Dedicated to Professor N. Suita on his 60th birthday

THE CONSERVATIVE-DISSIPATIVE DICHOTOMY FOR GEOMETRIC COVERS OF RIEMANN SURFACES

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We characterize Riemann surfaces which admit non-constant bounded harmonic functions by a certain property of their normal covers. It is concerning the degree of growth of the hyperbolic area within the distance r as r tends to infinity and indicated by the conservative and dissipative part for the action of the Fuchsian models. For this characterization, we study inheritance of this property to geometric normal covers of Riemann surfaces.

Throughout this note, a Riemann surface W is assumed to be hyperbolic, that is, W is represented by a quotient space of the unit disk Δ with the hyperbolic metric by a discrete isometry group (= Fuchsian group) Γ.

Let us denote by O_{HB} the class of Riemann surfaces which do not admit non-constant bounded harmonic functions. In[M], we have seen the following result about normal covers of Riemann surfaces which belong to O_{HB}.

PROPOSITION ([M, Th.5.4]). *If* $W \in O_{HB}$, *then any normal cover of W either belongs to the class* C *or does not belong to the class* U_C.

Here, we will explain the classes C and U_C . Let Γ be the Fuchsian model acting on Δ of a hyperbolic Riemann surface W. The action of Γ divides the circle at infinity $S^1 = \partial \Delta$ into conservative part $\mathscr{R}(\Gamma)$ and dissipative part $\mathscr{D}(\Gamma)$ up to null sets, where S^1 is equipped with the Lebesgue measure. Let ω be a Dirichlet fundamental region of Γ. Then we may define $\mathscr{D}(\Gamma) = \cup_{\gamma \in \Gamma} \gamma\left(\mathrm{cl}(\omega) \cap S^1\right)$ and $\mathscr{R}(\Gamma) = S^1 - \mathscr{D}(\Gamma)$. It is known that up to null sets $\mathscr{R}(\Gamma)$ is equal to the horocyclic limit set of Γ [S] and $\mathscr{D}(\Gamma)$ is equal to the set where Green's function of Γ has an angular derivative [P]. We define C(U_C) as a class of Riemann surfaces whose Fuchsian model has full (positive) measure conservative part, respectively. There is the following inclusion relation: $O_{HB} \subset C \subset U_C$.

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Next, we introduce a similar dichotomy for the family of all geodesic rays departing from a point in W. Let $\pi : \Delta \to W$ be the universal covering map. For a given point $p \in W$, we may assume that $\pi(0) = p$. Then the family $\mathscr{S}(p, W)$ of all geodesic rays from p is identified with S^1 and a sub-family E of $\mathscr{S}(p, W)$ is measured by the Lebesgue measure on S^1 (we call this measure the visual measure at p). Let $E(p,r)$ be the set of points in E which are within the distance r from p along the rays. We define $\mathscr{D}(p, W)$ as the maximal measurable sub-family of $\mathscr{S}(p, W)$ that has the following property (it is well- defined up to null sets):

for any sub-family $E \subset \mathscr{S}(p, W)$ of positive visual measure,
Area $(E(p, r))$ / Area (disk of radius r) $\not\to 0$ as $r \to \infty$.

And we set $\mathscr{R}(p, W) = \mathscr{S}(p, W) - \mathscr{D}(p, W)$. The condition $W \in$ C is equivalent to that $\mathscr{R}(p, W)$ is of full visual measure (see [S]). We can also prove that $W \in \mathrm{U_C}$ if $\mathscr{R}(p, W)$ is of positive visual measure.

We apply this to subregions of Riemann surfaces and classify them. In this note, we treat only subregions X of hyperbolic Riemann surfaces W with totally geodesic boundary ∂X. Take a point p of X and let $\mathscr{S}(p,X)$ be the sub-family of geodesic rays in $\mathscr{S}(p, W)$ which do not exit from X. When $\mathscr{S}(p,X)$ is of null measure, we say that X belongs to SO (= $\mathrm{SO_{HB}}$ in the usual notation). Moreover we say that a subregion X which does not belong to SO is in the class SC if $\mathscr{S}(p,X) \cap \mathscr{R}(p, W) = \mathscr{S}(p, X)$ a.e. for some $p \in X$, and in the class $\mathrm{SU_C}$ if $\mathscr{S}(p,X) \cap \mathscr{R}(p, W)$ is of positive measure for some $p \in X$.

These definitions enable us to localize the conservative-dissipative property of Riemann surfaces as follows.

LEMMA (localization lemma). *Let W be a hyperbolic Riemann surface and X a subregion with totally geodesic boundary. We have*

(1) *If* $X \in \mathrm{SU_C}$, *then* $W \in \mathrm{U_C}$, *and*

(2) *If* $X \notin$ SC, *then* $W \notin$ C.

Now, we will consider inheritance to covering surfaces. In this note, we focus on the following geometric setting. Let W be a Riemann surface with the complementary subregions X and Y whose fundamental groups are non-trivial. We define the geometric normal cover W^* with respect to X as a normal cover of W whose fundamental group $\pi_1(W^*)$ is the normal closure of $\pi_1(X)$ in $\pi_1(W)$, that is, the minimal normal subgroup generated by $\pi_1(X)$. Then, W^* consists of subregions which are copies of X and the planar subregion which is the preimage of Y under the covering map. The following theorem is the key to our investigation.

THEOREM 1. *Let W be a hyperbolic Riemann surface which has the complementary subregions X and Y whose boundary $\partial X = \partial Y$ is totally geodesic. Let f: $W^* \to W$ be the geometric normal cover with respect to X and denote the subregion $f^{-1}(Y)$ by Y^*. Further, we assume that (#) the hyperbolic lengths of loop components of $\partial X = \partial Y$ are bounded from below by some positive constant.*

Under these circumstances, if $Y \notin$ SO, then $Y^* \notin \mathrm{SU_C}$. In addition, if $X \in$ SO, then $W^* \notin \mathrm{U_C}$.

Remark. If $Y \in$ SO, then $Y^* \in$ SO, which is always valid.

Proof of Theorem 1. Let $\{\gamma_i\}_{i=1,2,\ldots}$ be the set of geodesic loops in $\partial X = \partial Y$. For each γ_i ,we consider the widest collar G_i of it. Since the lengths of $\{\gamma_i\}_{i=1,2,\ldots}$ are bounded from below, the widths of collars are bounded from above by a constant. We denote $Y - \bigcup \mathrm{cl}(G_i)$ by Y' and take a point p in Y'. Consider the family $\mathcal{S}(p, W)$ of geodesic rays. Since $Y \notin$ SO, $\mathcal{S}(p, Y)$ is of positive visual measure. Further, we can prove that almost every ray in $\mathcal{S}(p, Y)$ is eventually far away sufficiently from ∂Y (e.g. by McMillan's twist point theorem, cf. [P,§3]). Thus, we have a family $\mathcal{F}(= \mathcal{S}(p, Y)$ a.e.) each of whose rays stays in Y' except for a bounded set of intervals.

Let $\pi : \Delta \to W$ and $g : \Delta \to W^*$ be the universal covering maps such that $\pi = f \circ g$ and $\pi(0) = p$. We identify $\mathcal{S}(p, W)$, $\mathcal{S}(p^*, W^*)$ and $\mathcal{S}(0, \Delta) \cong S^1$, where $p^* = g(0)$. In the set $\mathcal{F}$, consider any set E of positive measure. When we regard E as the family of rays in Δ, we can choose a sub-family E' of E of positive measure and a component Ω of $\pi^{-1}(Y')$ such that all the rays in E' eventually stay in Ω. Remark that the map g is homeomorphic on each component of $\pi^{-1}(Y')$. Hence each ray of E' projected to W^* does not intersect with oneself or one another in $g(\Omega)$. On E' ($\subset W^*$), the hyperbolic area within the distance r from p^* grows as in the hyperbolic space, thus we have proved that for any family E of positive measure in $\mathcal{F}$ ($= \mathcal{S}(p^*, Y^*)$ a.e.)

$$\text{Area}\,(E(p^*,r)) / \text{Area}\,(\text{disk of radius } r) \not\to 0 \text{ as } r \to \infty.$$

This implies that $\mathcal{S}(p^*, Y^*) = \mathcal{D}(p^*, Y^*)$ a.e., that is $Y^* \notin \mathrm{SU_C}$.

In addition, assume that $X \in$ SO. Then, almost every ray of $\mathcal{S}(p, W)$ eventually stays in Y sufficiently far away from ∂Y. By the same argument as in the first part of the proof, it follows that for any positive measure set E of $\mathcal{S}(p^*, W^*)$, there exists E' ($\subset E$) of positive measure such that the rays of E' are eventually "parallel". Thus, E' determines a wandering set on the circle at infinity $S^1 \cong \mathcal{S}(p^*, W^*)$ for the action of the Fuchsian model of W^*. Since E is arbitrary, this implies that $W^* \notin U_C$. □

By the above Lemma and Theorem 1 combined with Proposition, we obtain the following result:

THEOREM 2. *For a Riemann surface W which satisfies that (##) the lengths of closed geodesics in W are bounded from below by a positive constant, W belongs to* $U_C - O_{HB}$ *if and only if it has a normal cover which belongs to* $U_C - C$.

Proof. The "if part" is known from Proposition and the fact that any cover of a Riemann surface not in U_C is not in U_C. We will show the "only if part". Suppose that $W \in U_C - O_{HB}$. If $W \in U_C - C$, we have nothing to prove. Thus, we may assume that $W \in C - O_{HB}$. Since $W \notin O_{HB}$, the two region test (cf. [SN, p.242]) gives the two complementary subregions X and Y such that $X, Y \notin SO$. Further, we may take them so that $\partial X = \partial Y$ is totally geodesic. Then by Theorem 1, $Y^* \notin SU_C$, in particular, $Y^* \notin SC$. Using Lemma (2), we have $W^* \notin C$. On the other hand, since $W \in C$, we have $X \in SC$ again by Lemma (2), in particular, $X \in SU_C$. The normal cover W^* has a subregion which is a copy of X, hence we have $W^* \in U_C$ by Lemma (1). Therefore, W^* belongs to $U_C - C$. □

Remarks. 1. Theorem 1 gives another proof of [M, Th.5.2].

2. By Theorem 1, we can see that the Riemann surface R' constructed in [M, Th.6.1, cf.§6. Remark (1)] does not belong to U_C.

3. We do not know whether the condition (#) is needed for Theorem 1, or (##) for Theorem 2.

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