The Petersson series for short geodesics

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Abstract. We consider the relative Poincaré series of a certain automorphic form with respect to a hyperbolic cyclic subgroup, and estimate the difference of the sum from the first term in terms of the translation length of the generator of the subgroup. As an application we explicitly construct an integrable but unbounded holomorphic automorphic $(2, 0)$-form for any Fuchsian group containing hyperbolic elements of arbitrarily small translation length.

1991 Mathematics Subject Classification: 30F40.

1. Introduction

The holomorphic quadratic differentials on a closed Riemann surface R of genus $g \geq 2$ form a vector space of dimension $3g-3$. This number is the same as the maximal number of non-trivial simple closed curves on R which are mutually disjoint and not freely homotopic to each other. For any non-trivial simple closed curve σ, we consider the annular covering of R with respect to σ, namely, a holomorphic covering $A_\sigma \to R$ which induces an injection of the fundamental group of A_σ onto $\langle\sigma\rangle \subset \pi_1(R)$. On the annulus A_σ, there is a canonical holomorphic quadratic differential $\varphi(z)dz^2$ associated with the euclidean structure of A_σ. The relative Poincaré series operator with respect to the covering $A_\sigma \to R$ projects $\varphi(z)dz^2$ to a holomorphic quadratic differential $\Theta_\sigma(z)dz^2$ on R. We call this the Petersson series operator.

Wolpert [5] proved that the first variation of the Bers embedding for the Fenchel–Nielsen deformation about σ is a constant multiple of Θ_σ. The differential of the geodesic length function for a homotopy class σ at the point R of the Teichmüller space is also given by Θ_σ. Using these facts, he showed that for the maximal curve system $\{\sigma_1, \ldots, \sigma_{3g-3}\}$, the set $\{\Theta_{\sigma_1}, \ldots, \Theta_{\sigma_{3g-3}}\}$ is a basis system of the vector space of the holomorphic quadratic differentials on R which is regarded as the cotangent space of the Teichmüller space at R. Thus the Petersson scalar product gives the Weil–Petersson metric on the Teichmüller space in terms

* This research was supported by the scientist exchange program between the Academy of Finland and JSPS.

of the Petersson series. These results link differential geometry on the Teichmüller space with hyperbolic geometry on the Riemann surface.

A certain estimate of the Petersson series is utilized in the above context by Wolpert [6]. Appreciating its importance, we prove his estimate in a different way and a little more generally in this note. Our technique is based on an idea of Ahlfors [1]. Our result is valid for any Fuchsian group, possibly with an infinite number of generators and with torsion. As an application, we give a method of constructing an integrable but unbounded holomorphic automorphic $(2,0)$-form for any Fuchsian group which contains hyperbolic elements of arbitrarily small translation length.

The author would like to express his sincere thanks to Dr. Lesley Ward for reading the manuscript carefully and making a number of helpful suggestions.

2. Preliminaries

In this section, we give notation and facts which are used later.

We use the upper half plane model of the hyperbolic plane $\mathbb{H}$ equipped with the hyperbolic metric $ds = \rho(\zeta)|d\zeta|$, where $\rho(\zeta) = 1/\operatorname{Im}\zeta$. Let G be a non-elementary Fuchsian group acting on $\mathbb{H}$, not necessarily finitely generated. By conjugation, we may assume that G contains a hyperbolic element with repelling fixed point 0 and attracting fixed point ∞. Let $\gamma(\zeta) = e^l \zeta$ be a primitive element of this form; l is the translation length of γ.

A *collar* of γ is a neighborhood of the axis α_γ of the form

$$\left\{ \zeta \in \mathbb{H} \;\middle|\; \frac{\pi}{2} - \theta \le \arg\zeta \le \frac{\pi}{2} + \theta \right\}$$

that is invariant under the normalizer of $\langle\gamma\rangle$ and equivariant under G. It is known that there is a universal constant $L_0 > 0$ (the Margulis constant) such that γ has a collar whenever l is not greater than L_0. Furthermore, the collar lemma (see [3]) asserts that there is a collar of γ whose area m modulo $\langle\gamma\rangle$ is $l/(2\sinh\frac{l}{2})$ (then $\frac{\pi}{2} - \theta = \arctan(l/m)$). Letting this collar be $\hat{C}_\gamma$, we denote by C_γ a smaller collar of half the area of $\hat{C}_\gamma$. We call the geodesic line β joining -1 and 1 the *transverse line*. Along this line, the euclidean distance from $\partial\hat{C}_\gamma$ to the real axis $\mathbf{R}$ is $O(l)$ ($\sim \frac{\pi}{2} - \theta$) as $l \to 0$, and so the euclidean distance from ∂C_γ to $\mathbf{R}$ has the same property.

We choose a positive constant L_1 as follows: for $\gamma(\zeta) = e^{L_1}\zeta$, the collar C_γ has width (that is, distance between the axis α_γ and the boundary ∂C_γ) equal to $2\log(\sqrt{2}+1)$. The meaning of this value will become clearer in the proof of Proposition 2. Hereafter, we always assume that l is not greater than $L = \min\{L_0, L_1\}$.

The normalizer of $\langle\gamma\rangle$ is either the cyclic group itself or the dihedral group with an additional generator of order 2. We denote the normalizer of $\langle\gamma\rangle$ by Γ.

We transfer this discussion to the unit disk $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ via the Möbius transformation

$$F : \mathbb{H} \ni \zeta \mapsto z = \frac{\zeta - i}{\zeta + i} \in \mathbb{D}.$$

The hyperbolic metric on $\mathbb{D}$ is $F_*(ds) = \rho(z)|dz|$, where $\rho(z) = 2/(1-|z|^2)$. The Fuchsian group G is conjugated by F to a Fuchsian group, and the element γ is conjugated by F to a hyperbolic element

$$z \mapsto \frac{(e^l + i)z + (e^l - i)}{(e^l - i)z + (e^l + i)},$$

with fixed points -1 and 1. We denote these by the same symbols, G and γ, as before. We do the same for the axis $F(\alpha_\gamma)$, the collar $F(C_\gamma)$ and the transverse line $F(\beta) = \mathbb{D} \cap \{ z \mid \operatorname{Re} z = 0\}$. Since $|F'(\zeta)|$ is bounded away from 0 and ∞ in a neighborhood of the transverse line, we also see that the euclidean distance from ∂C_γ to $\partial \mathbb{D}$ along β is $O(l)$ as $l \to 0$.

We say a function f on $\mathbb{H}$ or $\mathbb{D}$ is an automorphic (m,n)-form (m, $n \in \mathbf{Z}$) for a Fuchsian group G if it satisfies $f(g(z))g'(z)^m\overline{g'(z)^n} = f(z)$ for every $g \in G$. If $\varphi(z)$ is an automorphic $(2,0)$-form for G, then $|\varphi(z)|$ is a $(1,1)$-form. We say $\varphi(z)$ is *integrable* if the integral of $|\varphi(z)|$ over a fundamental region of G is finite.

Given a subgroup Γ of a Fuchsian group G and an integrable automorphic $(2,0)$-form φ for Γ, we define the relative Poincaré series operator $\Theta_{\Gamma\backslash G}$ by

$$\Theta_{\Gamma\backslash G}\varphi(z) = \sum_{[h]\in\Gamma\backslash G} \varphi(h(z))h'(z)^2.$$

The sum is well-defined since it is independent of the choice of representatives of the right cosets $[h] \in \Gamma\backslash G$. It converges to an integrable automorphic $(2,0)$-form for G. We can also apply the relative Poincaré series operator to an integrable $(1,1)$-form ψ for Γ,

$$\Theta_{\Gamma\backslash G}\psi(z) = \sum_{[h]\in\Gamma\backslash G} \psi(h(z))h'(z)\overline{h'(z)},$$

obtaining a $(1,1)$-form for G.

The holomorphic function $1/\zeta^2$ on $\mathbb{H}$ is an integrable automorphic $(2,0)$-form for Γ, for any translation length l, where Γ is either a subgroup of G generated by $\gamma(\zeta) = e^l\zeta$ or an index 2 extension of such a subgroup. We call the relative Poincaré series $\Theta_{\Gamma\backslash G}(1/\zeta^2)$ the Petersson series.

Again we consider a Fuchsian group G acting on $\mathbb{D}$. We regard the constant function $\equiv 1$ as an integrable $(1,1)$-form for $\{\mathrm{id}\}$ and apply the (relative) Poincaré series operator Θ_G to it. Then we have an automorphic $(1,1)$-form for G,

$$\lambda_G^2(z) = \sum_{g\in G} |g'(z)|^2.$$

It is obvious that for any subgroup Γ of G,

$$\Theta_{\Gamma\backslash G}\lambda_\Gamma^2(z) = \lambda_G^2(z).$$

The square of the hyperbolic density $\rho^2(z)$ is an automorphic $(1,1)$-form for the group of all Möbius transformations. Thus for an automorphic $(2,0)$-form φ for G, $\rho^{-2}(z)|\varphi(z)|$ is an $(0,0)$-form (in other words, an automorphic function for G), and so is $\rho^{-2}(z)\lambda_G^2(z)$.

3. Statement of theorems

First we estimate the relative Poincaré series $\Theta_{\Gamma\backslash G}$ of an automorphic form φ for Γ. We need not assume that φ is holomorphic, though we require that φ is bounded relative to λ_Γ^2. Hence our estimate is actually for $\Theta_{\Gamma\backslash G}\lambda_\Gamma^2 = \lambda_G^2$. Ahlfors [1] proved that $\rho^{-2}\lambda_G^2$ is bounded by a constant depending only on G. Our result gives a bound depending only on the translation length of γ in certain regions.

Theorem 1. *Let G be a Fuchsian group acting on $\mathbb{D}$ which contains a primitive hyperbolic element γ of translation length l $(\leq L)$ with fixed points -1 and 1. Suppose that $|\varphi(z)|$ is an automorphic $(1,1)$-form for Γ such that $\lambda_\Gamma^{-2}(z)|\varphi(z)|$ is bounded by some positive constant M. Then there is a constant $K_1 > 0$ depending neither on l nor on G such that*

(1) $\rho^{-2}(z)\Theta_{\Gamma\backslash G}|\varphi(z)| \leq K_1 M l$ *for* $z \in \mathbb{D} - G(C_\gamma)$*; and*

(2) $\rho^{-2}(z)\Theta_{\Gamma\backslash G}|\varphi(z)| - \rho^{-2}(z)|\varphi(z)| \leq K_1 M l$ *for* $z \in C_\gamma$*.*

Next we apply Theorem 1 to the estimate of the Petersson series. We have only to compare $1/\zeta^2$ with the pull-back of λ_Γ^2 to $\mathbb{H}$ by $F : \mathbb{H} \to \mathbb{D}$. The following theorem asserts that the Petersson series with weight ρ_H^{-2} is approximated by its first term in the collar C_γ, and converges to zero outside the collar, as l tends to zero. This result has been obtained by Wolpert [6] (Lemma 2.2) with a hyperbolic geometry proof, whereas our proof has a flavor of complex analysis.

Theorem 2. *Let G be a Fuchsian group acting on $\mathbb{H}$ which contains a primitive hyperbolic element $\gamma(\zeta) = e^l\zeta$ $(l \leq L)$. Then there is a constant $K_2 > 0$ depending neither on l nor on G such that*

(1) $\rho^{-2}(\zeta)\Theta_{\Gamma\backslash G}|1/\zeta^2| \leq K_2 l^2$ *for* $\zeta \in \mathbb{H} - G(C_\gamma)$*; and*

(2) $\rho^{-2}(\zeta)\Theta_{\Gamma\backslash G}|1/\zeta^2| - \rho^{-2}(\zeta)|1/\zeta^2| \leq K_2 l^2$ *for* $\zeta \in C_\gamma$*.*

4. Preparation for proof

In this section, we prove two propositions used in the proof of Theorem 1.

The first one is based on the mean value property of holomorphic functions. We also use a certain property of the collar.

Proposition 1. *Let G be a Fuchsian group acting on $\mathbb{D}$ which contains a primitive hyperbolic element γ of translation length l $(\leq L)$ with fixed points -1 and 1. Then there is a constant $K > 0$ depending neither on l nor on G such that*

$$\rho^{-2}(z)\lambda_G^2(z) \leq Kl$$

for any z on ∂C_γ.

Proof. By the mean value theorem for a holomorphic function f, we have

$$f(z) = \frac{1}{\pi r^2}\iint_{U(z,r)} f(w)\,du\,dv,$$

where $U(z,r)$ is an euclidean disk with center z and euclidean radius r. We apply this to $g'(z)^2$ at $z \in \partial C_\gamma$. We also require that the orbits $G(U(z,r))$ are mutually disjoint. The following lemma (cf. Bers [2], Lemma 4) enables us to choose a positive constant δ depending neither on l nor on G such that $G(U(z,r))$ are mutually disjoint for $z \in \partial C_\gamma$ and $r = \delta\rho^{-1}(z)$.

Lemma 1. *Let G be a Fuchsian group which contains a primitive hyperbolic element γ of translation length l $(\leq L)$. Then there is a constant $d > 0$ depending neither on l nor on G such that every hyperbolic disk B with hyperbolic radius d and center on ∂C_γ satisfies $g(B) \cap B = \emptyset$ for all $g \in G - \{\mathrm{id}\}$.*

Now setting $U = U(z, \delta\rho^{-1}(z))$, we have

$$g'(z)^2 = \frac{1}{\pi\delta^2\rho^{-2}(z)}\iint_U g'(w)^2\,du\,dv.$$

Then taking the absolute value and the sum over $g \in G$, we have

$$\rho^{-2}(z)\lambda_G^2(z) \leq \frac{1}{\pi\delta^2}\iint_U \sum_{g\in G} |g'(w)|^2\,du\,dv = \frac{1}{\pi\delta^2}\iint_{G(U)} du\,dv.$$

Here the orbits $G(U)$ are outside the collar C_γ, except for $\Gamma(U)$. The euclidean area of $\mathbb{D} - C_\gamma$ is $O(l)$ as $l \to 0$ because the distance from ∂C_γ to $\partial\mathbb{D}$ along the transverse line β is $O(l)$. Therefore the last integral is bounded by a constant multiple Kl of l. □

The next proposition is a crucial point in our argument, which is based on Ahlfors' argument in [1].

Proposition 2. *Let G be a Fuchsian group acting on $\mathbb{D}$ which contains a primitive hyperbolic element γ of translation length l not greater than L and with fixed points -1 and 1. Then the automorphic function $\rho^{-2}(z)\lambda_G^2(z)$ restricted to $\mathbb{D} - G(C_\gamma)$ is subharmonic, and it takes its maximum value on ∂C_γ. Moreover, $\rho^{-2}(z)(\lambda_G^2(z) - \lambda_\Gamma^2(z))$ restricted to C_γ is subharmonic and takes its maximum value on ∂C_γ.*

Proof. We know

$$\rho^{-2}(z)\lambda_G^2(z) = \tfrac{1}{4}\sum_{g\in G}(1-|g(z)|^2)^2.$$

We calculate the Laplacian of $(1-|g(z)|^2)^2$:

$$\Delta(1-|g(z)|^2)^2 = 8(2|g(z)|^2-1)|g'(z)|^2.$$

This is positive outside $g^{-1}(B)$, where $B = \{\, z \mid |z| \le 2^{-1/2}\,\}$. By the definition of L, we see that if $l \le L$, then $B \subset C_\gamma$. Hence each $(1-|g(z)|^2)^2$ is subharmonic outside $G(C_\gamma)$, and so the sum $\sum_{g\in G}(1-|g(z)|^2)^2$ is also subharmonic outside $G(C_\gamma)$.

Suppose that there is a point $z_0 \in \mathbb{D} - G(C_\gamma)$ where $\rho^{-2}(z)\lambda_G^2(z)$ takes a greater value than on $G(\partial C_\gamma)$. Then we can take a finite approximation $\sum(1-|g(z)|^2)^2$ whose value at z_0 is still greater than on $G(\partial C_\gamma)$. However, the finite sum is also subharmonic and identically zero on $\partial\mathbb{D}$, so it satisfies the maximum principle. Thus we have a contradiction. Since $\rho^{-2}(z)\lambda_G^2(z)$ is an automorphic function for G, the maximum value on $G(\partial C_\gamma)$ is the same as the maximum value on ∂C_γ.

Next we have

$$\rho^{-2}(z)(\lambda_G^2(z)-\lambda_\Gamma^2(z)) = \tfrac{1}{4}\sum_{g\in G-\Gamma}(1-|g(z)|^2)^2,$$

Since $g^{-1}(B)$ is disjoint from C_γ for every $g \in G - \Gamma$, we see that $\rho^{-2}(z)(\lambda_G^2(z) - \lambda_\Gamma^2(z))$ is subharmonic in C_γ, where the maximum principle holds. Thus it takes its maximum value on ∂C_γ. □

5. Proof of theorems

Proof. (Theorem 1) By the assumption $\lambda_\Gamma^{-2}(z)|\varphi(z)| \le M$, we have

$$\Theta_{\Gamma\backslash G}|\varphi(z)| \le M\Theta_{\Gamma\backslash G}\lambda_\Gamma^2(z) = M\lambda_G^2(z);$$
$$\Theta_{\Gamma\backslash G}|\varphi(z)| - |\varphi(z)| \le M(\Theta_{\Gamma\backslash G}\lambda_\Gamma^2(z) - \lambda_\Gamma^2(z)) = M(\lambda_G^2(z) - \lambda_\Gamma^2(z)).$$

Thus we have only to estimate $\rho^{-2}(z)\lambda_G^2(z)$ and $\rho^{-2}(z)(\lambda_G^2(z) - \lambda_\Gamma^2(z))$.

By Proposition 2, it suffices to estimate them on ∂C_γ. By Proposition 1, they are bounded by Kl. Thus, setting $K_1 = K$, we have the theorem. □

Proof. (Theorem 2) We transfer the automorphic $(2,0)$-form $1/\zeta^2$ to $\mathbb{D}$ via the Möbius transformation $F : \mathbb{H} \to \mathbb{D}$, and then apply Theorem 1. Let

$$\varphi(z) = F_*\left(\frac{1}{\zeta^2}\right) = \frac{(F^{-1})'(z)^2}{F^{-1}(z)^2}.$$

This is an automorphic $(2,0)$-form for $F\Gamma F^{-1}$. We consider the ratio of $|\varphi(z)|$ to $\lambda^2_{F\Gamma F^{-1}}(z)$:

$$\begin{aligned}\frac{|\varphi(z)|}{\lambda^2_{F\Gamma F^{-1}}(z)} &= \frac{|\varphi(F(\zeta))|\,|F'(\zeta)|^2}{\sum_{\gamma\in\Gamma} |(F\gamma F^{-1})'(F(\zeta))|^2 |F'(\zeta)|^2} \\ &= \frac{|1/\zeta^2|}{\sum_{\gamma\in\Gamma} |F'(\gamma(\zeta))\gamma'(\zeta)|^2} = \Big[\sum_{\gamma\in\Gamma} \frac{4|\zeta\gamma'(\zeta)|^2}{|\gamma(\zeta)+i|^4}\Big]^{-1}.\end{aligned}$$

Now we estimate this value. We may assume that $\zeta \in \mathbb{H}$ is in a fundamental region $\{\, 1 \le |\zeta| \le e^l \,\}$ of $\langle\gamma\rangle$. Then

$$\begin{aligned}\sum_{\gamma\in\langle\gamma\rangle} \frac{4|\zeta\gamma'(\zeta)|^2}{|\gamma(\zeta)+i|^4} &= \sum_{n\in\mathbf{Z}} \frac{4|e^{nl}\zeta|^2}{|e^{nl}\zeta+i|^4} \ge \sum_{n\in\mathbf{Z}} \frac{4e^{2nl}}{(e^{(n+1)l}+1)^4} \\ &\ge \sum_{n\ge 0} \frac{4e^{2nl}}{(2e^{(n+1)l})^4} + \sum_{n\le -2} \frac{4e^{2nl}}{2^4} = \frac{1}{2e^{2l}(e^{2l}-1)}.\end{aligned}$$

Therefore

$$\frac{|\varphi(z)|}{\lambda^2_{F\Gamma F^{-1}}(z)} \le e^{2l}(e^{2l}-1) \le kl$$

for some positive constant k.

Setting $M = kl$, we apply Theorem 1. We have

$$\rho^{-2}(z)\Theta_{F(\Gamma\backslash G)F^{-1}}|\varphi(z)| \le K_1 k l^2 \qquad \text{for } z \in \mathbb{D} - G(C_\gamma)$$

and

$$\rho^{-2}(z)\Theta_{F(\Gamma\backslash G)F^{-1}}|\varphi(z)| - \rho^{-2}(z)|\varphi(z)| \le K_1 k l^2 \qquad \text{for } z \in C_\gamma.$$

Pulling back by F, we get the required inequalities (1) and (2) with $K_2 = K_1 k$. □

6. Application

As an application of Theorem 2, we give an explicit construction of an integrable holomorphic automorphic $(2,0)$-form $\varphi(z)$ for a Fuchsian group G such that $\rho^{-2}(z)|\varphi(z)|$ is not bounded. Such examples were first constructed by Pommerenke, and recently, by Ohsawa. In this section, which is motivated by Ohsawa's preprint, we show the construction is possible whenever G contains hyperbolic elements of arbitrarily small translation length. The existence of such a $(2,0)$-form was a problem raised by Lehner. Niebur and Sheingorn [4] answered the question completely, but they did not give an explicit construction.

Our construction is as follows. Let G be a Fuchsian group of the type just described. We can choose a sequence $\{\gamma_n\}_{n\in\mathbf{N}}$ of primitive hyperbolic elements whose translation lengths l_n satisfy $l_n \leq L/n^4$. For each n, we construct an automorphic $(2,0)$-form $\psi_n(z)$ for G by the Petersson series for $\Gamma_n\backslash G$, and set $\hat{\psi}_n(z) = {l_n}^{-1/2}\psi_n(z)$. Then take the sum

$$\Psi(z) = \sum_{n\in\mathbf{N}} \hat{\psi}_n(z),$$

which is also an automorphic $(2,0)$-form for G if it converges.

Theorem 3. *For a Fuchsian group G which contains hyperbolic elements of arbitrarily small translation length, the above $\Psi(z)$ is an integrable holomorphic automorphic $(2,0)$-form for G such that $\rho^{-2}(z)|\Psi(z)|$ is not bounded.*

Proof. First we show $\Psi(z)$ is integrable. Let ω be a fundamental region of G. For each n, we have

$$\iint_\omega |\hat{\psi}_n(z)|\, dx\, dy \leq {l_n}^{-1/2} \iint_{1\leq|\zeta|\leq e^{l_n}} \frac{1}{|\zeta|^2}\, d\xi\, d\eta = \pi {l_n}^{1/2}.$$

Since we have chosen $l_n \leq L/n^4$, we see $\iint_\omega |\Psi(z)|\, dx\, dy \leq \pi\sqrt{L}\sum(1/n^2)$, which converges.

Next we show $\rho^{-2}(z)|\Psi(z)|$ is unbounded. By Theorem 2, we have $\rho^{-2}(z)|\psi_n(z)| \leq K_2{l_n}^2$ outside $G(C_{\gamma_n})$ and $\rho^{-2}(z)|\psi_n(z)| \geq 1 - K_2{l_n}^2$ on $G(\alpha_{\gamma_n})$, where α_{γ_n} is the axis of γ_n. Since the collars are disjoint for distinct n, we see

$$\rho^{-2}(z)|\Psi(z)| \geq {l_n}^{-1/2} - K_2 \sum_{m\in\mathbf{N}} {l_m}^{3/2}$$

on $G(\alpha_{\gamma_n})$. The sum on the right hand side converges. Letting $n\to\infty$, we see that $\rho^{-2}(z)|\Psi(z)|$ is unbounded. □

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