ belongs to $L_0(G)$ if, for any $\varepsilon > 0$, there exists a finite subset $V \subset G$ such that $\sup_{g \in G-V} |\xi(g)| < \varepsilon$.

A canonical action of G on $L(G)$ is defined by $(\gamma^* \xi)(g) := \xi(\gamma g)$ $(g \in G)$ for any $\xi \in L(G)$ and for any $\gamma \in G$. This action is isometric with respect to the norm on $L(G)$.

Let S be an invertible generating system of G $(S^{-1} = S)$. We number the elements in S as $\{g_1^{\pm 1}, g_2^{\pm 1}, g_3^{\pm 1}, \ldots\}$. For each generator $g_n^{\pm 1}$ $(n \in \mathbb{N})$, we give the integer weight n. The weighted word length $\ell(g)$ for an element $g \in G$ with respect

to S is defined by the minimum of the sum of their weights when we represent g as a word of the generators. This is equal to the weighted path metric between $g \in G$ and the identity on the Cayley graph of G with respect to S. Here each edge assigned for $g_n^{\pm 1}$ has length n. The triangle inequality $\ell(g_1 g_2) \leq \ell(g_1) + \ell(g_2)$ is satisfied for any g_1 and g_2 in G. This weighted word length ℓ induces a distance on the Cayley graph of G having a property that each bounded subset contains only finitely many vertices (elements of G). The definition of vanishing at infinity of $\xi \in L(G)$ is equivalent to saying that $\xi(g)$ converges to 0 as $\ell(g)$ tends to ∞.

We use the following specific function and its obvious property.

PROPOSITION 3.1. *Let $\eta : \mathbb{R} \to [0,1]$ be a piecewise-linear continuous function defined by*

$$\eta(x) = \begin{cases} 0 & (x \leq -1) \\ (x+1)/2 & (-1 \leq x \leq 1) \\ 1 & (1 \leq x) \end{cases}.$$

If x and x' satisfy either $|x - x'| < 2\varepsilon$, $\min\{x, x'\} \geq 1$ or $\max\{x, x'\} \leq -1$ for a positive constant $\varepsilon > 0$, then $|\eta(x) - \eta(x')| < \varepsilon$.

We construct a function $\xi \in L(G)$ from a quasi-homomorphism φ of G, the weighted word length ℓ and the piecewise-linear function η, and show that this function serves as a discrete model for our desired quasiconformal deformation of G.

LEMMA 3.2. *Suppose that a countable group G has a non-trivial homogenous quasi-homomorphism $\varphi : G \to \mathbb{R}$ with a normalization condition $\varphi(a) = 1$ for some $a \in G$. For an invertible generating system $S = \{g_1^{\pm 1}, g_2^{\pm 1}, g_3^{\pm 1}, \dots\}$ of G with $g_1 = a$, let $\ell(g)$ be the weighted word length for $g \in G$ with respect to S. Let $\eta : \mathbb{R} \to [0,1]$ be the piecewise-linear function given in Proposition 3.1. Define a function $\xi : G \to [0,1]$ by*

$$\xi(g) = \eta\left(\frac{\varphi(g)}{\ell(g)}\right) \qquad (g \in G).$$

Then the following properties are satisfied:
  (1) *For every $\gamma \in G$, the function $\gamma^* \xi - \xi$ belongs to $L_0(G)$;*
  (2) *For every $g \in G$, the values $((a^n)^* \xi)(g)$ converge to 1 as $n \to +\infty$ and converge to 0 as $n \to -\infty$.*

PROOF. For property (1), we prove that, for each fixed $\gamma \in G - \{1\}$ and for any small ε ($0 < \varepsilon < 1/2$), there exists ℓ_0 such that $|(\gamma^* \xi - \xi)(g)| < \varepsilon$ for all $g \in G$ with $\ell(g) \geq \ell_0$. Since φ is a quasi-homomorphism, there exists a constant $D \geq 0$ such that

$$|\varphi(\gamma g) - \varphi(g)| \leq |\varphi(\gamma)| + D$$

for every $g \in G$. Then we will show that

$$\ell_0 := \frac{2\ell(\gamma) + |\varphi(\gamma)| + D}{2\varepsilon} > 0$$

is appropriate for proving the assertion above. Having

$$(\gamma^* \xi - \xi)(g) = \eta\left(\frac{\varphi(\gamma g)}{\ell(\gamma g)}\right) - \eta\left(\frac{\varphi(g)}{\ell(g)}\right),$$

we apply Proposition 3.1 to $x_g := \varphi(\gamma g)/\ell(\gamma g)$ and $x_g' := \varphi(g)/\ell(g)$. Then it suffices to see that either $|x_g - x_g'| < 2\varepsilon$, $\min\{x_g, x_g'\} \geq 1$ or $\max\{x_g, x_g'\} \leq -1$ is satisfied for every $g \in G$ with $\ell(g) \geq \ell_0$.

Suppose that neither $\min\{x_g, x_g'\} \geq 1$ nor $\max\{x_g, x_g'\} \leq -1$ is satisfied for some $g \in G$ with $\ell(g) \geq \ell_0$. Then either $|x_g| < 1$ or $|x_g'| < 1$ are satisfied for this g. Indeed, if $x_g \geq 1$ and $-1 \geq x_g'$, then

$$|\varphi(\gamma g) - \varphi(g)| \geq \ell(\gamma g) + \ell(g) \geq \ell_0.$$

However, the left-hand side is bounded by $|\varphi(\gamma)| + D$, which contradicts the definition of ℓ_0. Similarly, we can rule out the case where $x_g' \geq 1$ and $-1 \geq x_g$.

In the case where $|x_g| = |\varphi(\gamma g)|/\ell(\gamma g) < 1$ for $g \in G$ with $\ell(g) \geq \ell_0$, we have

$$\begin{aligned}
|x_g - x_g'| &= \frac{|\varphi(\gamma g)\ell(g) - \varphi(g)\ell(\gamma g)|}{\ell(\gamma g)\ell(g)} \\
&\leq \frac{|\varphi(\gamma g)| \cdot |\ell(g) - \ell(\gamma g)| + \ell(\gamma g) \cdot |\varphi(\gamma g) - \varphi(g)|}{\ell(\gamma g)\ell(g)} \\
&< \frac{|\ell(g) - \ell(\gamma g)| + |\varphi(\gamma g) - \varphi(g)|}{\ell(g)} \\
&\leq \frac{\ell(\gamma) + |\varphi(\gamma)| + D}{\ell_0} < 2\varepsilon.
\end{aligned}$$

Similar calculation can be applied to the case where $|x_g'| < 1$. Thus we complete the proof of (1).

Property (2) is shown as follows. Consider $((a^n)^*\xi)(g) = \eta(\varphi(a^n g)/\ell(a^n g))$ for each fixed $g \in G$ and for every $n \in \mathbb{Z}$. Since φ is a homogenous quasi-homomorphism and $\varphi(a) = 1$, we see that

$$\begin{aligned}
n + \varphi(g) - D = \varphi(a^n) + \varphi(g) - D \\
\leq \varphi(a^n g) \\
\leq \varphi(a^n) + \varphi(g) + D = n + \varphi(g) + D.
\end{aligned}$$

Also, since $\ell(a^{\pm 1}) = 1$ (recall $g_1 = a$), we have $0 \leq \ell(a^n g) \leq |n| + \ell(g)$. And since η is a continuous increasing function, we conclude that

$$\lim_{n \to +\infty} ((a^n)^*\xi)(g) \geq \lim_{n \to +\infty} \eta\left(\frac{n + \varphi(g) - D}{n + \ell(g)}\right) = \eta(1) = 1;$$

$$\lim_{n \to -\infty} ((a^n)^*\xi)(g) \leq \lim_{n \to -\infty} \eta\left(\frac{n + \varphi(g) + D}{-n + \ell(g)}\right) = \eta(-1) = 0.$$

This shows property (2). $\qquad\square$

4. Asymptotically conformal automorphisms

Let Bel be the complex Banach space of all measurable functions μ on the unit disk Δ endowed with the supremum norm $\|\mu\|_\infty = \operatorname{ess.sup}_{z \in \Delta} |\mu(z)|$, where we regard μ as a Beltrami differential $\mu(z)d\bar{z}/dz$. Let M be the unit ball of Bel whose elements are called Beltrami coefficients. The Teichmüller projection $\Phi : M \to T$ onto the universal Teichmüller space T is denoted by $\Phi(\mu) = [f_\mu]$, where $[f_\mu] \in T$ is a Teichmüller class of the quasiconformal homeomorphism f_μ of Δ whose

complex dilatation $\bar{\partial} f_\mu / \partial f_\mu$ is μ. The Teichmüller projection Φ is a holomorphic split submersion with respect to the complex structure on T.

Let Bel_0 be the subspace of Bel consisting of all Beltrami differentials vanishing at infinity. Here we say that $\mu \in \mathrm{Bel}$ vanishes at infinity if, for every $\varepsilon > 0$, there exists a compact subset $V \subset \Delta$ such that $\mathrm{ess.\,sup}_{z \in \Delta - V} |\mu(z)| < \varepsilon$. We say that a quasiconformal homeomorphism f_μ of Δ is *asymptotically conformal* if μ vanishes at infinity. The *asymptotic Teichmüller space AT* is the set of all asymptotic equivalence classes of quasiconformal homeomorphisms of Δ and the asymptotic equivalence is defined similarly to the Teichmüller equivalence by using asymptotically conformal homeomorphisms. This is a quotient space of the Teichmüller space T and the quotient map $\alpha : T \to AT$ is holomorphic with respect to the complex structure endowed with AT. The asymptotic Teichmüller projection $\hat{\Phi} : M \to AT$ is defined by $\hat{\Phi} = \alpha \circ \Phi$. See [3] and [7].

We define the pull-back of $\mu \in \mathrm{Bel}$ by a conformal automorphism γ of Δ as

$$(\gamma^* \mu)(z) = \mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)}.$$

For a Fuchsian group G, let $\mathrm{Bel}(G)$ denote the subspace of Bel consisting of all Beltrami differentials invariant under G, that is, all such μ that $\gamma^* \mu = \mu$ for every $\gamma \in G$. Let $M(G)$ be the unit ball of $\mathrm{Bel}(G)$. The closed submanifold $\Phi(M(G))$ in T can be identified with the Teichmüller space of the orbifold $R = \Delta/G$ and it is denoted by $T(G)$.

Take a disk $\check{W}$ in R avoiding cone singularities and consider the inverse image of $\check{W}$ on Δ, which can be represented by the disjoint union $\bigsqcup_{g \in G} W_g$, where $W_g = g(W_1)$ and W_1 is a lift biholomorphically equivalent to $\check{W}$. For an arbitrary $\tilde{\mu}_0 \in \mathrm{Bel}(G)$, we obtain a Beltrami differential $\mu_0 \in \mathrm{Bel}(G)$ by the restriction of $\tilde{\mu}_0$ to $\bigsqcup_{g \in G} W_g$.

Using the function $\xi \in L(G)$ given in Lemma 3.2, we define a Beltrami differential μ on Δ by

$$\mu(z) = \sum_{g \in G} \xi(g) 1_{W_g}(z) \mu_0(z),$$

where $1_W(z)$ is the characteristic function of W on Δ. Then, the pull-back of μ by $\gamma \in G$ is

$$\begin{aligned}
(\gamma^* \mu)(z) &= \sum_{g \in G} \xi(g) 1_{W_g}(\gamma(z)) \mu_0(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} \\
&= \sum_{g \in G} (\gamma^* \xi)(g) 1_{W_g}(z) \mu_0(z).
\end{aligned}$$

For all $t \in \mathbb{R}$ with $\|t\mu\|_\infty < 1$, we consider a curve $p(t)$ in the Teichmüller space $T(R)$, where $p(t) = \Phi(t\mu)$ is the Teichmüller class determined by a quasiconformal homeomorphism $f_{t\mu} : \Delta \to \Delta$ having the complex dilatation $t\mu$. Also define $G_t = f_{t\mu} G f_{t\mu}^{-1}$, which is a group of quasiconformal automorphisms of Δ.

LEMMA 4.1. *Every quasiconformal automorphism in the group G_t is asymptotically conformal.*

PROOF. Fix $t \in \mathbb{R}$ with $\|t\mu\|_\infty < 1$ and set $f = f_{t\mu}$. For every $\gamma \in G$, we consider the complex dilatation $\mu_{f\gamma f^{-1}}$ of $f\gamma f^{-1} \in G_t$. It satisfies

(i)
$$|\mu_{f\gamma f^{-1}}(f(z))| = \frac{|\mu_{f\gamma}(z) - \mu_f(z)|}{|1 - \overline{\mu_{f\gamma}(z)}\mu_f(z)|} \leq \frac{|\mu_{f\gamma}(z) - \mu_f(z)|}{1 - \|t\mu\|_\infty^2}.$$

Here we have

$$\mu_f(z) = \sum_{g \in G} \xi(g) 1_{W_g}(z) t\mu_0(z);$$

$$\mu_{f\gamma}(z) = (\gamma^*\mu_f)(z) = \sum_{g \in G} (\gamma^*\xi)(g) 1_{W_g}(z) t\mu_0(z).$$

Hence the numerator of the right side fraction in inequality (i) is estimated as

$$|\mu_{f\gamma}(z) - \mu_f(z)| \leq \sum_{g \in G} |(\gamma^*\xi - \xi)(g)| 1_{W_g}(z) \|t\mu_0\|_\infty.$$

When $f(z)$ tends to the boundary $\partial\Delta$ at infinity, so does z. If $z \in W_g$ and z tends to $\partial\Delta$, then such $g \in G$ tends to the infinity, that is, $\ell(g) \to \infty$. Since $\gamma^*\xi - \xi$ vanishes at infinity by property (1) in Lemma 3.2, this implies that $\mu_{f\gamma f^{-1}}$ also vanishes at infinity. Therefore $f\gamma f^{-1}$ is asymptotically conformal. □

5. Asymptotically non-trivial Beltrami differentials

Let N denote a subspace of Bel consisting of all infinitesimally trivial Beltrami differentials. To define this space precisely, let Q be the Banach space of all integrable holomorphic functions φ on Δ endowed with the L^1-norm $\|\varphi\|_1 = \int_\Delta |\varphi|$, where we regard φ as a quadratic differential $\varphi(z)dz^2$. Then the tangent space of the universal Teichmüller space T at the base point $o = [\text{id}]$ is identified with the dual space Q^*. Each element $\mu \in$ Bel induces a bounded linear functional $v_\mu \in Q^*$ by $v_\mu(\varphi) = \int_\Delta \mu\varphi$. We say that $\mu \in$ Bel is *infinitesimally trivial* if $v_\mu = 0$, that is, $\int_\Delta \mu\varphi = 0$ for every $\varphi \in Q$. For the Teichmüller projection $\Phi : M \to T$, the kernel of the derivative $d\Phi_o$ at the base point is coincident with N.

A *degenerating sequence* is a sequence $\{\varphi_n\} \subset Q$ such that $\|\varphi_n\|_1 = 1$ and φ_n converge locally uniformly to zero. We say that $\mu \in$ Bel is *infinitesimally asymptotically trivial* if $\lim_{n\to\infty} |v_\mu(\varphi_n)| = 0$ for every degenerating sequence $\{\varphi_n\} \subset Q$. Let $\hat{N}$ denote the subspace of Bel consisting of all infinitesimally asymptotically trivial Beltrami differentials. For the asymptotic Teichmüller projection $\hat{\Phi} : M \to AT$, the kernel of the derivative $d\hat{\Phi}_o$ at the base point is coincident with $\hat{N}$. This is shown in [3] and [7]. It is clear that N is contained in $\hat{N}$. Actually, we know that $\hat{N} = N + \text{Bel}_0$.

The pull-back of $\varphi \in Q$ by a conformal automorphism γ of Δ is defined as

$$(\gamma^*\varphi)(z) = \varphi(\gamma(z))\gamma'(z)^2.$$

The push-forward $(\gamma_*\varphi)(z)$ is the pull-back by γ^{-1}. Let G be an infinite non-rigid Fuchsian group. Non-rigidity of G is equivalent to a property that there exists a non-trivial holomorphic quadratic differential ψ on Δ invariant under G such that

$$\int_{\Delta/G} |\psi| < \infty \quad \text{and} \quad \sup_{z \in \Delta} \rho^{-2}(z)|\psi(z)| < \infty,$$

where ρ is the hyperbolic density on Δ.

The harmonic Beltrami differential for this ψ is defined by $\tilde{\mu}_0 = \rho^{-2}\overline{\psi} \in \mathrm{Bel}(G)$. As in Section 4, we restrict $\tilde{\mu}_0$ to $\bigsqcup_{g \in G} W_g$ to obtain the Beltrami differential $\mu_0 \in \mathrm{Bel}(G)$. We first see that μ_0 does not belong to N. Indeed, by the surjectivity of the Poincaré series operator, there exists an integrable holomorphic quadratic differential $\tilde{\psi} \in Q$ such that $\sum_{\gamma \in G} \gamma^* \tilde{\psi} = \psi$. Hence

$$\int_\Delta \mu_0 \tilde{\psi} = \int_{\Delta/G} \mu_0 \psi = \int_{W_1} \rho^{-2}|\psi|^2 > 0,$$

which shows that $\mu_0 \notin N$.

By using the function $\xi \in L(G)$, we define the Beltrami differential

$$\mu(z) = \sum_{g \in G} \xi(g) 1_{W_g}(z) \mu_0(z)$$

as in Section 4. This satisfies the following.

LEMMA 5.1. *The Beltrami differential μ does not belong to $\mathrm{Bel}(G) + \hat{N}$.*

PROOF. Suppose to the contrary that we can write $\mu = \nu + \lambda$ for $\nu \in \mathrm{Bel}(G)$ and $\lambda \in \hat{N}$. Let $a \in G$ be the element of G chosen in Lemma 3.2. Take any $\varphi \in Q$ with $\|\varphi\|_1 = 1$ and set $\varphi_n = (a^n)_* \varphi$ for every $n \in \mathbb{N}$. Then $\{\varphi_n\}$ is a degenerating sequence.

Using the facts that the action of the conformal automorphism a preserves the integral of a $(1,1)$-form $u(z)dz d\bar{z}$ and that ν is G-invariant, we have

(ii) $$\int_\Delta (a^n)^* \mu \cdot \varphi = \int_\Delta \mu \varphi_n = \int_\Delta \nu \varphi + \int_\Delta \lambda \varphi_n,$$

where

$$((a^n)^* \mu)(z) = \sum_{g \in G} ((a^n)^* \xi)(g) 1_{W_g}(z) \mu_0(z).$$

Since $((a^n)^* \xi)(g) \to 1$ as $n \to \infty$ by property (2) in Lemma 3.2, we see that $((a^n)^* \mu)(z) \to \mu_0(z)$ pointwise. Hence the left side of equality (ii) converges to $\int_\Delta \mu_0 \varphi$ by the dominated convergence theorem. On the other hand, $\int_\Delta \lambda \varphi_n$ converges to 0 because $\lambda \in \hat{N}$ and $\{\varphi_n\}$ is a degenerating sequence. Hence $\int_\Delta \mu_0 \varphi = \int_\Delta \nu \varphi$ for every $\varphi \in Q$, which implies that $\mu_0 - \nu \in N \subset \hat{N}$. From this and $\mu = \nu + \lambda$, we have $\mu_0 - \mu = \lambda'$ for another $\lambda' \in \hat{N}$.

Next, we set $\varphi_{-n} = (a^{-n})_* \varphi$ for every $n \in \mathbb{N}$ and consider another degenerating sequence $\{\varphi_{-n}\}$. Similar to the above paragraph and by the fact that μ_0 is G-invariant, it satisfies

(iii) $$\int_\Delta (\mu_0 - (a^{-n})^* \mu) \varphi = \int_\Delta (\mu_0 - \mu) \varphi_{-n} = \int_\Delta \lambda' \varphi_{-n},$$

where

$$((a^{-n})^* \mu)(z) = \sum_{g \in G} ((a^{-n})^* \xi)(g) 1_{W_g}(z) \mu_0(z).$$

Since $((a^{-n})^* \xi)(g) \to 0$ as $n \to \infty$ again by property (2) in Lemma 3.2, we see that $((a^{-n})^* \mu)(z) \to 0$ pointwise. The left side of equality (iii) converges to $\int_\Delta \mu_0 \varphi$ and the right side converges to 0 as $n \to \infty$. Hence $\int_\Delta \mu_0 \varphi = 0$ for every $\varphi \in Q$, which

implies that $\mu_0 \in N$. However, this contradicts the fact that μ_0 is chosen so that $\mu_0 \notin N$. $\qquad\square$

Summing up all the above arguments, we have the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Consider the arc $p(t) = \Phi(t\mu)$ in T and its projection $\hat{\Phi}(t\mu)$ on AT. By Lemma 4.1, G_t is a group of asymptotically conformal automorphisms of Δ for all sufficiently small $t > 0$. Hence the boundary extension of G_t to $\partial\Delta$ is a symmetric group. On the other hand, since $\mu \notin \mathrm{Bel}(G) + \hat{N}$ by Lemma 5.1, the tangent vector $d\hat{\Phi}_o(\mu)$ of the arc $\hat{\Phi}(t\mu)$ at $\alpha(o) \in AT$ does not belong to the subspace defined by the submanifold $\alpha(T(G))$. Hence, for some sufficiently small $t > 0$, $\hat{\Phi}(t\mu)$ does not belong to $\alpha(T(G))$. This means that the boundary extension G_* of this G_t, which is the conjugate by the quasisymmetric homeomorphism $\bar{f}$ corresponding to the quasiconformal homeomorphism $f = f_{t\mu}$, is conjugate to a Fuchsian group by no symmetric homeomorphism of $\partial\Delta$. $\qquad\square$

References

[1] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), 441–456.

[2] A. Douady and C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.

[3] C. Earle, F. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part I: the complex structure*, In the tradition of Ahlfors and Bers, Contemp. Math., vol. 256, AMS, 2000, pp. 17–38.

[4] D. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), 1275–1289.

[5] D. Epstein and V. Markovic, *Extending homeomorphisms of the circle to quasiconformal homeomorphisms of the disk*, Geom. Topol. **11** (2007), 517–595.

[6] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992), 447–510.

[7] F. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, SURV 76, American Mathematical Society, 2000.

[8] F. Gardiner and D. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), 683–736.

[9] F. Gehring and G. Martin, *Discrete quasiconformal groups*, Proc. London Math. Soc. **55** (1987), 331–358.

[10] V. Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. **19** (2006), 673–715.

[11] K. Matsuzaki, *Elliptic modular transformations of infinite dimensional Teichmüller spaces*, Proceedings of the International Workshop on Teichmüller Theory and Moduli Problems, Ramanujan Math. Soc. Lecture Notes Series (to appear).

[12] Y. Mitsumatsu, *Bounded cohomology and ℓ^1-homology of surfaces*, Topology **23** (1984), 465–471.

[13] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics, Ann. Math. Studies, vol. 97, Princeton Univ. Press, 1981, pp. 465–496.

[14] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 73–78.

[15] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 1–54.

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