

The Patterson-Sullivan measure and proper conjugation for Kleinian groups of divergence type

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ABSTRACT. A Kleinian group (a discrete subgroup of conformal automorphisms of the unit ball) G is said to have proper conjugation if it contains the conjugate $\alpha G \alpha^{-1}$ by some conformal automorphism α as a proper subgroup in it. We show that a Kleinian group of divergence type cannot have proper conjugation. Uniqueness of the Patterson-Sullivan measure for such a Kleinian group is crucial to our proof.

§1. INTRODUCTION

The co-Hopf problem on an abstract group G asks whether any injective homomorphism of G into itself should be surjective or not. As a variation of this problem, we consider conditions under which there is no proper conjugation for Kleinian groups.

A Kleinian group G is a discrete subgroup of the group $\text{Isom}^+(\mathbb{H}^{n+1})$ of all orientation-preserving isometric automorphisms of the hyperbolic space \mathbb{H}^{n+1} for $n \geq 1$. We say that G has *proper conjugation* if there exists $\alpha \in \text{Isom}^+(\mathbb{H}^{n+1})$ such that the conjugate $\Gamma = \alpha G \alpha^{-1}$ is a proper subgroup of G .

For $n = 1$, namely, when G is a Fuchsian group, Heins [4] proved that, if G uniformizes a Riemann surface that does not admit the Green function, then G has no proper conjugation. On the other hand, Jørgensen, Marden and Pommerenke [5] gave a systematic construction of Fuchsian groups having proper conjugation. For higher dimensional Kleinian groups, the problem of proper conjugation has been studied topologically by Wang and Zhou [15] and Ohshika and Potyagailo [11] among others.

For a Kleinian group Γ , the Poincaré series $P_\Gamma^s(x, z)$ of dimension $s \geq 0$ is defined by the sum of e^{-sd} , where d runs over all the hyperbolic distances of the orbits $\Gamma(x)$ from z in \mathbb{H}^{n+1} . Its *critical exponent of convergence* $\delta(\Gamma)$ is defined by the infimum of the dimension s where the Poincaré series $P_\Gamma^s(x, z)$ converges. If $P_\Gamma^s(x, z)$ diverges at the critical dimension $s = \delta(\Gamma)$, we say that Γ is of *divergence type*. The hyperbolic manifold \mathbb{H}^{n+1}/Γ admits no Green function if and only if Γ is of divergence type and $\delta(\Gamma) = n$.

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The main result of this paper is the following, which generalizes the aforementioned theorem due to Heins [4] in two directions: one is to higher dimension $n \geq 2$ and the other is to general divergence type with $\delta(G) < n$.

Theorem 1.1. *A Kleinian group of divergence type cannot have proper conjugation.*

To prove this theorem, we use a G -invariant conformal density of dimension $\delta(G)$ called the Patterson-Sullivan measure. The uniqueness of this measure for a Kleinian group G of divergence type is crucial to our arguments. According to Culler and Shalen [1], we decompose the Patterson-Sullivan measure into parts corresponding to the coset decomposition of G modulo the conjugate $\Gamma = \alpha G \alpha^{-1}$. Then we estimate the ratio of the decomposed measures to the original one. For this purpose, we analyze an ascending sequence of the conjugated groups to G given by the iteration of α and the Patterson-Sullivan measure of its geometric limit. This system has appeared in a work by McMullen and Sullivan [8] for the case of Fuchsian groups.

Since every geometrically finite Kleinian group is of divergence type, the following result is easily obtained from our main theorem. This result has been proved by Wang and Zhou [15].

Corollary 1.2. *A geometrically finite Kleinian group has no proper conjugation.*

§2. THE POINCARÉ SERIES

In this section, we introduce the Poincaré series for Kleinian groups and give its basic properties. In particular, Kleinian groups of divergence type are defined.

A Kleinian group Γ is a discrete group of orientation-preserving isometric automorphisms of the $(n+1)$ -dimensional hyperbolic space (\mathbb{H}^{n+1}, d) for $n \geq 1$. It acts on \mathbb{H}^{n+1} properly discontinuously. We always assume that Γ is *non-elementary*. The unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$ with the metric $2|dx|/(1-|x|^2)$ is a model of the hyperbolic space and the group $\text{Isom}^+(\mathbb{H}^{n+1})$ of all orientation-preserving isometric automorphisms of \mathbb{H}^{n+1} is identified with the group $\text{Conf}(\mathbb{B}^{n+1})$ of all conformal transformations preserving \mathbb{B}^{n+1} . The euclidean boundary S^n of the model \mathbb{B}^{n+1} is located at infinity of the hyperbolic space and the action of Γ extends to S^n .

For a Kleinian group Γ , the Poincaré series of dimension $s \geq 0$ with respect to the base point $z \in \mathbb{B}^{n+1}$ and to the orbit point $x \in \mathbb{B}^{n+1}$ is defined by

$$P_\Gamma^s(x, z) = \sum_{\gamma \in \Gamma} \exp(-sd(\gamma(x), z)).$$

The *critical exponent of convergence* for Γ is defined by

$$\delta(\Gamma) = \inf \{s \geq 0 \mid P_\Gamma^s(x, z) < \infty\},$$

which is independent of the choices of z and x . It is known that $\delta(\Gamma)$ satisfies $0 < \delta(\Gamma) \leq n$. We say that Γ is of *divergence type* (at the critical exponent $\delta(\Gamma)$) if

$P_\Gamma^{\delta(\Gamma)}(x, z) = \infty$ and of *convergence type* if $P_\Gamma^{\delta(\Gamma)}(x, z) < \infty$. It is known that the hyperbolic manifold $N_\Gamma = \mathbb{H}^{n+1}/\Gamma$ does not admit the Green function if and only if $\delta(\Gamma) = n$ and Γ is of divergence type. In particular, if N_Γ is of finite volume, then this condition is satisfied. More generally, every geometrically finite Kleinian group Γ is of divergence type.

Here we introduce simple but important facts on the Poincaré series for our work. They easily follow from the definition; we skip the proof.

Proposition 2.1. *The Poincaré series $P_\Gamma^s(x, z)$ satisfies the following properties:*

- (1) $P_\Gamma^s(x, z) = P_\Gamma^s(z, x)$ for any x and z in \mathbb{B}^{n+1} ;
- (2) $P_\Gamma^s(g(x), g(z)) = P_\Gamma^s(x, z)$ for every element g in the normalizer $N(\Gamma)$ of Γ in $\text{Conf}(\mathbb{B}^{n+1})$.

§3. THE PATTERSON-SULLIVAN MEASURE

A fundamental tool for proving our main theorem is the Patterson-Sullivan measure. Uniqueness of such measures for Kleinian groups of divergence type is the key to our proof. In this section, we explain basic facts concerning invariant conformal densities.

We consider a family $\{\mu_z\}_{z \in \mathbb{B}^{n+1}}$ of positive finite Borel measures on S^n , more precisely, a map $\mu : \mathbb{B}^{n+1} \rightarrow M(S^n)$ where $M(S^n)$ is the set of such measures.

Definition. We say that $\{\mu_z\}_{z \in \mathbb{B}^{n+1}}$ is a *conformal density* of dimension $s \geq 0$ if μ_z and $\mu_{z'}$ are absolutely continuous to each other for any z and z' in \mathbb{B}^{n+1} and the Radon-Nikodym derivative satisfies

$$\frac{d\mu_z}{d\mu_{z'}}(\xi) = \left(\frac{k_z(\xi)}{k_{z'}(\xi)} \right)^s \quad (\xi \in S^n),$$

where $k_z(\xi)$ is the Poisson kernel, which is explicitly given by

$$k_z(\xi) = \frac{1 - |z|^2}{|\xi - z|^2}.$$

For any element $h_z \in \text{Conf}(\mathbb{B}^{n+1})$ sending z to the origin 0, the Poisson kernel $k_z(\xi)$ is coincident with the linear stretch factor $|h'_z(\xi)|$ of h_z at $\xi \in S^n$.

Definition. We say that the measure family $\{\mu_z\}_{z \in \mathbb{B}^{n+1}}$ is *invariant* under a subgroup $\Gamma \subset \text{Conf}(\mathbb{B}^{n+1})$ if

$$\mu_{\gamma(z)}(\gamma(E)) = \mu_z(E)$$

for any Borel set $E \subset S^n$, any element $\gamma \in \Gamma$ and any point $z \in \mathbb{B}^{n+1}$. If we define the pull-back of a measure μ by γ as $(\gamma^*\mu)(E) = \mu(\gamma(E))$, then the Γ -invariance can be expressed by $\gamma^*\mu_z = \mu_{\gamma^{-1}(z)}$.

Let $\Gamma \subset \text{Conf}(\mathbb{B}^{n+1})$ be a Kleinian group. It is known that, if a Γ -invariant conformal density of dimension s exists, then the dimension s is not less than

the critical exponent $\delta(\Gamma)$ (see Corollary 4.5.3 of Nicholls [10] or [6]). Here, we consider a Γ -invariant conformal density of dimension $\delta(\Gamma)$. When μ_z has support on the limit set $\Lambda(\Gamma)$, we call $\{\mu_z\}_{z \in \mathbb{B}^{n+1}}$ the *Patterson-Sullivan measure* for Γ . In the sequel, under the assumption that Γ is of divergence type, we review the construction (existence) of the Patterson-Sullivan measure.

For any $s > \delta = \delta(\Gamma)$ and for any x and z in \mathbb{B}^{n+1} , we consider a measure

$$\mu_{x,z}^s := \frac{1}{P_\Gamma^s(x, 0)} \sum_{\gamma \in \Gamma} e^{-sd(\gamma(x), z)} D_{\gamma(x)}$$

on the compact space $\overline{\mathbb{B}^{n+1}} = \mathbb{B}^{n+1} \cup S^n$. Here D_x is the Dirac measure at x . Let $\sigma(\mu_{x,z}^s)$ denote the total mass of $\mu_{x,z}^s$, which is equal to $P_\Gamma^s(x, z)/P_\Gamma^s(x, 0)$. It can be estimated as $e^{-sd(0,z)} \leq \sigma(\mu_{x,z}^s) \leq e^{sd(0,z)}$. In particular, $\sigma(\mu_{x,0}^s) = 1$ for $z = 0$, that is, $\mu_{x,0}^s$ is a probability measure.

Fix x and z . For a sequence $\{s_i\}$ of real numbers ($s_i > \delta$) converging to δ as $i \rightarrow \infty$, consider the corresponding sequence $\{\mu_{x,z}^{s_i}\}$ of the measures. Since the space of normalized Radon measures on a compact Hausdorff space is compact in the weak topology, we can choose a subsequence $\{\mu_{x,z}^{s_{i'}}\}$ that converges weakly to a measure $\mu_{x,z}$ on $\overline{\mathbb{B}^{n+1}}$. Since $\lim_{s \rightarrow \delta+} P_\Gamma^s(x, 0) = \infty$, we see that the support of $\mu_{x,z}$ is on the limit set $\Lambda(\Gamma) \subset S^n$. Moreover, $\mu_{x,z}$ is a Γ -invariant conformal density of dimension δ , namely, it is the Patterson-Sullivan measure for Γ . The total mass $\sigma(\mu_{x,z})$ is given by

$$\sigma(\mu_{x,z}) = \lim_{i' \rightarrow \infty} \frac{P_\Gamma^{s_{i'}}(x, z)}{P_\Gamma^{s_{i'}}(x, 0)}.$$

In particular, we have $\sigma(\mu_{x,0}) = 1$ for every $x \in \mathbb{B}^{n+1}$.

For a Kleinian group Γ of divergence type, it is known that the Patterson-Sullivan measure is unique up to constant multiples (see Corollaire 1.8 of Roblin [12] or [6]). From this fact, we can deduce that the measures $\mu_{x,z}$ constructed above are depending neither the orbit point $x \in \mathbb{B}^{n+1}$ nor the choice of the subsequence $s_{i'}$. Giving the normalization at $z = 0$ as a probability measure, we denote the Patterson-Sullivan measure for Γ by μ_z^Γ .

In particular, we have the following fact, which is a crucial observation in the rest of this work.

Proposition 3.1. *For a Kleinian group Γ of divergence type,*

$$\lim_{s \rightarrow \delta+} \frac{P_\Gamma^s(x, z)}{P_\Gamma^s(x, 0)} = \sigma(\mu_z^\Gamma)$$

is satisfied independently of $x \in \mathbb{B}^{n+1}$.

§4. INVARIANCE OF THE PATTERSON-SULLIVAN MEASURE UNDER THE NORMALIZER

As a consequence of Propositions 2.1 and 3.1, we will see in this section that, the Patterson-Sullivan measure for a Kleinian group Γ of divergence type is invariant

under its normalizer $N(\Gamma)$ in $\text{Conf}(\mathbb{B}^{n+1})$. This result is also contained in our other work [7], as a more general statement applicable to a Kleinian group whose Patterson-Sullivan measure is unique up to constant multiples.

The following fact has been used in Nayatani [9], which shows homothety of the Patterson-Sullivan measure under the normalizer. We give its proof here, for it helps us to understand the idea of our arguments.

Lemma 4.1. *Let μ_z be the Patterson-Sullivan measure for a Kleinian group Γ and assume that it is unique up to constant multiples. Then, for every element g in the normalizer $N(\Gamma)$, there exists a constant $c = c(g) > 0$ such that $g^*\mu_z = c\mu_{g^{-1}(z)}$.*

Proof. We define a conformal density $\nu_z := g^*\mu_{g(z)}$. This is invariant under Γ . Indeed, for any $\gamma \in \Gamma$, we have $g\gamma = \tilde{\gamma}g$ for some $\tilde{\gamma} \in \Gamma$. Hence

$$\gamma^*\nu_z = \gamma^*g^*\mu_{g(z)} = g^*\tilde{\gamma}^*\mu_{g(z)}.$$

Here, by the Γ -invariance of μ_z , we conclude that

$$g^*\tilde{\gamma}^*\mu_{g(z)} = g^*\mu_{\tilde{\gamma}^{-1}g(z)} = g^*\mu_{g\gamma^{-1}(z)} = \nu_{\gamma^{-1}(z)}.$$

This shows the invariance of ν_z under Γ .

Since $g \in N(\Gamma)$ keeps the limit set $\Lambda(\Gamma)$ invariant, the support of ν_z is also on $\Lambda(\Gamma)$. Then, by the uniqueness of the Patterson-Sullivan measure, there exists a constant $c > 0$ such that $\nu_z = c\mu_z$. This proves that $g^*\mu_{g(z)} = c\mu_z$, equivalently, $g^*\mu_z = c\mu_{g^{-1}(z)}$. \square

The following result in this section promotes the homothety in the above sense to the invariance. The point of our proof is that we trace back to the construction of the Patterson-Sullivan measure and utilize the properties of the Poincaré series.

Theorem 4.2. *Let Γ be a Kleinian group of divergence type and $N(\Gamma)$ the normalizer of Γ . Then the Patterson-Sullivan measure μ_z^Γ for Γ is invariant under $N(\Gamma)$.*

Proof. By Lemma 4.1, there exists a constant $c(g) > 0$ for every $g \in N(\Gamma)$ such that $g^*\mu_z^\Gamma = c(g)\mu_{g^{-1}(z)}^\Gamma$. We will show that $c(g) = 1$ for every $g \in N(\Gamma)$, which yields the assertion of the theorem.

By taking the total masses in the above equation on the measures, we have

$$c(g) = \frac{\sigma(g^*\mu_z^\Gamma)}{\sigma(\mu_{g^{-1}(z)}^\Gamma)} = \frac{\sigma(\mu_z^\Gamma)}{\sigma(\mu_{g^{-1}(z)}^\Gamma)}$$

for every $z \in \mathbb{B}^{n+1}$. In particular, we obtain

$$c(g) = \sigma(\mu_{g(0)}^\Gamma); \quad c(g)^{-1} = \sigma(\mu_{g^{-1}(0)}^\Gamma)$$

by setting $z = g(0)$ and $z = 0$ respectively. Here, Proposition 3.1 says that

$$\sigma(\mu_{g(0)}^\Gamma) = \lim_{s \rightarrow \delta+} \frac{P_\Gamma^s(x, g(0))}{P_\Gamma^s(x, 0)}; \quad \sigma(\mu_{g^{-1}(0)}^\Gamma) = \lim_{s \rightarrow \delta+} \frac{P_\Gamma^s(x, g^{-1}(0))}{P_\Gamma^s(x, 0)}$$

for every $x \in \mathbb{B}^{n+1}$. When we set $x = 0$, we see $P_\Gamma^s(0, g(0)) = P_\Gamma^s(0, g^{-1}(0))$ by Proposition 2.1. Hence $c(g) = c(g)^{-1}$. \square

Corollary 4.3. *Let Γ be a Kleinian group of divergence type and G a Kleinian group that contains Γ as a normal subgroup. Then $\delta(\Gamma) = \delta(G)$ and G is also of divergence type.*

Proof. We have only to show that $\delta(\Gamma) \geq \delta(G)$. By Theorem 4.2, the Patterson-Sullivan measure μ_z^Γ for Γ is invariant under G . Then μ_z^Γ is a G -invariant conformal density of dimension $\delta(\Gamma)$. Since the dimension of such a density is not less than $\delta(G)$, we have the required inequality $\delta(\Gamma) \geq \delta(G)$. \square

Remark. If we drop the condition that Γ is of divergence type, Theorem 4.2 or Corollary 4.3 is not valid any more. Indeed, if Γ is a normal subgroup of a convex cocompact Kleinian group G with $\delta(G) > n/2$ and the quotient G/Γ is non-amenable, then $\delta(\Gamma) < \delta(G)$ by Theorem 3 of Brooks [2].

§5. THE STRUCTURE OF PROPER CONJUGATION

If a Kleinian group G has proper conjugation, it imposes certain constraint on G in both senses of algebraic structure and invariance of conformal densities. In this section, we summarize these facts in two folds. We first consider the algebraic structure. Note that, in our work [3] with Fujikawa and Taniguchi, we have studied the proper conjugation of Fuchsian groups (in the case of $n = 1$) based on the following fact.

Proposition 5.1. *For a Kleinian group $G \subset \text{Conf}(\mathbb{B}^{n+1})$ and an element $\alpha \in \text{Conf}(\mathbb{B}^{n+1})$, assume that $\Gamma = \alpha G \alpha^{-1}$ is properly contained in G . Set $\Gamma_m = \alpha^{-m} \Gamma \alpha^m$ for every integer $m \geq 0$ ($\Gamma_0 = \Gamma$, $\Gamma_1 = G$). Then the following are satisfied:*

- (1) $\Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots$ and α is of infinite order;
- (2) $\Gamma_\infty = \bigcup_{m \geq 0} \Gamma_m$ is a Kleinian group;
- (3) α belongs to the normalizer $N(\Gamma_\infty)$ of Γ_∞ .

Proof. Statement (1) is evident. Statement (2) is based on a fact that Γ_∞ is discrete if every two-generator subgroup of Γ_∞ is discrete. This is a generalization of the Jørgensen theorem to higher dimension, which can be seen by Theorem 1.1 of Wang and Yang [16]. Since any two-generator subgroup of Γ_∞ is contained in some Γ_m , it is discrete. Statement (3) follows from the inclusion relations $\alpha \Gamma_\infty \alpha^{-1} \subset \Gamma_\infty$ and $\alpha^{-1} \Gamma_\infty \alpha \subset \Gamma_\infty$, which are both clear by the definition of Γ_∞ . \square

Next, we show that this structure of the proper conjugation yields the coincidence of the Patterson-Sullivan measures for all the Kleinian groups Γ_m if they are of divergence type.

Lemma 5.2. *In the same circumstances as in Proposition 5.1, we further assume that the Kleinian group G is of divergence type and hence so are all Γ_m ($0 \leq m < \infty$). Set $\delta = \delta(G) = \delta(\Gamma_m)$. Then the following are satisfied:*

- (1) $\delta(\Gamma_\infty) = \delta$ and Γ_∞ is of divergence type;
- (2) The Patterson-Sullivan measures $\mu_z^{\Gamma_m}$ for Γ_m are all coincident with $\mu_z^{\Gamma_\infty}$ for Γ_∞ ;

(3) $\mu_z^{\Gamma_\infty}$ is invariant under $N(\Gamma_\infty)$.

Proof. (1) It is clear that $\delta(\Gamma_\infty) \geq \delta$. On the other hand, since Γ_∞ is the limit of the increasing sequence $\{\Gamma_m\}$, we have $\delta(\Gamma_\infty) \leq \lim_{m \rightarrow \infty} \delta(\Gamma_m) = \delta$ (see Corollary 6 of Sullivan [13] or [6]). Hence $\delta(\Gamma_\infty) = \delta$. Since $P_{\Gamma_\infty}^\delta(x, z) \geq P_{\Gamma_m}^\delta(x, z) = \infty$, we see that Γ_∞ is of divergence type.

(2) The Patterson-Sullivan measure $\mu_z^{\Gamma_\infty}$ is of course invariant under its subgroups Γ_m ($m \geq 0$). Since they are all of divergence type and of the same critical exponent δ , the uniqueness of the Patterson-Sullivan measure concludes that $\mu_z^{\Gamma_\infty}$ is the Patterson-Sullivan measure for all these subgroups.

(3) Since Γ_∞ is of divergence type, Theorem 4.2 implies that $\mu_z^{\Gamma_\infty}$ is invariant under $N(\Gamma_\infty)$. \square

Lemma 5.2 is clearly true when the conjugate $\Gamma = \alpha G \alpha^{-1}$ is coincident with G . In this case, we can regard all Γ_m ($m = 0, 1, \dots, \infty$) as equal to G . Then, including this case, we summarize the result in this section as follows.

Theorem 5.3. *Let $G \subset \text{Conf}(\mathbb{B}^{n+1})$ be a Kleinian group of divergence type. If the conjugate $\Gamma = \alpha G \alpha^{-1}$ is contained in G for some $\alpha \in \text{Conf}(\mathbb{B}^{n+1})$, then the Patterson-Sullivan measure μ_z^G for G is coincident with μ_z^Γ for Γ and μ_z^G is invariant under α .*

Remark that the coincidence of the Patterson-Sullivan measures μ_z^G and μ_z^Γ implies the coincidence of the limit sets $\Lambda(G)$ and $\Lambda(\Gamma)$.

§6. PROOF OF THE MAIN THEOREM

Now we state our main theorem precisely as follows and complete its proof in this section.

Theorem 6.1. *Let $G \subset \text{Conf}(\mathbb{B}^{n+1})$ be a Kleinian group of divergence type. If the conjugate $\Gamma = \alpha G \alpha^{-1}$ is contained in G for some $\alpha \in \text{Conf}(\mathbb{B}^{n+1})$, then $\Gamma = G$.*

Proof. We consider the coset decomposition of G by Γ :

$$G = g_1\Gamma \sqcup g_2\Gamma \sqcup \cdots .$$

According to this decomposition, we also decompose the orbit $G(x)$ of any point $x \in \mathbb{B}^{n+1}$ into

$$G(x) = g_1\Gamma(x) \sqcup g_2\Gamma(x) \sqcup \cdots .$$

For each coset $g_k\Gamma(x)$ ($k = 1, 2, \dots$), we define a measure $\nu_{g_k\Gamma(x), z}^s$ ($s > \delta = \delta(G)$) on $\overline{\mathbb{B}^{n+1}}$ in the same manner as in the construction of the Patterson-Sullivan measure:

$$\nu_{g_k\Gamma(x), z}^s = \frac{1}{P_G^s(x, 0)} \sum_{\gamma \in \Gamma} e^{-sd(g_k\gamma(x), z)} D_{g_k\gamma(x)}.$$

Note that the measure $\mu_{x, z}^s$ for G can be represented by

$$\mu_{x, z}^s = \sum_k \nu_{g_k\Gamma(x), z}^s.$$

In what follows, we claim that, for any x and z in \mathbb{B}^{n+1} and for each $k \in \mathbb{N}$, the measure $\nu_{g_k \Gamma(x), z}^s$ converges weakly to the Patterson-Sullivan measure μ_z^Γ for Γ as $s \rightarrow \delta+$. Then, since $\mu_{x, z}^s$ in the left-hand side of the above equation converges to μ_z^G , we have $\mu_z^G \geq \sum_k \mu_z^\Gamma$. This is possible only for $k = 1$ since $\mu_z^G = \mu_z^\Gamma$ by Theorem 5.3. Thus the proof will complete.

We write

$$\begin{aligned} \nu_{g_k \Gamma(x), z}^s &= \frac{1}{P_G^s(x, 0)} \sum_{\gamma \in \Gamma} e^{-sd(g_k \gamma(x), z)} D_{g_k \gamma(x)} \\ &= \frac{P_\Gamma^s(x, 0)}{P_G^s(x, 0)} \left\{ \frac{1}{P_\Gamma^s(x, 0)} \sum_{\gamma \in \Gamma} e^{-sd(\gamma(x), g_k^{-1}(z))} (g_k^{-1})^* D_{\gamma(x)} \right\}. \end{aligned}$$

Here the measure in curly bracket above converges weakly to $(g_k^{-1})^* \mu_{g_k^{-1}(z)}^\Gamma$. Since the Patterson-Sullivan measure μ_z^Γ is invariant under G by Theorem 5.3, we see that this is equal to μ_z^Γ .

On the other hand,

$$\begin{aligned} \frac{P_\Gamma^s(x, 0)}{P_G^s(x, 0)} &= \frac{P_{\alpha G \alpha^{-1}}^s(x, 0)}{P_G^s(x, 0)} \\ &= \frac{P_G^s(\alpha^{-1}(x), \alpha^{-1}(0))}{P_G^s(x, 0)} \\ &= \frac{P_G^s(x, \alpha^{-1}(0))}{P_G^s(x, 0)} \cdot \frac{P_G^s(0, \alpha^{-1}(0))}{P_G^s(x, \alpha^{-1}(0))} \cdot \frac{P_G^s(\alpha^{-1}(x), \alpha^{-1}(0))}{P_G^s(0, \alpha^{-1}(0))} \\ &= \frac{P_G^s(x, \alpha^{-1}(0))}{P_G^s(x, 0)} \cdot \frac{P_G^s(\alpha^{-1}(0), 0)}{P_G^s(\alpha^{-1}(0), x)} \cdot \frac{P_G^s(\alpha^{-1}(0), \alpha^{-1}(x))}{P_G^s(\alpha^{-1}(0), 0)}, \end{aligned}$$

where we use Proposition 2.1 to obtain the last equality. Here Proposition 3.1 gives

$$\begin{aligned} \lim_{s \rightarrow \delta+} \frac{P_G^s(x, \alpha^{-1}(0))}{P_G^s(x, 0)} &= \sigma(\mu_{\alpha^{-1}(0)}^G); \\ \lim_{s \rightarrow \delta+} \frac{P_G^s(\alpha^{-1}(0), x)}{P_G^s(\alpha^{-1}(0), 0)} &= \sigma(\mu_x^G); \\ \lim_{s \rightarrow \delta+} \frac{P_G^s(\alpha^{-1}(0), \alpha^{-1}(x))}{P_G^s(\alpha^{-1}(0), 0)} &= \sigma(\mu_{\alpha^{-1}(x)}^G). \end{aligned}$$

Since the Patterson-Sullivan measure μ_z^G is invariant under α by Theorem 5.3, we have $\alpha^* \mu_z^G = \mu_{\alpha^{-1}(z)}^G$ for any $z \in \mathbb{B}^{n+1}$. This implies that $\sigma(\mu_{\alpha^{-1}(0)}^G) = \sigma(\mu_0^G) = 1$ and $\sigma(\mu_{\alpha^{-1}(x)}^G) = \sigma(\mu_x^G)$, from which $\lim_{s \rightarrow \delta+} \{P_\Gamma^s(x, 0)/P_G^s(x, 0)\} = 1$ follows. Thus we conclude that $\nu_{g_k \Gamma(x), z}^s$ converges weakly to μ_z^Γ . \square

§7. CERTAIN GENERALIZATION

In this section, we perform certain generalization of our main theorem, where one can loosen the divergence type condition a little. Recall that we have defined the Patterson-Sullivan measure μ_z for a Kleinian group $\Gamma \subset \text{Conf}(\mathbb{B}^{n+1})$ to be a Γ -invariant conformal density that has support on the limit set $\Lambda(\Gamma) \subset S^n$. For a Kleinian group of divergence type, its Patterson-Sullivan measure is known to be unique up to constants. On the other hand, there are Kleinian groups in a certain class that are not of divergence type but their Patterson-Sullivan measures are still unique (see Sullivan [14]). The statement of our main theorem can be extended to such a class of Kleinian groups.

Theorem 7.1. *Let $G \subset \text{Conf}(\mathbb{B}^{n+1})$ be a Kleinian group whose Patterson-Sullivan measure is unique up to constant multiples. If the conjugate $\Gamma = \alpha G \alpha^{-1}$ is contained in G for some $\alpha \in \text{Conf}(\mathbb{B}^{n+1})$ and the limit sets $\Lambda(\Gamma)$ and $\Lambda(G)$ are coincident, then $\Gamma = G$.*

Proof. Theorem 4.2 can be extended to Kleinian groups whose Patterson-Sullivan measures are unique. This is verified in [7]. Hence we have only to consider the extension of the results in Section 5. However, this is easy if we assume $\Lambda(\Gamma) = \Lambda(G)$. In this case, the Kleinian groups Γ_m and Γ_∞ also have the same limit set. Hence the uniqueness ensures that their Patterson-Sullivan measures are all coincident. Then the same arguments as in Section 6 yield the conclusion. \square

REFERENCES

1. M. Culler and P. Shalen, *Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **5** (1992), 231–288.
2. R. Brooks, *The bottom of the spectrum of a Riemannian covering*, J. Reine Angew. Math. **357** (1985), 101–114.
3. E. Fujikawa, K. Matsuzaki and M. Taniguchi, *Dynamics on Teichmüller spaces and self-covering of Riemann surfaces*, Math. Z. (to appear).
4. M. Heins, *On a problem of Heinz Hopf*, J. Math. pures et appl. **37** (1958), 153–160.
5. T. Jørgensen, A. Marden and C. Pommerenke, *Two examples of covering surfaces*, Riemann surfaces and related topics, Ann. Math. Studies, vol. 97, Princeton Univ. Press, 1978, pp. 305–319.
6. K. Matsuzaki, *Dynamics of Kleinian groups — the Hausdorff dimension of limit sets*, Selected papers on classical analysis, AMS Translation Series (2), vol. 204, The American Mathematical Society, 2001, pp. 23–44.
7. K. Matsuzaki and Y. Yabuki, *Invariance of the Nayatani metrics for Kleinian manifolds* (preprint).
8. C. McMullen and D. Sullivan, *Quasiconformal homeomorphisms and dynamics III. The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1998), 351–395.
9. S. Nayatani, *Patterson-Sullivan measure and conformally flat metrics*, Math. Z. **225** (1997), 115–131.
10. P. Nicholls, *The ergodic theory of discrete groups*, LMS Lecture Note Series 143, Cambridge University Press, 1989.
11. K. Ohshika and L. Potyagailo, *Self-embeddings of Kleinian groups*, Ann. Sci. Ecole Norm. Sup. **31** (1998), 329–343.
12. T. Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. 95, Soc. Math. France, 2003.

13. D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 171–202.
14. D. Sullivan, *Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension two*, Geometry, Proc. Symp., Utrecht 1980, Lecture Notes Math., vol. 894, Springer, 1981, pp. 127–144.
15. S. Wang and Q. Zhou, *On the proper conjugation of Kleinian groups*, Geom. Dedicata **56** (1995), 145–154.
16. X. Wang and W. Yang, *Discreteness criteria of Möbius groups of high dimensions and convergence theorems of Kleinian groups*, Adv. Math. **159** (2001), 68–82.

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