POLYCYCLIC QUASICONFORMAL MAPPING CLASS SUBGROUPS

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Abstract. For a subgroup of the quasiconformal mapping class group of a Riemann surface in general, we give an algebraic condition which guarantees its discreteness in the compact-open topology. Then we apply this result to its action on the Teichmüller space.

1. Introduction

We consider a Riemann surface R in general, not necessarily topologically finite, and a subgroup G consisting of quasiconformal mapping classes of R. Such a group usually appears as acting on the infinite dimensional Teichmüller space of R and in particular discreteness of its orbit is often discussed. In this case, the discreteness of G is understood through the action on the Teichmüller space. In this paper however, we first start from a more basic viewpoint on G as surface homeomorphisms and then look into its action on the Teichmüller space.

Throughout this introduction, we assume that a Riemann surface R has no ideal boundary at infinity ∂R for the sake of simplicity. The quasiconformal mapping class group $\mathrm{MCG}(R)$ of R is the group of all quasiconformal automorphisms g of R modulo homotopy equivalence. We introduce a topology for this group induced by the compact-open topology of homeomorphisms of R. Then a subgroup G of $\mathrm{MCG}(R)$ is defined to be *discrete* if it is discrete in this topology. Our main theorem refers to a certain algebraic condition under which G is always discrete. Here we say that a group G is *polycyclic* if G is solvable and if every subgroup of G is finitely generated.

Theorem 2.4. *If a subgroup G of $\mathrm{MCG}(R)$ is polycyclic, then G is discrete.*

This result is sharp in a sense that there is a counter-example for either a finitely generated solvable group or an infinitely generated abelian group.

In the first part of the application of this theorem, we deal with stationary mapping class subgroups and consider their action on Teichmüller spaces. The

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quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on the Teichmüller space $T(R)$ of a Riemann surface R biholomorphically and isometrically. A subgroup $G \subset \mathrm{MCG}(R)$ is called *stationary* if there exists a compact subsurface V of R such that every representative g of every mapping class $[g] \in G$ satisfies $g(V) \cap V \neq \emptyset$.

A basic nature of stationary subgroups in connection with their discreteness in the compact open topology and discontinuity of the action on the Teichmüller space is that, if $G \subset \mathrm{MCG}(R)$ is stationary and discrete, then G acts discontinuously on $T(R)$. Then we have the following consequence from the main theorem. Recall that we assume $\partial R = \emptyset$ until the end of this section.

Corollary 4.2. *If a polycyclic subgroup G of* $\mathrm{MCG}(R)$ *is stationary, then G acts discontinuously on $T(R)$.*

We expect that this result should be valid for every finitely generated stationary subgroup $G \subset \mathrm{MCG}(R)$.

In the second part, we apply our main theorem to asymptotically conformal mapping class subgroups. We say that a quasiconformal homeomorphism of a Riemann surface R is asymptotically conformal if its complex dilatation vanishes at infinity of R. We say that a subgroup $G \subset \mathrm{MCG}(R)$ is *asymptotically conformal* if there exists some $p \in T(R)$ such that every element of G can be realized as an asymptotically conformal automorphism of the Riemann surface R_p corresponding to p. We denote by $\mathrm{MCG}_p(R)$ the subgroup of $\mathrm{MCG}(R)$ having this property for $p \in T(R)$.

Theorem 5.1. *If an asymptotically conformal subgroup G of* $\mathrm{MCG}_p(R)$ *for $p \in T(R)$ is polycyclic, then the orbit $G(p)$ is a discrete set in $T(R)$.*

One may ask a question about how the algebraic assumption on G can be relaxed for this statement.

2. Discreteness of mapping class subgroups

We always assume that a Riemann surface R is hyperbolic, that is, R is represented by a Fuchsian group F acting on the unit disk $\mathbb{D}$ and is endowed with the hyperbolic metric. The quasiconformal mapping class group $\mathrm{MCG}(R)$ for R is the group of all homotopy classes $[g]$ of quasiconformal automorphisms g of R. Here the homotopy is considered to be relative to *the ideal boundary at infinity* ∂R of R, where $\partial R = (\partial \mathbb{D} - \Lambda(F))/F$ for the limit set $\Lambda(F)$ of F. This means that, when $\partial R \neq \emptyset$, two quasiconformal automorphisms g_0 and g_1 are regarded as homotopic if there is a homotopy $\Phi : R \times [0,1] \to R$ between $g_0 = \Phi(\cdot, 0)$ and $g_1 = \Phi(\cdot, 1)$ such that its extension to each $x \in \partial R$ is constant over $[0,1]$.

The compact-open topology on the space of all homeomorphic automorphisms of R induces a topology on $\mathrm{MCG}(R)$. More precisely, we say that a sequence of mapping classes $[g_n] \in \mathrm{MCG}(R)$ converges to a mapping class $[g] \in \mathrm{MCG}(R)$ in the compact-open topology if we can choose representatives $g_n \in [g_n]$ and $g \in [g]$ satisfying that g_n converge to g locally uniformly on R. When R has the ideal boundary at infinity ∂R, we further require that the extensions $\bar{g}_n$ of the quasiconformal automorphisms g_n to ∂R converge to the extension $\bar{g}$ of g in such a way that $\bar{g}_n$ is identical with $\bar{g}$ on a compact subset $W_n \subset \partial R$, where $\{W_n\}_{n=1}^{\infty}$ is some *compact exhaustion* of ∂R, that is, an increasing sequence of compact subsets of ∂R satisfying that the closure of the union of all W_n is ∂R. We call this topology on $\mathrm{MCG}(R)$ *compact-open topology relative to the boundary.* If $[g_n]$ converge to $[g]$ in the compact-open topology relative to the boundary, then there are quasisymmetric automorphisms $\tilde{g}_n$ and $\tilde{g}$ of the unit circle $\partial\mathbb{D}$ corresponding to $[g_n]$ and $[g]$ respectively such that $\tilde{g}_n$ converge uniformly to $\tilde{g}$.

Definition. We say that a subgroup G of $\mathrm{MCG}(R)$ is *discrete* if it is a discrete set in $\mathrm{MCG}(R)$ with respect to the compact-open topology relative to the boundary. The discreteness is equivalent to the condition that, if a sequence of mapping classes $\{[g_n]\}_{n=1}^{\infty} \subset \mathrm{MCG}(R)$ converges to $[\mathrm{id}]$, then $[g_n] = [\mathrm{id}]$ for all sufficiently large n.

Concerning the discreteness of the full mapping class group $\mathrm{MCG}(R)$, we have a simple characterization.

Proposition 2.1. *The quasiconformal mapping class group* $\mathrm{MCG}(R)$ *is discrete if and only if* R *is analytically finite, that is,* R *is a compact Riemann surface from which at most finitely many points are removed.*

Proof. Assume that R is analytically finite. In this case, there are a finite number of simple closed geodesics $\{c_i\}_{i=1}^{k}$ such that, if $[g] \in \mathrm{MCG}(R)$ satisfies that $g(c_i)$ is freely homotopic to c_i for every i, then $[g] = [\mathrm{id}]$. If a sequence of mapping classes $\{[g_n]\}_{n=1}^{\infty}$ converges to $[\mathrm{id}]$, then $g_n(c_i)$ is freely homotopic to c_i for every i and for all sufficiently large n. This implies that $\mathrm{MCG}(R)$ is discrete.

Conversely, assume that R is not analytically finite. If R is topologically finite, that is, the fundamental group $\pi_1(R)$ of R is finitely generated, then R should have the ideal boundary at infinity and clearly $\mathrm{MCG}(R)$ is not discrete in this case. If R is not topologically finite, then there is an infinite sequence of simple closed geodesics $\{c_n\}_{n=1}^{\infty}$ diverging to the infinity of R, in other words, escaping from any compact subset of R. Let $[\tau_n]$ be the mapping class caused by the Dehn twist along c_n. Then $[\tau_n] \neq [\mathrm{id}]$ and $\{[\tau_n]\}_{n=1}^{\infty}$ converges to $[\mathrm{id}]$. This implies that $\mathrm{MCG}(R)$ is not discrete. □

We will consider the discreteness of countable subgroups of $\mathrm{MCG}(R)$. Note that $\mathrm{MCG}(R)$ is uncountable in many cases when R is analytically infinite. See

[8]. An uncountable subgroup G of $\mathrm{MCG}(R)$ is not discrete as the following proposition asserts.

Proposition 2.2. *Assume that R has no ideal boundary at infinity ∂R. If a subgroup $G \subset \mathrm{MCG}(R)$ is uncountable, then G is not discrete.*

Proof. Let $\{c_i\}_{i=1}^{\infty}$ be the family of (free homotopy classes of) all simple closed geodesics on R. We first consider the images of c_1 under G. Since G is uncountable whereas $\{c_i\}$ is countable, there are uncountably many elements of G that map c_1 to simple closed curves freely homotopic to each other. Then, by composing the inverse of one of these elements, we have uncountably many elements of G that keep c_1 in its free homotopy class. Next we consider the images of c_2 under this uncountable subset of G and obtain an uncountably many elements of G that keep c_1 and c_2 in their free homotopy classes. By continuing this process and then by taking the diagonal, we can choose a sequence $\{[g_n]\}_{n=1}^{\infty}$ of elements in G such that $g_n(c_i)$ is freely homotopic to c_i for all $i = 1, 2, \ldots, n$ and for each n. This implies that $\{[g_n]\}$ converges to [id]. □

In this section, we investigate an algebraic condition on a countable subgroup G of $\mathrm{MCG}(R)$ under which G is always discrete. Our fundamental result is the following. The proof will be given in the next section.

Theorem 2.3. *If $G \subset \mathrm{MCG}(R)$ is a finitely generated abelian group, then G is discrete.*

Note that both assumptions that G is finitely generated and that G is abelian are necessary for the above theorem as examples below show. However, we cannot have the converse statement to the theorem. In fact, for any countable group G, there exists a discrete subgroup of $\mathrm{MCG}(R)$ for some Riemann surface R that is isomorphic to G. Indeed, we can construct R so that its conformal automorphism group, which is always discrete unless $\pi_1(R)$ is abelian, contains such a subgroup.

Example. (1) First we give an indiscrete $G \subset \mathrm{MCG}(R)$ that is abelian but not finitely generated. Let R be a Riemann surface with an infinite family of mutually disjoint simple closed geodesics $\{c_n\}_{n=1}^{\infty}$ and G a subgroup of $\mathrm{MCG}(R)$ generated by all the mapping classes $[\tau_n]$ caused by the Dehn twist along c_n for each integer $n \geq 1$. Since $[\tau_n]$ converge to [id], G is not discrete though G is abelian.

(2) Next we give an indiscrete $G \subset \mathrm{MCG}(R)$ that is finitely generated but not abelian. Assume that there are a simple closed geodesic c_0 on R and a mapping class $[g] \in \mathrm{MCG}(R)$ such that the images $\{g^n(c_0)\}_{n \in \mathbb{Z}}$ of c_0 under the iteration of a representative $g \in [g]$ are mutually disjoint. Define c_n to be the simple closed geodesic freely homotopic to $g^n(c_0)$ and $[\tau_n]$ to be the mapping classes caused by the Dehn twist along c_n. Let G be a subgroup of $\mathrm{MCG}(R)$

generated by two elements $[g]$ and $[\tau_0]$. Since $[g]^n[\tau_0] = [\tau_n][g]^n$ for every integer $n \in \mathbb{Z}$, we see that G contains the subgroup G' generated by all such $[\tau_n]$. Hence G is not discrete as Example (1) shows.

In the second example above, the group G is solvable since the commutator subgroup $[G, G]$ is contained in the abelian subgroup G'. Although G itself is finitely generated, G' is not, which makes G not to be discrete. Hence we consider the following stronger condition than solvability which requires all its subgroups to be finitely generated.

Definition. We say that a group G is *polycyclic* if G is solvable and if every subgroup of G is finitely generated.

See [12] for other equivalent conditions for G to be polycyclic. This name comes from the fact that G is polycyclic if and only if G has a finite normal chain of subgroups $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_m = \{1\}$ such that each quotient group G_{i-1}/G_i $(i = 1, \ldots, m)$ is cyclic. We can say that G is polycyclic when G is obtained in finitely many simple steps from finitely generated abelian groups.

Theorem 2.4. *If $G \subset \mathrm{MCG}(R)$ is a polycyclic group, then G is discrete.*

This extension of Theorem 2.3 is obtained by an inductive argument which is easily seen from the following assertion.

Lemma 2.5. *Assume that every subgroup of $G \subset \mathrm{MCG}(R)$ is finitely generated. If G is not discrete, then neither is the commutator subgroup $[G, G]$.*

Proof. Since G is not discrete, there is a sequence $\{[g_n]\}_{n=1}^{\infty}$ in G that converges to $[\mathrm{id}]$ as $n \to \infty$. Then we see that, for every $n_0 \geq 1$, there exist $m, n \geq n_0$ such that $[g_m]$ and $[g_n]$ are not commuting. Indeed, if not, there is n_0 such that $[g_m]$ and $[g_n]$ are commuting for any $m, n \geq n_0$. Then a subgroup G' of G generated by $\{[g_n]\}_{n \geq n_0}$ is abelian and G' is not discrete. By assumption, G' is finitely generated. However, this contradicts Theorem 2.3.

Fix some $n_0 \geq 1$. We choose $m_1, n_1 \geq n_0$ such that $[h_1] := [[g_{m_1}], [g_{n_1}]]$ is not the identity $[\mathrm{id}]$. Then we choose $m_2, n_2 \geq \max\{m_1, n_1\}$ such that $[h_2] := [[g_{m_2}], [g_{n_2}]]$ is not the identity. Inductively, for each $i \geq 1$, we choose $m_i, n_i \geq \max\{m_{i-1}, n_{i-1}\}$ such that $[h_i] := [[g_{m_i}], [g_{n_i}]]$ is not the identity. Then every $[h_i]$ belongs to the commutator subgroup $[G, G]$ of G and $[h_i]$ converge to $[\mathrm{id}]$ as $i \to \infty$. This implies that $[G, G]$ is not discrete. □

3. Restraint of mapping class groups

In this section, we will prove Theorem 2.3. The proof uses a certain property of mapping class groups, not necessarily satisfied for abstract groups in general. We first explain this situation by the following example.

Example. Let $\mathfrak{S}_\infty$ be the infinite symmetric group acting on a countable set $X = \{1, 2, \ldots\}$ as permutation. We consider an element $g = (1)(23)(456)\cdots$ of $\mathfrak{S}_\infty$ which gives a cyclic permutation on mutually disjoint subsets of n points in X where n runs over all positive integers. Then we see that $g^{n!}$ converge to id in the compact open topology with respect to the discrete topology on X. In particular, the cyclic subgroup $\langle g \rangle$ is not discrete.

Let $X = \{c_i\}_{i=1}^\infty$ be the family of (free homotopy classes of) all simple closed geodesics on a Riemann surface R. The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts faithfully on the countable set X by the correspondence of the free homotopy class $g(c)$ to $[g] \cdot c$ for any $[g] \in \mathrm{MCG}(R)$ and for any $c \in X$. In this way, we can represent $\mathrm{MCG}(R)$ as a subgroup of $\mathfrak{S}_\infty$. As the above example shows, an arbitrary subgroup of $\mathfrak{S}_\infty$ cannot have the required property which we want to prove in Theorem 2.3. The nature that $\mathrm{MCG}(R) \subset \mathfrak{S}_\infty$ is originated from R gives certain restriction to the action of $\mathrm{MCG}(R)$ and we must use such restraint in order to prove our theorem. The following lemma can be regarded as one of such properties of $\mathrm{MCG}(R)$.

Lemma 3.1. *For every element $[g] \in \mathrm{MCG}(R)$ of infinite order, there exists either a compact subsurface V in R or a compact subset V' in an arbitrarily given compact exhaustion of the ideal boundary at infinity ∂R such that either the restriction $g^n|_V$ is homotopic to $\mathrm{id}|_V$ on R or the extension $\bar{g}^n$ is the identity on V' for no positive integer $n \in \mathbb{N}$.*

Proof. Suppose to the contrary that there is no such compact subsurface V in R nor compact subset V' in the compact exhaustion of ∂R. Then, for any compact subsurface $V_1 \subset R$, there is $n_1 \in \mathbb{N}$ such that $g^{n_1}|_{V_1}$ is homotopic to $\mathrm{id}|_{V_1}$ on R. Also, for any compact subset V_1' in the compact exhaustion of ∂R, there is $n_1' \in \mathbb{N}$ such that $\bar{g}^{n_1'}$ is the identity on V'. Set $h = g^{n_1 n_1'}$. Since h is not homotopic to the identity on R relative to ∂R, there is either some compact subsurface $V_2 \subset R$ including V_1 such that $h|_{V_2}$ is not homotopic to $\mathrm{id}|_{V_2}$ on R or some compact subset V_2' in the compact exhaustion of ∂R including V_1' such that $\bar{h}$ is not the identity on V_2'. We assume that the first case occurs. The argument for the second case is similar.

For that compact subsurface V_2, there is $n_2 \in \mathbb{N}$ such that $g^{n_2}|_{V_2}$ is homotopic to $\mathrm{id}|_{V_2}$ on R. We may assume that n_2 is a proper multiple of $n_1 n_1'$, that is, $n_2 = k n_1 n_1'$ for some integer $k > 1$. Then $h|_{V_1} \sim \mathrm{id}|_{V_1}$, $h|_{V_2} \not\sim \mathrm{id}|_{V_2}$ and $h^k|_{V_2} \sim \mathrm{id}|_{V_2}$, where $\sim$ means that they are homotopic to each other on R. However, this is impossible, as we see in the following. Represent the Riemann surface R by a Fuchsian group F acting on the unit disk $\mathbb{D}$ and take a subgroup F_1 of F corresponding to the subsurface V_1. We choose a quasisymmetric automorphism $\tilde{h}$ of $\partial\mathbb{D}$ corresponding to h so that $\tilde{h}$ is the identity on the limit set $\Lambda(F_1) \subset \partial\mathbb{D}$ of F_1. We also take a subgroup F_2 of F corresponding to the subsurface V_2 which contains F_1. Then the quasisymmetric automorphism $\tilde{h}$ is not the identity on the limit set $\Lambda(F_2)$ containing $\Lambda(F_1)$. This implies that there is a point $x \in \Lambda(F_2) - \Lambda(F_1)$ that is moved by $\tilde{h}$. Since the movement of x is towards one direction in some interval of $\partial\mathbb{D} - \Lambda(F_1)$, it cannot return to the original place under the iteration of $\tilde{h}$. Thus $\tilde{h}^k(x) \neq x$, which violates the condition that $h^k|_{V_2} \sim \mathrm{id}|_{V_2}$. □

Although the following fact is not special for mapping class groups, the property of discreteness is shared with a subgroup of finite index as in usual arguments. We also use this fact in the proof of Theorem 2.3.

Proposition 3.2. *Let G' be a subgroup of $G \subset \mathrm{MCG}(R)$ of finite index. If G' is discrete, then so is G.*

Proof. If G is not discrete, there is a sequence of distinct elements $[g_n]$ of G that converges to [id]. Since the index of G' in G is finite, we may assume that $[g_n]$ are all in the same coset, say, $G'[h]$ for some $[h] \in G$. Then $[g'_n] = [g_n] \cdot [h]^{-1}$ belong to G' and converge to $[h]^{-1}$. This contradicts the assumption that G' is discrete. □

Now we are ready to prove our fundamental result.

Proof of Theorem 2.3. By Proposition 3.2, we may assume that G is isomorphic to a free abelian group $\mathbb{Z}^m$ of rank $m \geq 1$. We will prove the statement of the theorem by induction with respect to m. First, we show that the statement is valid when $m = 1$. Assume that $G \cong \mathbb{Z}$ is not discrete, that is, there is a sequence of elements in G converging to [id]. When R has the ideal boundary at infinity ∂R, some compact exhaustion of ∂R is associated to this converging sequence. For a generator $[g] \in \mathrm{MCG}(R)$ of G, Lemma 3.1 gives either a compact subsurface V of R or a compact subset V' in the exhaustion of ∂R as in its statement. However, since G is not discrete, there is some $n \in \mathbb{N}$ such that $g^n|_V$ is homotopic to $\mathrm{id}|_V$ on R and the extension $\bar{g}^n$ of g^n to ∂R is the identity on V'. This contradicts the choice of V and V'.

We assume that the statement is true for any subgroup of $\mathrm{MCG}(R)$ isomorphic to $\mathbb{Z}^j$ for every integer j with $1 \leq j \leq m-1$. Let G be a subgroup of $\mathrm{MCG}(R)$ isomorphic to $\mathbb{Z}^m$ and prove that G is discrete. Suppose to the contrary that G is not discrete. Then we have a sequence $[g_n] \in G$ converging to [id] as well as a compact exhaustion of ∂R associated with this sequence. We will choose a subsequence of $[g_n]$ so that any m elements in the subsequence generates a subgroup isomorphic to $\mathbb{Z}^m$. To this end, first we notice that all the elements $[g_n]$ in the convergent sequence cannot be contained in a finite union of subgroups of G that are isomorphic to $\mathbb{Z}^j$ with $1 \leq j \leq m-1$. This is because of the assumption of the induction. Then we choose a subsequence $[g_{n(i)}]$ in the following way. The first $m-1$ entries $[g_{n(1)}], \ldots, [g_{n(m-1)}]$ are chosen so that they are linearly independent over $\mathbb{Z}$. Suppose that we have already chosen l entries $G_l = \{[g_{n(1)}], \ldots, [g_{n(l)}]\}$ for $l \geq m-1$. Then we take the $(l+1)$-st entry $[g_{n(l+1)}]$ so that any $m-1$ elements of G_l together with $[g_{n(l+1)}]$ are linearly independent over $\mathbb{Z}$, in other words, $[g_{n(l+1)}]$ belongs to no maximal proper subgroup ($\cong \mathbb{Z}^{m-1}$) of G containing $m-1$ elements of G_l. The reason why we can choose such $[g_{n(l+1)}]$ is that, if not, all $[g_n]$ must be contained in the union of the finite number of subgroups of G determined by any $m-1$ elements of G_l. By this construction, it is clear that any m elements in the subsequence $[g_{n(i)}]$ generate a subgroup isomorphic to $\mathbb{Z}^m$.

We fix an arbitrary non-trivial element $[g_0] \in G$. By Lemma 3.1, we take either a compact subsurface V of R such that $g_0^n|_V \not\sim \mathrm{id}|_V$ or a compact subset V' in the exhaustion of ∂R such that $\bar{g}_0^n|_{V'} \neq \mathrm{id}|_{V'}$ for all $n \in \mathbb{N}$. We only consider the first case. The second case is similar. Since we are assuming that $[g_{n(i)}]$ converge to [id], there is some i_0 such that $g_{n(i)}|_V \sim \mathrm{id}|_V$ for every $i \geq i_0$. We take arbitrary m elements $[g_{n(i)}]$ with $i \geq i_0$ and rename them as $[g_i]$ $(i = 1, \ldots, m)$. Since they generate a subgroup of G isomorphic to $\mathbb{Z}^m$, a linear combination of $[g_i]$ $(i = 1, \ldots, m)$ over $\mathbb{Z}$ yields some multiple of any element of G. This implies that $[g_0]^n$ for some $n \in \mathbb{N}$ is represented by $[g_1]^{k_1} \cdots [g_m]^{k_m}$ for some $k_i \in \mathbb{Z}$. However, this forces $g_0^n|_V \sim \mathrm{id}|_V$, which contradicts the choice of V. □

4. Discontinuity of the action on the Teichmüller space

We apply our theorem to the action of mapping class subgroups on Teichmüller spaces. For a Riemann surface R, the Teichmüller space $T(R)$ is defined to be the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are equivalent if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R. Here the homotopy is considered to be relative to the ideal boundary at infinity ∂R. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$ which is called the Teichmüller distance. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R', where R' is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure. Consult [6], [7] and [11] for the theory of Teichmüller spaces.

Each element $[g] \in \mathrm{MCG}(R)$ acts on $T(R)$ from the left in such a way that $[g] \cdot [f] = [f \circ g^{-1}]$ for $[f] \in T(R)$. It is evident from the definition that $\mathrm{MCG}(R)$ acts on $T(R)$ isometrically with respect to the Teichmüller distance. It also acts biholomorphically on $T(R)$. Except for few cases where the dimension of

$T(R)$ is lower, the action of $\mathrm{MCG}(R)$ on $T(R)$ is faithful. Then $\mathrm{MCG}(R)$ can be represented in the group of all isometric biholomorphic automorphisms of $T(R)$.

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at $p = [f] \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that the number of the elements $g \in G$ satisfying $g(U) \cap U \neq \emptyset$ is finite. We denote the orbit of p under G by $G(p)$ and the stabilizer subgroup of G at p by $\mathrm{Stab}_G(p)$. Then G acts discontinuously at p if and only if $G(p)$ is a discrete set and $\mathrm{Stab}_G(p)$ is a finite group. If G acts discontinuously at every point p in $T(R)$, then we say that G acts discontinuously on $T(R)$. When R is analytically finite, $\mathrm{MCG}(R)$ itself acts discontinuously on $T(R)$. However, for a Riemann surface in general, this is not always true. See [4] for the discontinuity of the action of mapping class groups on Teichmüller spaces.

We consider mapping class subgroups by imposing a stationary property on them in the following sense.

Definition. We call a subgroup G of $\mathrm{MCG}(R)$ *stationary* if there exists a compact subsurface V of R such that every representative g of every mapping class $[g] \in G$ satisfies $g(V) \cap V \neq \emptyset$.

The stationary property puts certain normalization on a family of quasiconformal automorphisms of R. Under this condition, the discreteness of G in the compact open topology affects a behavior of its orbit on the Teichmüller space.

Lemma 4.1. *Let G be a stationary subgroup of $\mathrm{MCG}(R)$ for a Riemann surface R with $\partial R = \emptyset$. If G is discrete then the orbit $G(p)$ for any $p \in T(R)$ diverges to the infinity of $T(R)$, and in particular, G acts discontinuously on $T(R)$.*

Proof. Compactness of a family of normalized quasiconformal homeomorphisms with uniformly bounded dilatations yields that, if there is a sequence $[g_n]$ in a stationary subgroup G of $\mathrm{MCG}(R)$ such that $[g_n](p)$ is bounded in $T(R)$, then a subsequence of some representatives $g_n \in [g_n]$ converges to some quasiconformal automorphism of R locally uniformly. However, if G is discrete in the compact-open topology, then there is no such sequence. This implies that $[g_n](p)$ is bounded in $T(R)$ for no sequence $[g_n] \in G$, that is, the orbit $G(p)$ diverges to the infinity of $T(R)$. □

The combination of Theorem 2.4 and Lemma 4.1 immediately yields the following.

Corollary 4.2. *Let G be a stationary subgroup of $\mathrm{MCG}(R)$ for a Riemann surface R with $\partial R = \emptyset$. If G is polycyclic, then G acts discontinuously on $T(R)$.*

We expect that this corollary is valid for every finitely generated stationary subgroup G of $\mathrm{MCG}(R)$. This will be a consequence of the following conjecture for the discreteness of a stationary subgroup of the mapping class group.

Conjecture. If a finitely generated subgroup $G \subset \mathrm{MCG}(R)$ is stationary, then G is discrete.

If R is analytically finite, then $\mathrm{MCG}(R)$ is finitely generated and stationary. In this case, $\mathrm{MCG}(R)$ is discrete and acts on $T(R)$ discontinuously. The above conjecture can be regarded as a generalization of this property for mapping class groups of analytically finite Riemann surfaces.

Note that there is an example of an infinitely generated (countable) stationary subgroup G such that G does not act discontinuously on $T(R)$. This is obtained similarly to Example (1) in Section 2 but we must further assume that the lengths of the simple closed geodesic c_n there tend to zero as $n \to \infty$.

Remark. If we assume a bounded geometry condition concerning the hyperbolic metric on R, then we do not have to impose any algebraic condition on a stationary subgroup G for the discontinuity of its action on $T(R)$. This result has been proved in [4] and [5]. See also these papers for the definition of the bounded geometry condition.

5. Discreteness of the orbit on a fiver over the asymptotic Teichmüller space

In this section, we impose a certain analytic condition on a subgroup of the quasiconformal mapping class group and show the discreteness of its orbit in the Teichmüller space. Our condition also generalizes certain properties of the mapping class group of an analytically finite Riemann surface.

A quasiconformal homeomorphism f of a Riemann surface R is called *asymptotically conformal* if, for every $\varepsilon > 0$, there exists a compact subsurface V of R such that the maximal dilatation of f restricted to $R - V$ is less than $1 + \varepsilon$. The asymptotic Teichmüller space $AT(R)$ of R is defined by replacing the words "conformal automorphisms" with "asymptotically conformal automorphisms" in the definition of the Teichmüller space $T(R)$. Since a conformal automorphism is asymptotically conformal, there is a projection $\alpha : T(R) \to AT(R)$. We denote the fiber of α containing $p \in T(R)$ by T_p, that is, $T_p = \alpha^{-1}(\alpha(p))$. Consult [1], [2], [3] and [6] for the theory of asymptotic Teichmüller spaces.

The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $T(R)$ preserving the fiber structure of α. Hence it acts on $AT(R)$. We define $\mathrm{MCG}_p(R)$ to be the subgroup of $\mathrm{MCG}(R)$ consisting of all elements keeping the fiber T_p invariant. Every element of $\mathrm{MCG}_p(R)$ can be realized as an asymptotically conformal automorphism of the Riemann surface R_p corresponding to p. We say that a subgroup G of $\mathrm{MCG}(R)$ is asymptotically conformal if G is a subgroup of

$\mathrm{MCG}_p(R)$ for some $p \in T(R)$. When R is analytically finite, $AT(R)$ consists of a single point and $\mathrm{MCG}_p(R)$ coincides with the full $\mathrm{MCG}(R)$ for every $p \in T(R)$.

We will show the following theorem concerning the discreteness of the orbit of an asymptotically conformal subgroup.

Theorem 5.1. *For a Riemann surface R with $\partial R = \emptyset$, if an asymptotically conformal subgroup G of $\mathrm{MCG}_p(R)$ is polycyclic, then the orbit $G(p)$ is a discrete set in $T(R)$.*

We first prove this theorem in the case that G is a finitely generated abelian group. Before the proof, we give the definition of an escaping sequence of mapping classes. A sequence $\{[g_n]\}_{n=1}^{\infty}$ of mapping classes in $\mathrm{MCG}(R)$ is stationary if there exists a compact subsurface V of R such that every representative g_n of each mapping class $[g_n]$ satisfies $g_n(V) \cap V \neq \emptyset$. If a subgroup G of $\mathrm{MCG}(R)$ is stationary in the previous sense, then every sequence in G is stationary in this sense. On the contrary, a sequence $\{[g_n]\}_{n=1}^{\infty}$ is called *escaping* if, for every compact subsurface V of R, there exists some representative g_n of each mapping class $[g_n]$ such that $\{g_n(V)\}$ diverges to the infinity of R (that is, escapes from every compact subset of R) as $n \to \infty$. Remark that a sequence $\{[g_n]\} \subset \mathrm{MCG}(R)$ can be neither stationary nor escaping, but we can always choose a subsequence either stationary or escaping.

The following lemma is crucial for considering an escaping sequence in an asymptotically conformal mapping class group. The proof has been given in [9] and [10, Theorem 5.6].

Lemma 5.2. *Assume that the fundamental group $\pi_1(R)$ of R is non-cyclic. Let G be an abelian subgroup of $\mathrm{MCG}_p(R)$ having an escaping sequence $[g_n]$ such that $[g_n](p) \to p$ as $n \to \infty$. Then $[g](p) = p$ for every $[g] \in G$.*

Then the following inductive step gives the full statement of Theorem 5.1 as we have done in Section 2.

Lemma 5.3. *Assume that $\partial R = \emptyset$ and every subgroup of $G \subset \mathrm{MCG}_p(R)$ is finitely generated. If the orbit $G(p)$ is not a discrete set, then neither is the orbit $G_1(p)$ of the commutator subgroup $G_1 = [G, G]$.*

Proof of Theorem 5.1. Let G be a finitely generated abelian subgroup of $\mathrm{MCG}_p(R)$. If G is stationary, then Corollary 4.2 gives that G acts discontinuously on $T(R)$, and in particular, the orbit $G(p)$ is a discrete set in $T(R)$. If G contains an escaping sequence, then Lemma 5.2 implies that $G(p) = \{p\}$ is a discrete set. Hence, if G is a finitely generated abelian subgroup, then the statement of the theorem is valid. For the general case that G is polycyclic, we apply Lemma 5.3 to obtain the statement. □

Proof of Lemma 5.3. If $G(p)$ is not a discrete set, then we find a sequence $\{[g_n]\}_{n=1}^{\infty} \subset G$ such that $[g_n](p) \neq p$ converge to p as $n \to \infty$. Then we can apply the same arguments as in the proof of Lemma 2.5. Namely, for every $n_0 \geq 1$, there exist $m, n \geq n_0$ such that $[g_m]$ and $[g_n]$ are not commuting. Indeed, if not, there is n_0 such that $[g_m]$ and $[g_n]$ are commuting for any $m, n \geq n_0$. Then the finitely generated subgroup G' of G generated by $\{[g_n]\}_{n \geq n_0}$ is abelian and $G'(p)$ is not a discrete set. However, this contradicts Theorem 5.1 in the finitely generated abelian case. Note that this case has been proved without Lemma 5.3.

Fix some $n_0 \geq 1$. We choose $m_1, n_1 \geq n_0$ such that $[h_1] := [g_{m_1}, g_{n_1}] \neq [\mathrm{id}]$. Then we choose $m_2, n_2 \geq \max\{m_1, n_1\}$ such that $[h_2] := [g_{m_2}, g_{n_2}] \neq [\mathrm{id}]$. Inductively, for each $i \geq 1$, we choose $m_i, n_i \geq \max\{m_{i-1}, n_{i-1}\}$ such that $[h_i] := [g_{m_i}, g_{n_i}] \neq [\mathrm{id}]$. Then every $[h_i]$ belongs to the commutator subgroup $[G, G]$ of G. Note that all $[h_i]$ are not necessarily distinct. We see that $[h_i](p) \to p$ as $i \to \infty$. Indeed,

$$d([h_i](p), p) \leq 2d([g_{m_i}](p), p) + 2d([g_{n_i}](p), p) \to 0$$

as $i \to \infty$. If $[h_i](p) \neq p$ for infinitely many i, then we have done by passing to a subsequence. Hence we have only to consider the case that all but finitely many $[h_i] \neq [\mathrm{id}]$ belong to the stabilizer subgroup $H = \mathrm{Stab}_G(p)$ of G for p, and in particular the case that H is not trivial.

We may assume that p is the base point of the Teichmüller space $T(R)$. Then there is a conformal automorphism group of R identified with H. Let $\mathrm{Fix}(H)$ be the fixed point locus of H in $T(R)$, which can be identified with the Teichmüller space $T(R/H)$ of the orbifold R/H. If $[g_n](p)$ does not lie in $\mathrm{Fix}(H)$, then there is some $[e_n] \in H$ such that $[e_n][g_n](p) \neq [g_n](p)$. We set $[h_n] = [e_n]^{-1}[g_n]^{-1}[e_n][g_n]$ for such n, which belongs to $[G, G]$ and satisfies $[h_n](p) \neq p$. If there are infinitely many such n, we have $[h_n](p) \to p$, which is the desired consequence. Hence we have only to consider the case that $[g_n](p)$ lie in $\mathrm{Fix}(H)$ for all but finitely many n.

The condition $[g_n](p) \in \mathrm{Fix}(H)$ is equivalent to that $[g_n]^{-1}[e][g_n] \in H$ for every $[e] \in H$. This is satisfied if and only if the mapping class $[g_n] \in \mathrm{MCG}(R)$ descends to a mapping class $[\hat{g}_n]$ of R/H. We consider the subgroup of the mapping class group $\mathrm{MCG}(R/H)$ generated by all $\{[\hat{g}_n]\}_{n=1}^{\infty}$. Here $[\hat{g}_n]$ belongs to $\mathrm{MCG}_p(R/H)$ for $p \in T(R/H) = \mathrm{Fix}(H) \subset T(R)$. In the case where H is a finite group, this is easily seen. In the case where H is an infinite group, this is possible only when $[g_n]$ belongs to H. Indeed, this follows from the fact that $T_p \cap \mathrm{Fix}(H) = \{p\}$ for the infinite group H [10, Theorem 4.2]. However, since we are dealing with the elements $[g_n] \in G$ satisfying $[g_n](p) \neq p$, this is not in our case. Hence, by the same reason as before, we can choose a sequence $\{[h_i]\}$ in $[G, G]$ such that $[h_i](p) \to p$ as $i \to \infty$ and in addition that none of

$[h_i]$ belongs to $H = \mathrm{Stab}_G(p)$. This implies that $[h_i](p) \neq p$ converge to p as $i \to \infty$ and thus completes the proof. □

In the remark of the previous section, we have mentioned that, when R satisfies the bounded geometry condition, we do not have to impose any algebraic condition on G. Especially, G is not necessarily finitely generated. The corresponding statement for the discreteness of the orbit of an asymptotically conformal mapping class subgroup will be the following.

Proposition 5.4. *Assume that a Riemann surface R satisfies the bounded geometry condition. If a subgroup G of $\mathrm{MCG}_p(R)$ is solvable, then the orbit $G(p)$ is a discrete set in $T(R)$.*

However, if $G \subset \mathrm{MCG}_p(R)$ is an infinitely generated (countable) group for instance, then the orbit is not necessarily a discrete set. Our problem asks for some algebraic conditions upon G that guarantee this discreteness.

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