Dynamics of Kleinian groups
—the Hausdorff dimension of limit sets—

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1 Introduction

A Kleinian group is a discrete group of fractional linear transformations of the complex plane. In the mathematics of the 19th century, particular discrete groups such as Fuchsian groups became attracting attention in the theory of ordinary differential equations or even before in the theory of modular functions. At the end of the 19th century, Fricke and Klein (see [Mg]) created a general theory of discrete groups of fractional linear transformations and obtained some fundamental results such as complexity of the limit set (the set of accumulation points of the orbit) in the era when the word "fractal" had not been born yet. On the other hand, Poincaré [Po] extended the concept of Fuchsian groups to three dimension and regarded a Kleinian group as a discrete group of isometries of the 3-dimensional hyperbolic space (although there had been no much progress from this point of view for more than half a century).

Classical 2-dimensional theories of Kleinian groups after Fricke and Klein developed gradually in the first half of the 20th century. At the same time, the iteration of rational functions was studied by Fatou and Julia around 1920 and some analogies to Kleinian groups were recognized. Then these brothers of complex dynamics had spent their pupal stage until the time was ripe for the next. The first metamorphosis of the theory of Kleinian groups was brought by the completion of fundamental theorems on quasiconformal maps after 1960. In virtue of this, in addition to the conventional analysis of the phase space of dynamics, giving deformation to the dynamics became a powerful tool, and further it developed to the analysis of the parameter space itself, namely, to a branch of the theories of Teichmüller spaces. Ahlfors and Bers established the foundation of complex analytic methods for the study of Kleinian groups.

However, the real nature of Kleinian groups become clearer only when we obverse them as isometry groups of the 3-dimensional hyperbolic space. As 3-dimensional topology developed, the recurrence to Poincaré's viewpoint had appeared in their works of Marden and Maskit in 1970s, and at last the theory of 3-dimensional hyperbolic manifolds due to Thurston [T] brought a revolutionary turning point to Kleinian groups and created a big current of mathematics which has lasted until now. Moreover, several fields of mathematics have been involved with this current and as a consequence the range of the theory of Kleinian groups has been expanded.

Among these modern theories of Kleinian groups, another event which seems no less important was Sullivan's work so to say the renascence of classical analysis and the unification of complex dynamics. He generalized the Hopf-Tsuji ergodicity theorem, the Mostow rigidity theorem and the Ahlfors finiteness theorem by dynamical methods [S2], and imported quasiconformal maps to the study of the iteration of rational maps and tried to make a unified interpretation with Kleinian groups. Also he utilized a measure on the limit set which was introduced by Patterson to clarify how the Hausdorff dimension, which had been a primitive index measuring complexity of the limit set as a fractal until then, reflected geometric features of the hyperbolic manifold well [S1].

This expository paper is concentrated on topics on the Hausdorff dimension of limit sets among the complex dynamical theories of Kleinian groups. Sullivan's theories appeared in 1980s and since then a lot of good expositions have been already published, among which the monograph by Nicholls [N] is popular, where several propositions in this paper have their detailed proofs even if they are not specifically referred in the context. However certain results in recent papers (in particular a paper by Bishop and Jones [BJ]) succeeded in simplifying and completing Sullivan's original work. In addition, a satisfactory answer was given to a problem asking the change of the Hausdorff dimension of the limit sets under deformation of Kleinian groups, a typical problem on dynamics (McMullen [Mc]). In this paper, including these topics, we try to reorganize important results and to explain them in such a way that it seems most simple and clear (for the author). Basic terminology and facts on Kleinian groups will be stated minimally when they are necessary; even if they might be omitted, we can consult monographs of Kleinian groups, say [MT].

2 The limit sets of Kleinian groups

A conformal automorphism of the Riemann sphere $\hat{\mathbf{C}}$ is a fractional linear transformation (an orientation preserving Möbius transformation), and the group of all these transformations is identified with $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\pm I$ as a Lie group. A Kleinian group is a discrete subgroup of it. In this paper we always assume that *Kleinian groups have no elements of finite order* other than the identity. As a Möbius transformation is originally defined as a composition of reflections with respect to circles or lines on the complex plane, its action is naturally extended to the upper half-space

$$H^3 = \{(x, y, t) \mid t > 0\}$$

(the Poincaré extension). Moreover if we provide the hyperbolic metric

$$d\rho^2 = (dx^2 + dy^2 + dt^2)/t^2$$

with H^3 and regard it as a model of the 3-dimensional hyperbolic space, Möbius transformations are isometric automorphisms of (H^3, ρ). Furthermore, mapping the upper half-space to the unit ball by the Cayley transformation, we obtain the unit ball model (B^3, ρ) of the hyperbolic space. In this model, the boundary S^2 of B^3 is the sphere located at infinity of the hyperbolic space.

A Kleinian group Γ acts freely and properly discontinuously on the hyperbolic space as a group of orientation preserving isometries, and the quotient space $N_\Gamma = B^3/\Gamma$ is a hyperbolic manifold. On the other hand, if we consider the maximal open subset $\Omega(\Gamma)$ (which is called the region of discontinuity) of S^2 where Γ acts properly discontinuously, its quotient space $\Omega(\Gamma)/\Gamma$ is a complex manifold (Riemann surface). We assume the manifold $(B^3 \cup \Omega(\Gamma))/\Gamma$ with boundary to have the hyperbolic structure inside and to have the complex structure on the boundary and call it a Kleinian manifold.

The complement of the region of discontinuity $\Omega(\Gamma)$ of S^2 is called the limit set and denoted by $\Lambda(\Gamma)$. This can be alternatively defined as follows:

Definition (Limit set) Let $\Gamma(0)$ be the orbit of the origin $0 \in B^3$ by a Kleinian group Γ. Then the set of accumulation points of $\Gamma(0)$ in the Euclid topology, which is on S^2, is defined as the limit set of Γ and denoted by $\Lambda(\Gamma)$.

If the limit set $\Lambda(\Gamma)$ contains more than 2 points then it is an uncountable perfect set consisting of infinitely many points. Otherwise Γ is called elementary. In this paper we always assume that *Kleinian groups are non-elementary.* Another characterization of the limit set is that it is the minimal, non-empty, Γ-invariant, closed subset of S^2. We consider the closed convex hull of the union of all geodesic line with the end points in $\Lambda(\Gamma)$ and take the quotient by Γ, which is a subset of N_Γ. This is called the *convex core* and denoted by C_Γ. The convex core C_Γ is the minimal, convex, closed subset of N_Γ such that the inclusion map into N_Γ is homotopy equivalence.

As a generalization of finiteness of the volume of N_Γ, we define the following property of N_Γ, which means "the finiteness on the limit set".

Definition (Geometric finiteness) We say that a Kleinian group Γ (or a hyperbolic manifold N_Γ) is geometrically finite if Γ is finitely generated and the volume of the convex core C_Γ is finite.

For a geometrically finite Kleinian group Γ, the limit set is decomposed into

$$\Lambda(\Gamma) = \Lambda_c(\Gamma) \cup \Lambda_p(\Gamma).$$

Here $\Lambda_c(\Gamma)$ is the *conical limit set* of Γ, meaning that $x \in S^2$ belongs to $\Lambda_c(\Gamma)$ if $\Gamma(0)$ accumulates on x within a bounded distance of a geodesic ray towards x. On the other hand, $\Lambda_p(\Gamma)$ is the set of all parabolic fixed points of Γ, which is a countable set. In fact, if G is geometrically finite then any point of $\Lambda_p(\Gamma)$ is a *bounded parabolic fixed point.*

Definition (Hausdorff measure, dimension) For a subset E of S^2, we take a covering $E \subset \bigcup \Delta(x_i, r_i)$ by countably many disks $\Delta(x_i, r_i)$ with respect to the spherical metric where the radius r_i is less than $\delta > 0$. For an arbitrary real number $s \geq 0$, we set $\mathcal{H}_s^\delta(E) = \inf \sum r_i^s$, where the infimum is taken over all such covering of E. Next we take the limit

$$\mathcal{H}_s(E) = \lim_{\delta \to 0} \mathcal{H}_s^\delta(E) = \sup_{\delta > 0} \mathcal{H}_s^\delta(E)$$

as $\delta \to 0$. Then $\mathcal{H}_s$ is an outer measure on S^2 and the restriction to $\mathcal{H}_s$-measurable sets (which includes the Borel sets) is, by definition, the s-dimensional Hausdorff measure.

For any set $E \subset S^2$, if $s < t$ then $\mathcal{H}_s^\delta(E) \geq \delta^{s-t}\mathcal{H}_t^\delta(E)$. Hence there exists a unique number d such that if $0 \leq s < d$ then $\mathcal{H}_s(E) = \infty$ and if $d < s < \infty$ then $\mathcal{H}_s(E) = 0$. This number d is, by definition, the Hausdorff dimension of E and denoted by $\dim E$.

There are detailed arguments on the Hausdorff measure in monographs [Fa], [Fe] and [Ro].

Ahlfors proved that, for a geometrically finite Kleinian group Γ, the 2-dimensional Hausdorff measure satisfies $\mathcal{H}_2(\Lambda(\Gamma)) = 0$ if $\Lambda(\Gamma) \neq S^2$ (see Corollary 14). The statement that any finitely generated Kleinian group should satisfy this conclusion is called the Ahlfors conjecture, which is not completely solved yet. It is known that this follows from the Marden conjecture (see [O]) asserting that N_Γ is *topologically tame*, namely that N_Γ is homeomorphic to the interior of some compact manifold with boundary for any finitely generated Kleinian group Γ [C2].

3 Critical exponent of convergence

We define the following other classical index for Kleinian groups:

Definition (Critical exponent) For a Kleinian group Γ, we define

$$\delta(\Gamma) = \inf\{s \geq 0 \mid g_s := \sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0))) < \infty\}$$

as the critical exponent for Γ, where ρ denotes the hyperbolic distance. Here g_s is called the s-dimensional *Poincaré series*. If the $\delta(\Gamma)$-dimensional Poincaré series diverges then Γ is said to be of *divergence type*, and if it converges then *convergence type*.

In the unit ball model B^3, since $\exp(-\rho(0, \gamma(0)))$ is comparable with $1 - |\gamma(0)|$, the critical exponent can be defined by the convergence or divergence of

$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^s$. In addition, by an equation

$$|\gamma'(x)| = \frac{1+|\gamma(0)|}{|x-\gamma^{-1}(0)|^2}(1-|\gamma(0)|)$$

for $x \in S^2$, it can be also defined by the convergence or divergence of $\sum_{\gamma\in\Gamma}|\gamma'(x)|^s$ ($x \in \Omega(\Gamma)$), where $|\gamma'(x)|$ means the magnifying rate of a conformal map γ of $\mathbf{R}^3$ at x.

Let $n(r)$ denote the number of points of $\Gamma(0)$ within a distance r of the origin. Then it can be proved that

$$\delta(\Gamma) = \limsup_{r\to\infty} \frac{\log n(r)}{r}.$$

In particular this implies that $\delta(\Gamma) \leq 2$. Also $\delta(\Gamma) > 0$ can be proved in several ways.

Concerning the relationship between the critical exponent and the Hausdorff dimension, the following theorem due to Bishop and Jones [BJ] is of the final form:

Theorem 1 *Any Kleinian group* Γ *satisfies* $\dim \Lambda_c(\Gamma) = \delta(\Gamma)$.

First we show the easier estimate from above: $\dim \Lambda_c(\Gamma) \leq \delta(\Gamma)$. For any $\gamma \in \Gamma$, let $b(\gamma, t)$ denote the "*shadow*" on S^2 caused by the rays from the origin, of a ball with radius $t > 0$ and center $\gamma(0)$. Then, numbering all elements of Γ as $\{\gamma_n\}_{n=1}^{\infty}$, we can represent the conical limit set as

$$\Lambda_c(\Gamma) = \bigcup_{t>0} \Lambda_c^t(\Gamma), \quad \Lambda_c^t(\Gamma) := \bigcap_{N\geq 1} \bigcup_{n>N} b(\gamma_n, t).$$

Proposition 2 *If the s-dimensional Poincaré series converges then the s-dimensional Hausdorff measure of* $\Lambda_c(\Gamma)$ *is zero.*

Proof. It suffices to prove that the s-dimensional Hausdorff measure of $\Lambda_c^t(\Gamma)$ is zero for each fixed $t > 0$. This follows from the fact that the radius of the shadow $b(\gamma_n, t)$ is comparable with $1 - |\gamma_n(0)|$. ■

The assertion of Theorem 1 was already known for Fuchsian groups in general and for geometrically finite Kleinian groups [S1], [S4]. The value of their work lies in that they proved it for Kleinian groups in general. We will introduce an interpretation of the original proof by McMullen, who utilizes *quasigeodesics*. Here we say that a piecewise geodesic curve in the hyperbolic space is an (L, θ)-quasigeodesic if the length of each geodesic segment is greater than $L > 0$ and the angle between any two consecutive segments is greater than $\theta > 0$.

Proof. We will prove the other inequality $\dim \Lambda_c(\Gamma) \geq \delta(\Gamma)$. For any $\epsilon > 0$, set $s' = \delta(\Gamma) - \epsilon$ and $s = \delta(\Gamma) - 2\epsilon$. Then the s'-dimensional Poincaré series diverges. We will prove that the s-dimensional Hausdorff measure of $\Lambda_c(\Gamma)$ is not zero.

We construct a tree T rooted at the origin in B^3 as follows: The vertices of T are in a subset of the orbit $\Gamma(0)$; each edge of T is a geodesic segment of length greater than $L > 0$; and the angle between two edges at each vertex is greater than $\theta > 0$. Then, by a property of quasigeodesics, a piecewise geodesic curve from a vertex of T to a descendent vertex along the edges lies within a constant distance of the geodesic line connecting the two vertices, where the constant is depending only on L and θ. From this fact, we can see that a set $E(T) \subset S^2$ of the ends of T is contained in $\Lambda_c(\Gamma)$. Furthermore, we require the following additional properties for T: the sum taken over all vertices v' that are direct sons of any vertex v in T satisfies

$$\sum_{v'} \exp(-s\rho(v', 0)) \geq \exp(-s\rho(v, 0)) \quad \cdots \quad (*)$$

and the length of each edge e is bounded by a constant L'.

Assuming the existence of such a tree T, we prove that $\mathcal{H}_s(E(T)) > 0$. We take an arbitrary covering $\bigcup \Delta(x_i, r_i)$ of $E(T)$ by disks. For each path in T from the origin towards an end, if it terminates at an end contained in a disk $\Delta(x_i, r_i)$, we cut the path at the vertex where it first enter into the hemiball B_i in B^3 spanning $\Delta(x_i, 2r_i)$ and take off the far part from there. In this manner, we construct a finite tree T'. Then the sum taken over all vertices v' on the boundary of T' satisfies

$$\sum \exp(-s\rho(v', 0)) \geq \exp(-s\rho(0, 0)) = 1$$

by $(*)$. Since $\exp(-s\rho(v', 0))$ and $(1 - |v'|)^s$ are comparable, there exists a constant $C > 0$ such that if $v' \in B_i$ then $C(2r_i)^s \geq \exp(-s\rho(v', 0))$. The number of $v' \in \partial T'$ that are contained in a hemiball B_i is bounded by a constant K which is depending only on $a = \inf_{\gamma \in \Gamma - \{\mathrm{id}\}} \rho(\gamma(0), 0)$ and on L'. Hence

$$2^s CK \sum_i r_i^s \geq \sum_{v' \in \partial T'} \exp(-s\rho(v', 0)) \geq 1.$$

This implies that, for any covering of $E(T)$ by disks, the sum of their radii to the s-th power is not less than $(2^s CK)^{-1}$, and thus we obtain

$$\mathcal{H}_s(\Lambda_c(\Gamma)) \geq \mathcal{H}_s(E(T)) \geq (2^s CK)^{-1} > 0.$$

Next we state a method of constructing a tree T that satisfies the above requirements. Let A_1 be a cone consisting of the rays from the origin towards the points in a disk Δ_1 on S^2. We can choose Δ_1 so that a partial sum of the s'-dimensional Poincaré series $g_{s'}$ taken over the points of $\Gamma(0)$ that are contained in A_1 diverges. Moreover, since Γ is non-elementary, we can take another disk

Δ_2 on S^2 disjoint from Δ_1 so that the partial sum of $g_{s'}$ taken over the points in the corresponding cone A_2 to Δ_2 also diverges. Let θ be the half of the minimum of angles between a ray from the origin in A_1 and that in A_2. For a fixed $l > 0$, we choose $L > 0$ so that any two (L, θ)-quasigeodesics that connects the same points in B^3 lie within the distance l of each other.

We divide the cone A_i $(i = 1, 2)$ according to the hyperbolic distance from the origin and let $A_{i,n}$ $(n = 1, 2, \ldots)$ be a block in A_i where the distance is in the interval $[n, n+1)$. We consider a partial sum of the Poincaré series $g_{s'}$ taken over the points of $\Gamma(0)$ in $A_{i,n}$. Since $g_{s'}$ diverges, for any $M > 0$, there exist infinitely many blocks $\{A_{i,n(k)}\}_{k=1,2,\ldots}$ such that the partial sum of g_s taken over $A_{i,n(k)}$ is greater than M. On the other hand, we can find a way of dividing $\Gamma(0) \cap A_{i,n}$ into b groups, where b is an integer depending only on a and l, such that, if two (L, θ)-quasigeodesic rays from the origin pass through distinct points in the same subset, then they are at least l distant from each other in $A_{i,n}$. We take M sufficiently larger than b and choose a block $A_{i,n(k)}$ with $n(k) \geq L$. At least one group among the b groups satisfies that the partial sum of g_s taken over the group is greater than $M/b >> 1$.

We adopt all points in the group as the vertices $\{v\}$ in the first generation of the tree and connect the origin and each vertex v by a geodesic segment, which is an edge e. The way of choosing the vertices $\{v'\}$ in the second generation, which are sons of a vertex v, is as follows: We regard $v = \gamma(0)$ as the new origin and take two cones $\gamma(A_1)$ and $\gamma(A_2)$ with the vertex v. Then the angle between the edge e and at least one of the two cones is not less than θ. Choose certain points of $\Gamma(0)$ in a block of the cone exactly in the same manner as before and adopt them as the vertices $\{v'\}$. Connect v and each v' by a geodesic edge e'.

Repeating this process, we have a family of (L, θ)-quasigeodesics rays starting from the origin, branching at the vertices and extending to the infinity. This family of quasigeodesic rays forms a tree because, once they branch, they never meet again due to our construction. For each vertex v and its sons $\{v'\}$, we have

$$\sum_{v'} \exp(-s\rho(v', 0)) \approx \sum_{v'} \exp(-s\rho(v', v)) \cdot \exp(-s\rho(v, 0))$$

because the quasigeodesics satisfy $\rho(v', 0) \approx \rho(v', v) + \rho(v, 0)$. Then, since

$$\sum_{v'} \exp(-s\rho(v', v)) >> 1$$

by construction, we obtain the inequality $(*)$. ■

4 Measures on the limit set

We consider measures on the limit set that are invariant under the group action (in the sense of the following definition). The Hausdorff measure has the invariance for Möbius transformations. There is a standard construction of an invariant probability measure with the support on the limit set, which reflects the distribution of the orbit of a Kleinian group. Using this measure, we can estimate the Hausdorff measure and dimension of the limit set.

Definition (Invariant measure) For a Kleinian group Γ, a Borel measure μ on S^2 is said to be an s-dimensional, Γ-invariant measure if

$$\mu(\gamma E) = \int_E |\gamma'|^s d\mu$$

for any Borel measurable set E on S^2 and for any $\gamma \in \Gamma$.

We consider the range of dimensions where a Γ-invariant probability measure exists. We define the infimum of such dimensions (which is actually the minimum by taking the weak limit of a sequence of measures) as the *critical dimension* and denote it by $\alpha(\Gamma)$.

Patterson [Pt1] and Sullivan [S1] constructed a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure as follows: For $s > \delta(\Gamma)$, letting each term of the convergent Poincaré series g_s be a weight of the Dirac measure $\delta_{\gamma(0)}$, we consider the sum

$$\sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0))\delta_{\gamma(0)}.$$

Dividing it by the total mass, we have a probability measure on $\overline{B^3}$. Then, letting $s \downarrow \delta(\Gamma)$, we have a weak limit of a subsequence of the measures. We can prove that the probability measure constructed in this way has the support on the limit set and that it has $\delta(\Gamma)$-dimensional Γ-invariance. (Precisely speaking, this is true only in the case where Γ is of divergence type. If Γ is of convergence type, we have to adjust the given weight so that the weak limit measure has no support inside of B^3.)

Lemma 3 *For any Kleinian group Γ, there exists a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure. Hence $\alpha(\Gamma) \leq \delta(\Gamma)$.*

Next, we investigate the relationship between an s-dimensional, Γ-invariant probability measure μ_s and the s-dimensional Hausdorff measure. To this end, the following *shadow lemma* due to Sullivan [S1] is crucial, which asserts that the mass of the shadow $b(\gamma, t)$ (defined in Section 3) measured by μ_s is comparable with the radius $r(b(\gamma, t))$ of the shadow to the s-th power. Consult [Pt3] for its proof.

Lemma 4 *Let μ_s be an s-dimensional, Γ-invariant probability measure. Then, for a sufficiently large t, there exists a constant $A > 0$ such that*

$$\frac{1}{A}\mu_s(b(\gamma, t)) < r(b(\gamma, t))^s < A\mu_s(b(\gamma, t))$$

for all $\gamma \in \Gamma$ except a finite number of elements.

Applying this lemma directly to the proof of Proposition 2 and interpreting the conclusion as a statement on a Γ-invariant probability measure, we have the following:

Proposition 5 *If the s-dimensional Poincaré series converges then $\Lambda_c(\Gamma)$ is a null set for any s-dimensional, Γ-invariant measure μ_s.*

The restriction of the s-dimensional Hausdorff measure $\mathcal{H}_s$ to the conical limit set $\Lambda_c(\Gamma)$ is denoted by h_s. This is an s-dimensional, Γ-invariant measure (including the case where the total mass is zero or infinity). In order to estimate h_s from above in terms of μ_s, we use Lemma 4 as well as the fact that we can choose such a covering of $\Lambda_c(\Gamma)$ by shadows that they have no intersection (or few intersection) with each other. In general, this kind of statement is called the *covering theorem* and, for example, the Vitali covering theorem (see [Fa]) is applicable to our case.

Lemma 6 *If an s-dimensional, Γ-invariant probability measure μ_s exists then there is a constant $A' < \infty$ such that $h_s \leq A'\mu_s$. Hence $\dim \Lambda_c(\Gamma) \leq s$ and thus $\dim \Lambda_c(\Gamma) \leq \alpha(\Gamma)$.*

By Lemmas 3 and 6, the above three values $\dim \Lambda_c$, α and δ are in this order from low to high. Then, applying Theorem 1, we see that they are actually coincident with each other.

Theorem 7 *Any Kleinian group Γ satisfies $\dim \Lambda_c(\Gamma) = \alpha(\Gamma) = \delta(\Gamma)$.*

We consider the following more detailed conditions on convergence of the Poincaré series and on nullity of the conical limit set at the critical dimension $\delta = \delta(\Gamma)$.

(1) The δ-dimensional Poincaré series g_δ converges;

(2) Some/any δ-dimensional, Γ-invariant probability measure μ_δ satisfies $\mu_\delta(\Lambda_c(\Gamma)) = 0$;

(3) The δ-dimensional Hausdorff measure $\mathcal{H}_\delta$ satisfies $\mathcal{H}_\delta(\Lambda_c(\Gamma)) = 0$.

Although only the implication (1) $\Rightarrow$ (2) was proved by Proposition 5, the converse (2) $\Rightarrow$ (1) is also true. To see this, we take the unit tangent bundle T_1N_Γ of the hyperbolic manifold N_Γ and introduce a measure m on T_1N_Γ from the hyperbolic volume element and a Γ-invariant probability measure μ. Then we consider a condition that the *geodesic flow* $g(v,t) : T_1N_\Gamma \times \mathbf{R} \to T_1N_\Gamma$ is conservative with respect to the measure m. In other words, it is that for almost all points $v \in T_1N_\Gamma$ with respect to m, there exists an infinite sequence $t_n \to \infty$ such that $g(v,t_n)$ belong to a compact subset of T_1N_Γ. Clearly this is equivalent to the condition $\mu(\Lambda_c(\Gamma)) = 1$. Sullivan proved that, in the case that μ is n-dimensional, if the Poincaré series g_n diverges then the geodesic flow is conservative [S2], and that this is extendable to any dimension [S3], [N]. Further, Thurston gave an elementary proof directly without using a geodesic flow, of the fact that if g_n diverges then $\Lambda_c(\Gamma)$ has full n-dimensional measure (see [Ah], [N]). Tukia [Tu2] generalized this method to any dimension.

Theorem 8 *If Γ is of divergence type then $\Lambda_c(\Gamma)$ is positive with respect to any $\delta(\Gamma)$-dimensional, Γ-invariant probability measure.*

Next we show that a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure is unique for a Kleinian group Γ of divergence type. The uniqueness is equivalent to the *ergodicity* of the group action.

Proposition 9 *Let μ_s be an s-dimensional, Γ-invariant probability measure. Then it is the unique s-dimensional, Γ-invariant probability measure if and only if μ_s satisfies the following ergodic condition: any Γ-invariant measurable set A on S^2 satisfies either $\mu_s(A) = 0$ or $\mu_s(S^2 - A) = 0$.*

In general, as the following lemma shows, any Γ-invariant probability measure μ satisfies the ergodic condition on the conical limit set $\Lambda_c(\Gamma)$. Hence, if $\Lambda_c(\Gamma)$ is positive with respect to μ, which implies that μ must be $\delta(\Gamma)$-dimensional, then it is unique.

Lemma 10 *If a Γ-invariant measurable subset A of $\Lambda_c(\Gamma)$ satisfies $\mu(A) > 0$ for a Γ-invariant probability measure μ then $\mu(A) = 1$.*

Proof. We take a density point x of A and choose a sequence $\gamma_n(0)$ out of the orbit that converges to x conically. Passing to a subsequence, we may assume that $\gamma_n^{-1}(0) \to y$ for some $y \in S^2$. Since x is a density point, a sequence of shadows $b(\gamma_n, t)$ satisfies

$$\lim_{n\to\infty} \frac{\mu(b(\gamma_n, t) - A)}{\mu(b(\gamma_n, t))} \to 0.$$

We map the shadow $b(\gamma_n, t)$ by γ_n^{-1} and consider the ratio of A in the image. Estimating $|(\gamma_n^{-1})'|$, we have a constant C depending only on t such that

$$\frac{\mu(\gamma_n^{-1} b(\gamma_n, t) - A)}{\mu(\gamma_n^{-1} b(\gamma_n, t))} = \frac{\int_{b(\gamma_n,t)-A} |(\gamma_n^{-1})'| d\mu}{\int_{b(\gamma_n,t)} |(\gamma_n^{-1})'| d\mu} \leq C \frac{\mu(b(\gamma_n, t) - A)}{\mu(b(\gamma_n, t))} \ (\to 0).$$

Then, for any $\epsilon > 0$, there exists some t such that $\mu(\gamma_n^{-1} b(\gamma_n, t)) > 1 - \lambda - \epsilon$ for all sufficiently large n, where $\mu(\{y\}) = \lambda$. Hence

$$\frac{\mu(A \cap \gamma_n^{-1} b(\gamma_n, t))}{\mu(\gamma_n^{-1} b(\gamma_n, t))} < \frac{\mu(A) - \lambda}{1 - \lambda - \epsilon}.$$

Since the left hand side converges to 1 as $n \to \infty$, we see that $1 - \lambda - \epsilon \leq \mu(A) - \lambda$. Thus we have $\mu(A) = 1$, for ϵ is arbitrary. ■

Definition (The Patterson–Sullivan measure) For a Kleinian group Γ of divergence type, the unique $\delta(\Gamma)$-dimensional, Γ-invariant probability measure is defined to be the *Patterson–Sullivan measure.* Hereafter we call it the PS measure. The total mass of the PS measure is on the conical limit set.

Finally we consider the implication (3) $\Rightarrow$ (2). Note that the converse is true, which can be seen from Lemma 6 or alternatively from Proposition 2 if we use the equivalence of (1) and (2). In the definition of the Hausdorff measure, all the disks are candidates for a covering of a given set, however we restrict the disks to the family $\mathcal{F}_t = \{b(\gamma, t)\}_{\gamma \in \Gamma}$ of shadows of balls of radius t with center at the orbit and define a new Hausdorff measure $\mathcal{H}_s(\cdot\,;\mathcal{F}_t)$ in the same way. Then, by the shadow lemma and by the covering theorem, we can see that $\Lambda_c^t(\Gamma)$ is a null set for an s-dimensional, Γ-invariant probability measure μ_s if and only if it is a null set for $\mathcal{H}_s(\cdot\,;\mathcal{F}_t)$. Hence a problem remains of the comparison of $\mathcal{H}_s(\cdot\,;\mathcal{F}_t)$ and $\mathcal{H}_s(\cdot)$. Although an inequality $\mathcal{H}_s(\cdot) \leq \mathcal{H}_s(\cdot\,;\mathcal{F}_t)$ is clear by definition, the converse is the problem; if $\Lambda_c^t(\Gamma)$ is a null set for $\mathcal{H}_s(\cdot)$ then so is for $\mathcal{H}_s(\cdot\,;\mathcal{F}_t)$. However this is not true in general.

Remark Assume that Γ is a geometrically finite Kleinian group without a parabolic element. Then, for any disk $\Delta(x, r)$ with sufficiently small radius that contains a point of $\Lambda_c^t(\Gamma)$, there exists a shadow $b(\gamma, t) \subset \Delta(x, 2r)$ such that if the radius is multiplied by a uniform constant then the resulting shadow contains $\Delta(x, 2r)$. Hence, in this case, both Hausdorff measures are comparable on $\Lambda_c^t(\Gamma)$ and thus (3) $\Rightarrow$ (2) follows. However, even if Γ is geometrically finite, this is not true in certain cases where Γ contains a parabolic element. Sullivan [S4] gave a condition for this implication to be satisfied in terms of the inequalities between $\delta(\Gamma)$ and the maximal rank of parabolic subgroups of Γ.

We have seen that, for any Kleinian group Γ, there exists a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure and if Γ is of divergence type in addition then it is unique, which is the PS measure. Although the uniqueness is not satisfied in general, a measure that is constructed in a standard manner has the following property [S4]:

Theorem 11 *For any Kleinian group Γ, there exists a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure μ whose support is on the limit set and which has no atom (point mass) on bounded parabolic fixed points.*

Indeed, since the measure μ constructed in Lemma 3 has its support on the limit set, it suffice only to prove that μ has no atom (point mass) on bounded parabolic fixed points. The PS measure automatically satisfies this condition.

Definition (Change of base points) For an s-dimensional, Γ-invariant probability measure μ and a point $x \in B^3$, we define a measure μ_x, which is absolutely continuous with respect to μ and vice versa, as

$$\mu_x(E) = \int_E P(x, \zeta)^s d\mu(\zeta), \quad P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^2}.$$

Here $P(x,\zeta)$ is the Poisson kernel, which is the magnifying rate $|h_x'(\zeta)|$ at $\zeta \in S^2$ of a Möbius transformation h_x of B^3 that maps x to the origin 0. Then the Radon–Nikodym derivative is

$$\frac{d\mu_x}{d\mu_y}(\zeta) = \left(\frac{P(x,\zeta)}{P(y,\zeta)}\right)^s.$$

We consider a Γ-invariant function $\varphi_\mu(x) = \mu_x(S^2)$, where the total mass of μ_x is regarded as a function of $x \in B^3$. We will see that the growth order of $\varphi_\mu(x)$ as x tends conically to ξ has a difference according to whether ξ is an atom for μ or not.

In general, when $\xi \in S^2$ is an atom for an s-dimensional, Γ-invariant probability measure μ, we can see from an estimate of the Poisson kernel that, for a sequence of points $x \in B^3$ that converges conically to ξ, there exists a constant $C_1 > 0$ such that

$$\varphi_\mu(x) \geq \mu_x(\{\xi\}) = P(x,\xi)^s \mu(\{\xi\}) \geq C_1 e^{s\rho(0,x)}.$$

This in particular implies that a conical limit point ξ of Γ cannot be an atom for μ. Contrarily to this case, when ξ is a bounded parabolic fixed point of Γ and it is not an atom, we have the following result. For the sake of simplicity of notation, ξ is regarded as a bounded parabolic fixed point of rank 0 if $\xi \in \Omega(\Gamma)$.

Lemma 12 *Let μ be an s-dimensional, Γ-invariant probability measure with support on $\Lambda(\Gamma)$. Assume that ξ is a bounded parabolic fixed point of Γ of rank k ($k = 0,1,2$) that is not an atom for μ. Then, for a sequence $x \in B^3$ that converges conically to ξ, there exists a constant $C_2 < \infty$ that satisfies*

$$\varphi_\mu(x) \leq C_2 e^{(k-s)\rho(0,x)}.$$

Proof. Using the upper half-space model H^3, we may assume that $\xi = \infty$ and that μ_x is a measure on $\mathbf{C}$. Setting $P(x,\zeta) = 2t/|x-\zeta|^2$ for $x = (z,t) \in H^3$ (which is the magnifying rate $|h_x'(\zeta)|$ of a Möbius transformation $h_x : H^3 \to B^3$ that maps x to 0), we have

$$\frac{d\mu_x}{d\mu_{x_0}}(\zeta) = \left(\frac{P(x,\zeta)}{P(x_0,\zeta)}\right)^s = \left(\frac{t(1+|\zeta|^2)}{t^2+|\zeta-z|^2}\right)^s,$$

where $x_0 = (0,1)$. Let J be the stabilizer of ∞ in Γ and E a fundamental domain for J on $\mathbf{C}$. Then

$$\begin{aligned}\mu_x(\mathbf{C}) = \sum_{j\in J} \mu_x(j(E)) &= \sum_{j\in J} \mu_{j^{-1}(x)}(E) \\ &= \sum_{j\in J} \int_E \frac{d\mu_{j^{-1}(x)}}{d\mu_{x_0}}(\zeta)\, d\mu_{x_0}(\zeta) \\ &= t^s \sum_{j\in J} \int_E \left(\frac{1+|\zeta|^2}{t^2+|\zeta-j^{-1}(z)|^2}\right)^s d\mu_{x_0}(\zeta).\end{aligned}$$

Since the support $\Lambda(\Gamma)$ of μ_{x_0} restricted to E is bounded, the sum of the integrals in the last term is comparable with

$$\int_{\mathbf{R}^k} \frac{dx_1 \cdots dx_k}{(t^2 + x_1^2 + \cdots + x_k^2)^s} = t^{k-2s} \int_{\mathbf{R}^k} \frac{dx_1 \cdots dx_k}{(1 + x_1^2 + \cdots + x_k^2)^s}.$$

(This in particular implies $s > k/2$ for $k = 1, 2$.) We transfer these things to B^3. When x converges to ξ conically, t is comparable with $e^{\rho(x_0,x)}$. Hence we can take a required constant C_2 as in the assertion. ∎

Proof of Theorem 11. We take a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure μ with support on $\Lambda(\Gamma)$, which was constructed in Lemma 3. Suppose that Γ has a bounded parabolic fixed point ξ of rank $k = 1, 2$. Then $\delta(\Gamma) > k/2$, which can be seen from the proof of Lemma 12 above or more simply from Lemma 30 later. For a sequence $x \in B^3$ that converges conically to ξ, whether ξ is an atom or not makes such a difference of growth order as

$$\begin{aligned} \varphi_\mu(x) &\geq C_1 e^{\delta(\Gamma)\rho(0,x)}; \\ \varphi_\mu(x) &\leq C_2 e^{(k-\delta(\Gamma))\rho(0,x)} = C_2 e^{(\delta(\Gamma)-\epsilon)\rho(0,x)} \quad (\epsilon = 2\delta(\Gamma) - k > 0). \end{aligned}$$

The measure μ was constructed as a weak limit of a sequence of probability measures μ_s on $\overline{B^3}$ which consists of atoms on the orbit $\Gamma(0)$. We define $\varphi_{\mu_s}(x)$ for these μ_s similarly to the case of measures on S^2. Then, even if we take the arrangement of the weight on $\Gamma(0)$ into account (which is necessary when Γ is of convergence type), we have $\varphi_{\mu_s}(x) = O(e^{(\delta(\Gamma)-\epsilon)\rho(0,x)})$ as in the proof of Lemma 12. Hence their weak limit μ cannot have an atom on ξ. ∎

We conclude this section with a summary of facts on geometrically finite Kleinian groups [S1], [S4].

Theorem 13 *Let Γ be a geometrically finite Kleinian group. Then Γ is of divergence type and the unique $\delta(\Gamma)$-dimensional, Γ-invariant probability measure (PS measure) exists. Moreover, if Γ has no parabolic element then the PS measure is coincident with the Hausdorff measure $h_{\delta(\Gamma)} = \mathcal{H}_{\delta(\Gamma)}|_{\Lambda_c(\Gamma)}$ up to a multiplicative constant. Furthermore, if μ_s is an s-dimensional, Γ-invariant probability measure with support on the limit set for $s > \delta(\Gamma)$ then μ_s consists only of atoms on bounded parabolic fixed points.*

Proof. We take a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure μ as in Theorem 11. Since μ has no atom on bounded parabolic fixed points, the total mass lies on $\Lambda_c(\Gamma)$. Hence, by Proposition 5, the $\delta(\Gamma)$-dimensional Poincaré series diverges. Also the uniqueness follows. In addition, if Γ has no parabolic elements then $\mathcal{H}_{\delta(\Gamma)}(\Lambda_c(\Gamma)) > 0$ by the remark concerning the implication (3) $\Rightarrow$ (2). Hence the uniqueness implies that $h_{\delta(\Gamma)}$ coincides with μ modulo normalization. Next, by Proposition 5 again, if $s > \delta(\Gamma)$ then $\Lambda_c(\Gamma)$ is a null set for μ_s.

Hence it has the total mass on the bounded parabolic fixed points. Since they are countable, μ_s consists only of atoms. ■

Using the fact that $\dim \Lambda(\Gamma) = \dim \Lambda_c(\Gamma)$ for a geometrically finite Kleinian group Γ, we obtain the following corollary [S4], [Tu1].

Corollary 14 *The limit set $\Lambda(\Gamma)$ of a geometrically finite Kleinian group Γ has Hausdorff dimension 2 if and only if $\Lambda(\Gamma) = S^2$.*

Proof. If $\Lambda(\Gamma) \neq S^2$ then the 2-dimensional Poincaré series converges. This can be seen from the fact that $\sum_{\gamma \in \Gamma} |\gamma'(x)|^2 < \infty$ for $x \in \Omega(\Gamma)$. On the other hand, Γ is of divergence type by Theorem 13. Therefore $\delta(\Gamma) < 2$, which implies that the Hausdorff dimension of the limit set is less than 2. ■

5 Geometric indices

The Hausdorff dimension $\dim \Lambda_c(\Gamma)$ of the conical limit set can be estimated by certain geometrical indices of the hyperbolic manifold N_Γ. More precisely, the Hausdorff dimension is related to them via the relationship between the critical exponent and the bottom of the spectrum.

Definition (Bottom of spectrum) For a complete Riemannian manifold N in general, we define $\lambda_0(N)$ as the infimum of the *Rayleigh quotient*

$$\lambda_0(N) = \inf \left\{ \frac{\int_N |\nabla f|^2}{\int_N |f|^2} \;\middle|\; f \in C_0^\infty(N) \right\},$$

and call it the bottom of the spectrum. In particular we denote the bottom of the spectrum of a hyperbolic manifold N_Γ by $\lambda_0(\Gamma)$.

The $\lambda_0(N)$ is equal to the supremum (actually maximum) of the set of eigenvalues for smooth positive eigenfunctions with respect to the Laplace–Beltrami operator Δ on N. On the other hand, Δ is uniquely extendable to a semi-positive definite, self-adjoint operator on the Hilbert space of the square integrable functions on N, and $\lambda_0(N)$ is also equal to the infimum of the set of eigenvalues with respect to this operator. Geometrically, $\lambda_0(N)$ is related to the following constants:

Definition (Isoperimetric constant, volume growth) For a Kleinian group Γ, we set

$$h(\Gamma) = \inf_W \frac{\mathrm{Area}\,(\partial W)}{\mathrm{Vol}\,(W)},$$

where the infimum is taken over all relatively compact subregions W of N_Γ, which we call the isoperimetric constant of N_Γ. Also we set

$$\kappa(\Gamma) = \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{Vol}\,(B(p,r)),$$

where $B(p, r)$ is a subregion of N_Γ within a distance r of p.

Cheeger, Buser, Brooks proved the following results respectively:

Proposition 15

[Ch] $\lambda_0(\Gamma) \geq \frac{1}{4} h(\Gamma)^2$.

[Bu] *There exists a universal constant C such that $\lambda_0(\Gamma) \leq C h(\Gamma)$.*

[Br] $\lambda_0(\Gamma) \leq \frac{1}{4} \kappa(\Gamma)^2$.

For a topologically tame hyperbolic manifold N_Γ, we can estimate the isoperimetric constant by considering the ratio of the volume of the convex core C_Γ to the area of its boundary; a geometrically infinite end (i.e. an end contained in C_Γ) is parabolic whereas a geometrically finite end (i.e. an end facing to $\Omega(\Gamma)$) is hyperbolic. The area of the boundary ∂C_Γ is bounded by a constant depending on topological type of N_Γ (more concretely, depending on the minimal number of generators of Γ). Therefore, by such an estimate as Buser did, the following theorem due to Canary [C1] is obtained.

Theorem 16 *A topologically tame hyperbolic manifold N_Γ satisfies*

$$\lambda_0(\Gamma) \leq \frac{K}{\mathrm{Vol}\,(C_\Gamma)},$$

where K is a constant depending only on topological type of N_Γ. In particular if N_Γ is topologically finite but not geometrically finite then $\lambda_0(\Gamma) = 0$.

Hereafter in this section, we explain the following Elstrodt–Patterson–Sullivan theorem, which proves the relationship between the critical exponent and the bottom of the spectrum.

Theorem 17 *Any Kleinian group Γ satisfies*

$$\lambda_0(\Gamma) = \begin{cases} 1 & (\delta(\Gamma) \leq 1) \\ \delta(\Gamma)(2 - \delta(\Gamma)) & (\delta(\Gamma) > 1) \end{cases}.$$

A proof of this theorem for Kleinian groups was given by Sullivan [S5]. Here we introduce an easier proof by assuming the Marden conjecture to be true. In particular our proof is valid for Fuchsian groups.

For an s-dimensional, Γ-invariant probability measure μ, we consider a Γ-invariant function $\varphi_\mu(x) = \int_{S^2} P(x, \zeta)^s d\mu(\zeta)$. Since the Poisson kernel satisfies $\Delta P(x, \zeta)^s = s(2 - s) P(x, \zeta)^s$, the projection of $\varphi_\mu(x)$ to N_Γ defines a positive eigenfunction for an eigenvalue $s(2-s)$ with respect to Δ. In particular, taking a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure as μ, we obtain the following:

Proposition 18 *There exists a positive eigenfunction on N_Γ for the eigenvalue $\delta(\Gamma)(2-\delta(\Gamma))$ with respect to Δ. Hence $\lambda_0(\Gamma) \geq \delta(\Gamma)(2-\delta(\Gamma))$.*

Next, we prove that Theorem 17 is valid for particular Kleinian groups by showing that the eigenfunctions are square integrable in those cases (see [S4], [Pt3]).

Lemma 19 *If a geometrically finite Kleinian group Γ satisfies $\delta(\Gamma) > 1$ then the function φ_μ induced by the PS measure μ for Γ is square integrable on N_Γ.*

Proof. The closed ϵ-neighborhood of the convex core C_Γ is denoted by C_Γ^*. The complement of the cusp neighborhoods in C_Γ^* is compact since Γ is geometrically finite. Suppose that Γ has a bounded parabolic fixed point ξ of rank k. By Lemma 12, if x converges to ξ conically then $\varphi_\mu(x) = O(e^{(k-\delta(\Gamma))\rho(0,x)})$. By Lemma 30, which will be proved later, $\epsilon = 2\delta(\Gamma) - k$ is positive, and thus $\varphi_\mu(x) = O(e^{(k-\epsilon)\rho(0,x)/2})$. From this we can see that φ_μ is square integrable in the cusp neighborhoods.

Next we will see that $\varphi_\mu(x)$ is also square integrable on $W = N_\Gamma - C_\Gamma^*$. For any point p in W, we take the nearest point p' on $\Sigma = \partial C_\Gamma^*$ to p and define a map $p \mapsto (p', t)$, where $t = \rho(p, p')$. This map induces a homeomorphism $h : W \to W' = \Sigma \times (0, \infty)$. Moreover, if we provide a metric $e^{2t} ds_\Sigma^2 + dt^2$ with W' (where ds_Σ^2 is the hyperbolic metric on Σ) then h is quasi-isometric (see [C2]). Hence we may use this metric instead in order to see that the integral is finite. Since $\varphi_\mu = O(e^{-t\delta(\Gamma)})$ as $t \to \infty$ by Lemma 12, we can reduce the problem to seeing $\int_0^\infty e^{2t-2t\delta(\Gamma)} dt$. Then, by the assumption $\delta(\Gamma) > 1$, this is finite. ∎

Next we deal with a geometrically finite Kleinian group Γ with $\delta(\Gamma) \leq 1$. First of all, it is known that $\lambda_0(\{1\}) = 1$ for the trivial group. On the other hand, as we have seen above, if $\delta(\Gamma) \downarrow 1$ then $\lambda_0(\Gamma) \uparrow 1$. Hence it is natural to guess by interpolation that $\lambda_0(\Gamma)$ is identically equal to 1 whenever $0 < \delta(\Gamma) \leq 1$. Indeed, we will prove this assertion as follows. In general, if $\Gamma \subset G$ then $\lambda_0(\Gamma) \geq \lambda_0(G)$ because any positive eigenfunction on N_G with respect to Δ can be lifted to an eigenfunction on N_Γ for the same eigenvalue. Hence it suffice to prove that, for any $\epsilon > 0$, there exists a geometrically finite Kleinian group G that contains Γ and that satisfies $1 < \delta(G) < 1 + \epsilon$. To this end, relying on the following result due to Patterson [Pt2], we take the Klein combination G of Γ and a geometrically finite Kleinian group Γ' with $\delta(\Gamma') = 1$ (say a Fuchsian group Γ'). By Lemma 30, we can see that the critical exponent of G is greater than 1.

Proposition 20 *For Kleinian groups Γ and Γ' with non-empty regions of discontinuity, there exists a divergent sequence g_n of Möbius transformations such that Γ and $g_n \Gamma' g_n^{-1}$ generate a Kleinian group $G_n = \Gamma * g_n \Gamma' g_n^{-1}$ by the Klein combination and that*

$$\lim_{n\to\infty} \delta(G_n) = \max\{\delta(\Gamma), \delta(\Gamma')\}.$$

So far we have proved that Theorem 17 is valid for a geometrically finite Kleinian group. In addition, Theorem 16 and Proposition 18 assert that, even if it is not geometrically finite, $\lambda_0(\Gamma) = 0$ and $\delta(\Gamma) = 2$ for a topologically tame hyperbolic manifold, and thus Theorem 17 is satisfied also in this case. Therefore, if we assume the Marden conjecture, Theorem 17 is valid for any finitely generated Kleinian group. Finally, to extend the result to the case of infinitely generated groups, we use the following lemma. Here, for a sequence of Kleinian groups Γ_n, we set

$$\mathrm{Env}\,\{\Gamma_n\} = \{\gamma \in \mathrm{PSL}_2(\mathbf{C}) \mid \gamma = \lim_{n\to\infty} \gamma_n \ (\gamma_n \in \Gamma_n)\}.$$

Lemma 21 *Let Γ be a Kleinian subgroup of $\mathrm{Env}\,\{\Gamma_n\}$ for a sequence of Kleinian groups Γ_n. Then $\delta(\Gamma) \leq \liminf \delta(\Gamma_n)$. In particular, if a Kleinian group Γ is represented as $\Gamma = \bigcup_n \Gamma_n$ where $\Gamma_1 \subset \Gamma_2 \subset \cdots$, then $\delta(\Gamma) = \lim \delta(\Gamma_n)$ and $\lambda_0(\Gamma) = \lim \lambda_0(\Gamma_n)$.*

Proof. Let μ_n be a $\delta(\Gamma_n)$-dimensional, Γ_n-invariant probability measure. Taking a subsequence, we may assume that $\delta(\Gamma_n)$ converge to a dimension d, and taking a further subsequence, μ_n converge weakly to a measure μ on S^2. Then μ is a d-dimensional, Γ-invariant probability measure. Thus the first assertion follows from $d \geq \alpha(\Gamma) = \delta(\Gamma)$. Under the assumption of the second assertion, we have $\delta(\Gamma) \geq \delta(\Gamma_n)$, and thus $\delta(\Gamma) = \lim \delta(\Gamma_n)$. Concerning $\lambda_0(\Gamma)$, for any $\epsilon > 0$, we take a function $f \in C_0^\infty(N_\Gamma)$ such that the Rayleigh quotient of f is less than $\lambda_0(\Gamma)+\epsilon$. Since the compact support of f is approximated by domains of N_{Γ_n}, we have a function $f_n \in C_0^\infty(N_{\Gamma_n})$ whose Rayleigh quotient is less than $\lambda_0(\Gamma) + 2\epsilon$ for any sufficiently large n. Hence we see $\lambda_0(\Gamma_n) \leq \lambda_0(\Gamma) + 2\epsilon$. Combining this inequality with $\lambda_0(\Gamma) \leq \lambda_0(\Gamma_n)$, we obtain the assertion. ∎

6 Geometric convergence and continuity of the Hausdorff dimension

We consider the change of the Hausdorff dimension of the limit set under deformation of a Kleinian group. It is seen from the distortion theorem on quasiconformal maps (see [As]) that the Hausdorff dimension varies continuously (with respect to the Teichmüller distance) under quasiconformal deformation. Moreover, it is a real analytic function on the quasiconformal deformation space for certain kinds of Kleinian groups [Ru], [AR]. Hence the problem lies in the continuity of the Hausdorff dimension on the boundary of the quasiconformal deformation spaces and more generally for convergent sequences of Kleinian groups. To formulate this problem exactly, we begin with defining a couple of concepts on convergence of Kleinian groups.

Definition (Algebraic convergence) Let Γ_0 be a group in general and let $\theta_n : \Gamma_0 \to \Gamma_n$ be a $\mathrm{PSL}_2(\mathbf{C})$-representation of Γ_0 onto Γ_n. We say that a sequence θ_n converges to a $\mathrm{PSL}_2(\mathbf{C})$-representation $\theta : \Gamma_0 \to \Gamma$ of Γ_0 onto Γ if $\theta_n(\gamma)$ converge to $\theta(\gamma)$ for any $\gamma \in \Gamma_0$.

Hereafter, we always assume that Γ_0 is *finitely generated* when we consider the algebraic convergence. Under this assumption, if Γ_n are Kleinian then their algebraic limit Γ is also a Kleinian group.

Although the algebraic convergence preserves the algebraic structure of Kleinian groups, it does not necessarily imply convergence of the geometric structure of the corresponding hyperbolic manifolds. As we have seen in Section 5, the Hausdorff dimension of the limit set reflects the geometric nature of a hyperbolic manifold. Hence we have to take the following geometric convergence of a sequence of Kleinian groups into account in order to investigate the continuity of the Hausdorff dimension.

Definition (Geometric convergence) We say that a sequence of subgroups Γ_n of $\mathrm{PSL}_2(\mathbf{C})$ converges geometrically to G if Γ_n converge to G as closed subsets of $\mathrm{PSL}_2(\mathbf{C})$ in the *Hausdorff topology*; namely the following two conditions are satisfied:

- Any $g \in G$ is written as $g = \lim_{n\to\infty} \gamma_n$ ($\gamma_n \in \Gamma_n$).
- Any element written as $g = \lim_{i\to\infty} \gamma_{n_i}$ ($\gamma_{n_i} \in \Gamma_{n_i}$) belongs to G.

The image of the origin $0 \in B^3$ under the projection $B^3 \to N_\Gamma = B^3/\Gamma$ is denoted by o_Γ. The condition that Kleinian groups Γ_n converge geometrically to G is equivalent to saying that the sequence of corresponding hyperbolic manifolds with the base point $(N_{\Gamma_n}, o_{\Gamma_n})$ converges to (N_G, o_G) *in the sense of Gromov*. That is, as n grows, the larger neighborhoods of the base points in N_{Γ_n} and in N_G are mapped onto each other by diffeomorphisms that are the closer to an isometry.

In general, a sequence of hyperbolic manifolds with base points has a convergent subsequence in the sense of Gromov if the injectivity radii at the base points are uniformly bounded from below. Hence, under certain circumstances that imply this condition, we can choose a geometrically convergent subsequence from a sequence of Kleinian groups. The assumption that the sequence is already known to be algebraically convergent produces one of such situations.

Proposition 22 *If representations $\theta_n : \Gamma_0 \to \Gamma_n$ onto Kleinian groups Γ_n converge algebraically to $\theta : \Gamma_0 \to \Gamma$ then a subsequence of Γ_n converges geometrically to a Kleinian group G that contains Γ.*

Bishop and Jones [BJ] proved lower semi-continuity of the Hausdorff dimension of the limit sets of finitely generated Kleinian groups under the algebraic convergence.

Theorem 23 *If representations $\theta_n : \Gamma_0 \to \Gamma_n$ onto Kleinian groups Γ_n converge algebraically to $\theta : \Gamma_0 \to \Gamma$ then*

$$\liminf_{n\to\infty} \dim \Lambda(\Gamma_n) \geq \dim \Lambda(\Gamma).$$

By Lemma 21, we can see that the Hausdorff dimension of the conical limit set has lower semi-continuity under either algebraic or geometric convergence. Hence the assertion of Theorem 23 immediately follows if the algebraic limit Γ is geometrically finite. From now on, in the case where Γ is not geometrically finite, we will prove a stronger assertion, the continuity of the Hausdorff dimension. The essential step for it is the following result due to Bishop and Jones [BJ].

Theorem 24 *If a finitely generated Kleinian group Γ is not geometrically finite then* $\dim \Lambda(\Gamma) = 2$.

Sketch of proof. For the sake of simplicity, we explain this theorem in the case where the boundary ∂C_Γ of the convex core is totally geodesic. We take the double of C_Γ with respect to ∂C_Γ and denote it by N. Then N is represented as $N = N_G$ by a Kleinian group G. If $\lambda_0(\Gamma) = 0$ then $\dim \Lambda_c(\Gamma) = 2$ by Proposition 18. Thus we have only to consider the case $\lambda_0(\Gamma) > 0$. By the Ahlfors finiteness theorem, the hyperbolic area of ∂C_Γ is finite. By assumption, C_Γ has an infinite end. From these two facts, we can see that N_G satisfies $\lambda_0(G) > 0$, too. (We are convinced of this fact more easily if we consider the isoperimetric constant for example.) This implies in particular that the harmonic measure of the ideal boundary of N_G is positive, and thus so is the harmonic measure of the ideal boundary of C_Γ, which is the half of N_G divided along ∂C_Γ. This is equivalent to saying that the 2-dimensional measure of the limit set $\Lambda(\Gamma)$ is positive. In particular $\dim \Lambda(\Gamma) = 2$ follows. ∎

Note that if N_Γ is known to be topologically tame in addition, Theorem 16 already implied a stronger conclusion $\dim \Lambda_c(\Gamma) = 2$ than Theorem 24.

Remark For a geometrically finite Kleinian group Γ, we have obtained the condition for $\dim \Lambda(\Gamma) = 2$ (Corollary 14). Then, in the aid of the theorem above, we can extend it to all finitely generated Kleinian groups. However, for infinitely generated Kleinian groups, such a condition has not been completely investigated yet. As a necessary condition, we may think of unboundedness of the distance of points in the convex core from its boundary [Mt2]. As we can see from the proof of Theorem 24, a sufficient condition is, say, analytic finiteness of $\Omega(\Gamma)/\Gamma$, which is a consequence of the Ahlfors finiteness theorem [BJ].

Therefore, by the following lemma, we complete a proof of Theorem 23.

Lemma 25 *If representations $\theta_n : \Gamma_0 \to \Gamma_n$ onto Kleinian groups Γ_n converge algebraically to $\theta : \Gamma_0 \to \Gamma$ and if Γ is not geometrically finite then*

$$\lim_{n\to\infty} \dim \Lambda(\Gamma_n) = \dim \Lambda(\Gamma) \ (= 2)$$

Proof. Passing to a subsequence, we may assume that Γ_n are all geometrically finite or all geometrically infinite. In the latter case, the assertion is clear by Theorem 24. Hence we only consider the case where Γ_n are all geometrically finite. Passing to a subsequence again, we may assume that $\dim \Lambda(\Gamma_n)$ converge and, by Proposition 22, Γ_n converge geometrically to a Kleinian group G that contains Γ. If $\lim \dim \Lambda(\Gamma_n) < 2$ then, by Theorem 16, volumes of the convex cores C_{Γ_n} are uniformly bounded from above. In this case, the volume of C_G is finite (see [Ta]), and thus G is geometrically finite. By the lower semi-continuity, $\dim \Lambda(G) = \dim \Lambda_c(G) < 2$. However this contradicts $\dim \Lambda(\Gamma) = 2$. Hence $\lim \dim \Lambda(\Gamma_n) = 2$, which implies the assertion. ∎

Remark On the other hand, when a sequence of Kleinian groups Γ_n converges geometrically to G, we have a similar assertion; if G is finitely generated but not geometrically finite then $\lim \dim \Lambda(\Gamma_n) = \dim \Lambda(G)$. Indeed, if the geometric limit is finitely generated in general then there exist homomorphisms (not necessarily surjective) $\psi_n : G \to \Gamma_n$ for all sufficiently large n that converge algebraically to the identity isomorphism $\mathrm{id} : G \to G$ [JM]. Then applying Lemma 25, we obtain this assertion.

From these observations, we can see that the case where the limit is geometrically finite is essential as long as we consider the problem on the convergence of the Hausdorff dimension. Thus we restricted ourselves to this case hereafter.

First of all, we exhibit a simple example where the geometric convergence does not necessarily imply the convergence of the Hausdorff dimension of the limit set.

Example We take a sequence of finite index subgroups Γ_n of a Kleinian group Γ_0 such that $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \cdots$ converge geometrically to $\{1\}$. (For example, we can take such a sequence by letting the injectivity radii at the base point increase to ∞.) Then $\dim \Lambda(\Gamma_n) = \dim \Lambda(\Gamma_0) > 0$ but $\dim \Lambda(\{1\}) = 0$. The limit sets $\Lambda(\Gamma_n) = \Lambda(\Gamma_0)$ do not converge to $\Lambda(\{1\}) = \emptyset$ in the Hausdorff topology either.

In order to show the convergence of the Hausdorff dimension, we plan the following strategy: Let μ be a weak limit of the PS measures μ_n for geometrically finite Kleinian groups Γ_n. If we were able to determine that μ is the PS measure for the geometric limit G, then the critical exponents $\delta(\Gamma_n)$ would converge to $\delta(G)$ and thus $\dim \Lambda(\Gamma_n) \to \dim \Lambda(G)$ would be obtained. However, even though μ is a G-invariant probability measure, it may have a positive mass on $\Omega(G)$ in general. If we know at least that $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology, we can see that μ has its support on $\Lambda(G)$, which is a necessary condition for μ to be the PS measure. Hence, in order to keep this line, we have to impose such a condition that at least the Hausdorff convergence of the limit set is guaranteed.

Remark However, even if the limit sets do not converge in the Hausdorff topology, the Hausdorff dimension may converge. For example, we can take the Klein combination $G_n = \Gamma * g_n \Gamma' g_n^{-1}$ so that it converges geometrically to Γ as in Proposition 20, but $\Lambda(G_n)$ do not converge to $\Lambda(\Gamma)$ in the Hausdorff topology. (Remark that the assumption of Corollary 7.34 in [MT] is insufficient.) If $\delta(\Gamma) < \delta(\Gamma')$ then the Hausdorff dimension does not converge (and the weak limit μ of the PS measures has a positive mass on $\Omega(\Gamma)$) whereas if $\delta(\Gamma) \geq \delta(\Gamma')$ then it converges.

When the Hausdorff dimension of the limit set converges and the geometric limit G is a geometrically finite Kleinian group without a parabolic element, we can see the convergence of the Hausdorff dimension, because the only G-invariant probability measure with the support on the limit set is the PS measure. However, when G contains a parabolic element, the Hausdorff convergence of the limit sets does not necessarily implies the convergence of the Hausdorff dimension. In more details, the weak limit μ of the PS measures may consist of atoms on parabolic fixed points of G and in this case $\delta(G)$ must be less than the dimension of μ. For example, if we explain this phenomenon in terms of hyperbolic manifolds, it occurs in the case where a Gromov convergent sequence of hyperbolic manifolds collapses at a cusp so that the "main part" disappear in the limit and a mass of the measure which was in the main part is concentrated on the parabolic fixed points that correspond to the new cusp.

Therefore, to make the geometric structure of hyperbolic manifolds converge by the geometric convergence and make the structure of the fundamental group be preserved (i.e. make the manifold not collapse at the limit) by the algebraic convergence, we define the following convergence which includes both the properties:

Definition (Strong convergence) We say that a sequence of Kleinian groups Γ_n converges strongly to G if Γ_n converge geometrically to G and, for all sufficiently large n, there exist surjective homomorphisms $\psi_n : G \to \Gamma_n$ that converge algebraically to $\mathrm{id} : G \to G$.

Remark Usually we define strong convergence by a condition that an algebraically convergent sequence of faithful discrete representations is geometrically convergent at the same time. Clearly our definition above, which is due to McMullen [Mc], is wider than the usual one. Moreover, even in the case where the algebraic limit and the geometric limit differ, if the geometric limit is finitely generated then it is strongly convergent in our sense. Indeed, although we mentioned that there exist homomorphisms, not necessarily surjective, of a finitely generated geometric limit that converge algebraically to the identity isomorphism, they are actually surjective if we know in addition that the original sequence is algebraically convergent [JM].

If the limit G of a strongly convergent sequence is geometrically finite then Γ_n for all sufficiently large n are also geometrically finite and $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology [JM]. However, even in this case, there still exists an example, which was raised by McMullen [Mc], where the Hausdorff dimension does not converge.

Concerning convergence of the bottoms of the spectrum of hyperbolic manifolds under strong convergence, Canary and Taylor [CT2] showed the following result by a proof of geometric flavor independently of McMullen. Also Fan and Jorgenson [FJ] proved convergence of the heat kernels of hyperbolic manifolds and obtained a similar result.

Theorem 26 *If a sequence of Kleinian groups Γ_n converges strongly to G and if N_G is topologically tame then* $\lim_{n\to\infty} \lambda_0(\Gamma_n) = \lambda_0(G)$.

The reason why the convergence of the bottom of the spectrum does not necessarily imply the convergence of the Hausdorff dimension lies in their relationship (Theorem 17); as long as the critical exponent is not greater than 1, the bottom of the spectrum is identically 1. In fact, McMullen's counterexample is a phenomenon that occurs near the dimension 1. He proved the following theorem concerning the convergence of the Hausdorff dimension under strong convergence by an analytic method and obtained the theorem above contrarily by interpreting the Hausdorff dimension into the bottom of the spectrum [Mc].

Theorem 27 *If a sequence of Kleinian groups Γ_n converges strongly to a finitely generated Kleinian group G that satisfies* $\dim \Lambda(G) \geq 1$ *then*

$$\lim_{n\to\infty} \dim \Lambda(\Gamma_n) = \dim \Lambda(G).$$

7 Hausdorff dimension 1

Hausdorff dimension 1 stands at a special position among the dimensions of the limit sets of Kleinian groups. Actually, we have experienced certain arguments in which the Hausdorff dimension and the critical exponent have a different feature on different sides of 1. The Hausdorff dimension of the limit set of a Fuchsian group is clearly 1 and conversely Bowen [Bo] proved that only a Fuchsian group is a quasifuchsian group whose limit set has Hausdorff dimension 1. Generalizing this result, we completely determine finitely generated Kleinian groups whose limit sets have Hausdorff dimension not greater than 1 [CT1], [BJ].

Definition (Quasifuchsian group) A geometrically finite Kleinian group Γ such that $\Omega(\Gamma)$ consists of two Γ-invariant Jordan domains is called a quasifuchsian group. In particular, if the two Jordan domains are round disks, it is called a Fuchsian group.

Theorem 28 *If a quasifuchsian group has the limit set of Hausdorff dimension 1 then it is a Fuchsian group.*

Proof. Let Γ be a quasifuchsian group whose limit set has Hausdorff dimension 1. By Lemma 6, the 1-dimensional Hausdorff measure of $\Lambda(\Gamma)$ is finite, in other words, $\Lambda(\Gamma)$ is rectifiable. Then the following lemma implies that Γ is a Fuchsian group.

Lemma 29 *If the limit set of a quasifuchsian group Γ is rectifiable then Γ is a Fuchsian group.*

Proof. Let Ω_1 and Ω_2 be the two invariant components of $\Omega(\Gamma)$ and consider Riemann mappings $f_1 : \Delta \to \Omega_1$ and $f_2 : \Delta^* \to \Omega_2$ where Δ is the unit disk and $\Delta^* = \hat{\mathbf{C}} - \overline{\Delta}$. Then f_1^{-1} and f_2^{-1} induce conjugation of Γ onto Fuchsian groups Γ_1 and Γ_2 respectively. Since f_1 and f_2 extend to the boundary homeomorphically, a homeomorphism $(f_2)^{-1} \circ f_1$ of $\partial\Delta$ is defined. By the assumption that $\Lambda(\Gamma)$ is rectifiable, the 1-dimensional Hausdorff measure and the harmonic measure on $\Lambda(\Gamma)$ are mutually absolutely continuous with respect to each other. Therefore $(f_2)^{-1} \circ f_1$ is absolutely continuous with respect to the 1-dimensional measure on the unit circle that induces conjugation between Fuchsian groups Γ_1 and Γ_2. Here we apply the rigidity of Fuchsian groups: an automorphism of the unit circle that is compatible with a Fuchsian group is either (totally) singular with respect to the 1-dimensional measure or the restriction of a Möbius transformation. Then $(f_2)^{-1} \circ f_1$ is the restriction of a Möbius transformation, from which we see that Γ should be a Fuchsian group. ∎

Remark A quasifuchsian group is alternatively defined as a quasiconformal deformation of a Fuchsian group. For a quasiconformal deformation of a Fuchsian group in general (which is not necessarily finitely generated), we may consider the statement of Lemma 29, however this is not true in general. The reason why not is that the rigidity of Fuchsian groups, which was used in the above proof, is valid only for a Fuchsian group whose 1-dimensional Poincaré series diverges (see [Mt1]). Astala and Zinsmeister showed the necessity of this condition as well as an example where a non-trivial quasiconformal deformation of a Fuchsian group has a rectifiable limit set [AZ].

The following lemma is useful not only for the extension of Theorem 28 but also at several places where we have utilized it already.

Lemma 30 *If a Kleinian group Γ contains a Kleinian subgroup H of divergence type and if $\Lambda(\Gamma)$ properly contains $\Lambda(H)$, then $\delta(\Gamma) > \delta(H)$. In particular, if Γ contains a parabolic abelian subgroup of rank $k = 1, 2$ then $\delta(\Gamma) > k/2$.*

Proof. Let μ be a $\delta(\Gamma)$-dimensional, Γ-invariant probability measure. We can take a disk $\Delta = \Delta(x, r) \subset \Omega(H)$ such that $\mu(\Delta) > 0$ and $\Delta \cap h(\Delta) = \emptyset$ for any non-trivial $h \in H$. Since

$$1 > \sum_{h \in H} \mu(h(\Delta)) \approx \mu(\Delta) \cdot \sum_{h \in H} |h'(x)|^{\delta(\Gamma)},$$

the $\delta(\Gamma)$-dimensional Poincaré series for H converges. Since H is of divergence type, this implies that $\delta(\Gamma) > \delta(H)$.

For a parabolic abelian subgroup J of rank k, the Poincaré series $\sum_{j \in J} |j'(z)|^s$ ($z \in \Omega(J)$) converges for $s > k/2$ and diverges for $s \leq k/2$ (see Lemma 12). Hence we obtain the latter assertion. ■

Theorem 31 *For a finitely generated Kleinian group* Γ*, if* $\dim \Lambda(\Gamma) \leq 1$ *then* $\Lambda(\Gamma)$ *is either a circle or a totally disconnected set.*

Proof. In general, a finitely generated Kleinian group Γ has either a quasifuchsian subgroup, a totally degenerate subgroup or a totally disconnected limit set. However, it cannot have a totally degenerate subgroup because it has the limit set of Hausdorff dimension 2 by Theorem 16 or 24. If Γ has a quasifuchsian subgroup H, it follows from Theorem 28 that H is Fuchsian. Then Lemma 30 implies that $\Lambda(\Gamma) = \Lambda(H)$. ■

Corollary 32 *If* $\dim \Lambda(\Gamma) < 1$ *for a finitely generated Kleinian group* Γ *then the hyperbolic manifold* N_Γ *is homeomorphic to the interior of a handlebody.*

Proof. By Theorem 31, $\Lambda(\Gamma)$ is a totally disconnected set. If Γ contained a parabolic abelian subgroup of rank 2 then we would have $\dim \Lambda(\Gamma) > 1$ by Lemma 30. Thus Γ must be a Schottky-like group which may contain a parabolic cyclic subgroup. This is equivalent to the conclusion of the statement. ■

Form this result, we see that the Hausdorff dimension determines the topology of a hyperbolic manifold in some case. We conversely define a topological invariant of a 3-manifold from the Hausdorff dimension.

Definition For a compact 3-manifold M (with boundary) that admits a hyperbolic structure, we define $D(M)$ as the infimum of $\dim \Lambda(\Gamma)$ taken over all Kleinian groups Γ such that the hyperbolic manifold N_Γ is homeomorphic to the interior of M.

By Corollary 32, if $D(M) = 0$ then M is a handlebody. Conversely if M is a handlebody then $D(M) = 0$ by Proposition 20. Hence the next possible value of $D(M)$ is 1. Though it had been known that if M is an I-bundle (direct product or twisted bundle) over a surface then $D(M) = 1$, Canary, Minsky and Taylor [CMT] completely determined all other 3-manifolds M with $D(M) = 1$. Intuitively speaking, if M is a union of parts of the product structure, we can take the following sequence which shows $D(M) = 1$ for a similar reasoning to Proposition 20: the Hausdorff dimension of a quasifuchsian group that corresponds to each part of M converges to 1 but only one of them remains so that M converges geometrically to a Fuchsian product structure. Conversely they proved $D(M) > 1$ for the other 3-manifolds M. Here we show a proof of this fact only in the simplest case.

Proposition 33 *If M is acylindrical and not an I-bundle then $D(M) > 1$.*

Proof. Suppose that there exists a sequence of Kleinian groups Γ_n such that N_{Γ_n} is homeomorphic to the interior of M and $\dim \Lambda(\Gamma_n) \to 1$. By compactness of the set of discrete faithful representations of the fundamental group of an acylindrical M [Th], we may assume that $\theta_n : \pi_1(M) \to \Gamma_n$ converge algebraically to a discrete faithful representation $\theta : \pi_1(M) \to \Gamma$. Then $\dim \Lambda(\Gamma) \leq 1$ by Theorem 23, however this contradicts Theorem 31. ■

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(Translated by the author)

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