UNIFORM CONVEXITY, NORMAL STRUCTURE AND THE FIXED POINT PROPERTY OF METRIC SPACES

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Abstract. We say that a complete metric space X has the fixed point property if every group of isometric automorphisms of X with a bounded orbit has a fixed point in X. We prove that if X is uniformly convex then the family of admissible subsets of X possesses uniformly normal structure and if so then it has the fixed point property. We also show that from other weaker assumptions than uniform convexity, the fixed point property follows. Our formulation of uniform convexity and its generalization can be applied not only to geodesic metric spaces.

1. Introduction

In this paper, we consider certain properties of metric spaces which can be used in the geometric group theory. If a group acts on a complete metric space isometrically, one may ask a question about whether it has a fixed point in it or not. Usually we are interested in what kind of groups satisfy this property when the metric space in question is canonical such as a Hilbert space. A weaker problem than this is to see whether the group has a bounded orbit in it and thus we can reduce the original problem to finding a condition under which the bounded orbit implies the fixed point.

We extract this situation as a property of a metric space and define a metric space to have the fixed point property if this is always the case. A well-known sufficient condition for this property of a complete metric space is uniform convexity, which is a generalization of the concept of uniform convexity for L^p spaces. In this space, the circumcenter of an arbitrary bounded subset exists uniquely; for a bounded orbit of an isometry group, its unique circumcenter is a fixed point. In Section 5, we prove this property for a complete metric space of more general uniform convexity (Theorem 8). In Section 4, we will find a weaker condition than uniform convexity that still hold the fixed point property (Theorem 6), which we call uniform pseudo-convexity. An advantage of uniform pseudo-convexity is that it is invariant under a bi-Lipschitz homeomorphism with Lipschitz constant sufficiently close to one (Theorem 7).

On the other hand, for a non-expanding self-map of a metric space, the problem of finding its fixed point seems to have another history as the fixed point theory, which originates in the Brouwer fixed point theorem. Normal structure for a certain family of admissible subsets in a metric space is an important concept in this theory and we can

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borrow this condition for investigating the fixed point property of isometric group action. In Section 3, we verify that under a certain extra assumption the normal structure implies the fixed point property (Theorem 2). Before reviewing this claim, we will see that uniform convexity of a metric space implies uniformly normal structure (Theorem 1) in Section 2.

Our definition of uniform convexity in this paper is applicable to metric spaces in general, not necessarily to geodesic metric spaces. In Section 5, in order to present another example for which our formulation works, we generalize the concept of uniform convexity of metric spaces more widely in this manner and prove that it still holds the fixed point property via the normal structure condition (Theorem 9).

2. Uniformly convexity and normal structure

Let us begin with defining properties of a metric space X which we are concerned with. The first one is a numerical generalization of the concept of uniform convexity for L^p spaces to metric spaces. For $p = 2$, this is called the NC-inequality, which was first introduced in [4]. The following definition, which is inspired by [2, Proposition I.5.1], is slightly different from the usual one. In particular, we do not have to assume that X is a geodesic metric space (i.e., for any $x, y \in X$ there is a geodesic segment connecting them).

Definition. A metric space X with distance d is (p, c)-*uniformly convex* for $p \in [1, \infty)$ and $c > 0$ if for any $x, y \in X$ there is some $m \in X$ such that every $z \in X$ satisfies

$$d(z, m)^p \leq \frac{1}{2}\{d(z, x)^p + d(z, y)^p\} - c\, d(x, y)^p.$$

If we can find such p and c, then we simply say that $X = (X, d)$ is *uniformly convex*.

Remark. In the usual definition, one assumes that X is a geodesic metric space and the above condition is replaced with a similar condition for each point m on every geodesic segment connecting x and y. See [10] and [11]. The usual definition for uniform convexity implies ours and in this case the constants should be restricted to $p > 1$ and $c \leq 1/2^p$. However, when X is complete and when $p = 2$ and $c = 1/4$, our condition automatically implies that X is a uniquely geodesic metric space (i.e., for any $x, y \in X$ there is a unique geodesic segment connecting them) and X is contractible. These facts were shown in [2]. Moreover, a complete metric space X is $(2, 1/4)$-uniformly convex if and only if it is a CAT(0)-space.

In the above definition, the freedom of the constant c has the benefit of generalizing the concept of uniform convexity. This was already done in [11, Proposition 3.1] for geodesic metric spaces with $p = 2$. Actually, a CAT(1)-space (X, d) with diameter not greater than $\pi/2 - \varepsilon$ for $\varepsilon \in (0, \pi/2)$ is $(2, c)$-uniformly convex for $c = (\pi - 2\varepsilon)\sin\varepsilon/(2\cos\varepsilon)$. This in particular shows that the sphere with diameter less than $\pi/2$ is uniformly convex though its spherical distance function $d : X \times X \to [0, \infty)$ is not convex on geodesic segments.

Furthermore, as an example below shows, there is a complete metric space X which is uniformly convex but is not a geodesic metric space. Note that a complete metric space (X, d) is geodesic if and only if it is metrically convex, that is, for any two points $x, y \in X$ there is a midpoint $m \in X$ satisfying $d(x, m) = d(y, m) = d(x, y)/2$ (see [2, Proposition I.1.5] and [7, Theorem 2.16]). Although the following example is rather artificial, our definition of uniform convexity might have the advantage of treating a possible situation where a subspace X with relative distance d embedded in an infinite-dimensional Riemannian or Finsler manifold M is not known to be a geodesic metric space but (M, d) is uniformly convex.

Example. Let $S_\theta = \{w \in \mathbb{R}^2 \mid 0 \leq \arg w \leq \theta\}$ be the infinite circular sector with center at the origin 0 and angle $\theta \in (0, \pi/3)$. Then $X = X_\theta$ is given as the part of S_θ that is not contained in the open unit disk $\mathbb{D} = \{w \in \mathbb{R}^2 \mid |w| < 1\}$, that is, $X_\theta = S_\theta \setminus \mathbb{D}$. We provide X_θ with the restriction of the Euclidean distance d on $\mathbb{R}^2$. Clearly (X_θ, d) is not a geodesic metric space. However, we see that X_θ is (p, c)-uniformly convex for $p = 2$ and $c = (2\cos\theta - 1)/4$. Its proof is as follows.

For arbitrary points $x, y \in X_\theta$, take $m_0 \in \mathbb{R}^2$ as the midpoint of x and y with respect to the Euclidean distance on $\mathbb{R}^2$. If m_0 is in X_θ then just set $m = m_0$. Otherwise, take $m \in X_\theta$ on the unit circle so that m and m_0 are on the same radial ray ℓ from the origin. We have only to consider the latter case.

Take $z \in X_\theta$ arbitrarily. Let $z' \in \ell$ be the orthogonal projection of z onto ℓ. We may assume that $d(z, m) \geq d(z, m_0)$, which is equivalent to that $d(z', m) \geq d(z', m_0)$. Then

$$\begin{aligned} d(z, m)^2 - d(z, m_0)^2 &= d(z', m)^2 - d(z', m_0)^2 \\ &= (d(z', m) + d(z', m_0))(d(z', m) - d(z', m_0)) \\ &\leq 2d(z', m) \cdot d(m, m_0). \end{aligned}$$

Here, we see that $d(z', m) \leq 1 - \cos\theta$ and $d(m, m_0) \leq d(x, y)^2/4$ by elementary geometric observation. Indeed, for the first inequality, we consider an extremal case where z is at one of the two corners of X_θ and m is at the other. For the second inequality, let x' and y' be the intersections of the unit circle with the segment between x and y. We take the midpoint m_0' of x' and y', and take m' on the unit circle so that m' and m_0' are on the same radial ray ℓ' from the origin. Then $d(m, m_0) \leq d(m', m_0')$. Moreover,

$$d(m' m_0') = 1 - \left\{1 - \left(\frac{d(x', y')}{2}\right)^2\right\}^{1/2} \leq \frac{d(x', y')^2}{4} \leq \frac{d(x, y)^2}{4}.$$

Therefore

$$d(z, m)^2 - d(z, m_0)^2 \leq \frac{1 - \cos\theta}{2} d(x, y)^2$$

for every $z \in X_\theta$. Since $d(z, m_0)$ satisfies the inequality for $(2, 1/4)$-uniform convexity, we can conclude that

$$d(z, m)^2 \leq \frac{1}{2}\{d(z, x)^2 + d(z, y)^2\} - \frac{2\cos\theta - 1}{4}\, d(x, y)^2$$

for every $z \in X_\theta$.

Next, the relation between the radius and the diameter of certain admissible subsets gives another condition for a metric space. Here, for a subset A of a metric space (X, d), denote its diameter and *Chebyshev radius* by

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}; \quad \operatorname{rad}(A) = \inf\{r > 0 \mid A \subset B(z, r),\ z \in A\},$$

where $B(z, r) = \{x \in X \mid d(z, x) \leq r\}$ is the closed metric ball with center z and radius r. We regard a non-empty subset $A \subset X$ *admissible* if it is the intersection of some closed metric balls $\{B(z_i, r_i)\}_{i \in I}$ (I is an index set) of X. The family of all such non-empty bounded closed subsets A of X is denoted by $\mathcal{A}(X)$.

Definition. The family $\mathcal{A}(X)$ of admissible subsets of a metric space (X, d) possesses *normal structure* if every subset $A \in \mathcal{A}(X)$ with $\operatorname{diam}(A) > 0$ satisfies $\operatorname{rad}(A) < \operatorname{diam}(A)$. Moreover, $\mathcal{A}(X)$ possesses *uniformly normal structure* if there exists a positive constant $\alpha > 0$ such that this inequality is uniformly valid in the form

$$\operatorname{rad}(A) \leq (1 - \alpha) \operatorname{diam}(A).$$

Our first result shows the implication of the above two properties.

Theorem 1. *If a metric space (X, d) is (p, c)-uniformly convex, then $\mathcal{A}(X)$ has uniformly normal structure. More precisely,*

$$\operatorname{rad}(A) \leq (1 - c)^{1/p} \operatorname{diam}(A)$$

for every $A \in \mathcal{A}(X)$.

Proof. Take an arbitrary $A \in \mathcal{A}(X)$ with $d = \operatorname{diam}(A) > 0$. Choose an arbitrary $\varepsilon > 0$. Then there are $x, y \in A$ such that $d(x, y) \geq d - \varepsilon$. For these x and y, the definition of (p, c)-uniform convexity finds some point $m_\varepsilon \in X$ that satisfies

$$d(z, m_\varepsilon)^p \leq \frac{1}{2}\{d(z, x)^p + d(z, y)^p\} - c\, d(x, y)^p$$

for every $z \in X$.

First we check that m_ε belongs to A. Suppose that A is the intersection of closed metric balls $B(z_i, r_i)$ for all indices $i \in I$. Since $x, y \in A \subset B(z_i, r_i)$, we have $d(z_i, x) \leq r_i$ and $d(z_i, y) \leq r_i$ for each $i \in I$. Then, applying the above inequality to $z = z_i$, we obtain

$$d(z_i, m_\varepsilon)^p \leq \frac{1}{2}\{d(z_i, x)^p + d(z_i, y)^p\} \leq r_i^p.$$

This implies that $m_\varepsilon \in B(z_i, r_i)$ and hence $m_\varepsilon \in A$.

Consider an arbitrary $z \in A$. Then $d(z, x) \leq d$ and $d(z, y) \leq d$. Substituting these bounds and $d(x, y) \geq d - \varepsilon$ to the above inequality, we obtain $d(z, m_\varepsilon)^p \leq d^p - c(d - \varepsilon)^p$. This yields that

$$d(z, m_\varepsilon) \leq d\left(1 - c(1 - \varepsilon/d)^p\right)^{1/p},$$

and hence A is in the closed ball of center $m_\varepsilon \in A$ and radius $d\left(1 - c(1 - \varepsilon/d)^p\right)^{1/p}$. Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we see that $\operatorname{rad}(A) \leq (1 - c)^{1/p} \operatorname{diam}(A)$. □

3. The fixed point property

We investigate the following property of a metric space (X, d) concerning the action of its automorphism group. We denote by $\mathrm{Aut}(X, d)$ the group of isometric bijections of X onto itself with respect to distance d.

Definition. A metric space (X, d) has the *fixed point property* if every subgroup $G \subset \mathrm{Aut}(X, d)$ with a bounded orbit in X has a fixed point in X.

Remark that if the orbit $G(x)$ of $x \in X$ is bounded then the orbit $G(x')$ for any other $x' \in X$ is also bounded. In particular, if G has a fixed point in X then G has a bounded orbit $G(x)$ for every $x \in X$.

Hereafter, we use the following terminology: the family $\mathcal{A}(X)$ of admissible subsets of X is *compact* if every totally ordered sub-family $\{A_i\}_{i \in I} \subset \mathcal{A}(X)$ with respect to the inclusion relation satisfies $\bigcap_{i \in I} A_i \neq \emptyset$, that is, $\bigcap_{i \in I} A_i \in \mathcal{A}(X)$.

We will see that the properties introduced in the previous section imply the fixed point property. The following result can be proved by a similar argument to [7, Theorem 5.1], which has its origin in [8] and whose abstract formulation is due to [12]. Note that if $\mathcal{A}(X)$ is compact then (X, d) is complete (see [7, Proposition 5.1]).

Theorem 2. *If the admissible family $\mathcal{A}(X)$ of a metric space (X, d) possesses normal structure and $\mathcal{A}(X)$ is compact, then (X, d) has the fixed point property.*

Proof. Let $G \subset \mathrm{Aut}(X, d)$ with a bounded orbit $G(x)$ $(x \in X)$. For a closed metric ball containing $G(x)$, we take the intersection $A_* \neq \emptyset$ of all its images under G. Then $A_* \in \mathcal{A}(X)$ is invariant under G. We consider the G-invariant sub-family of $\mathcal{A}(X)$:

$$\mathcal{A}_G(X) = \{A \in \mathcal{A}(X) \mid g(A) = A \ (\forall g \in G)\}.$$

Since $A_* \in \mathcal{A}_G(X)$, this is not an empty family. Also it is clear that if $A_i \in A_G(X)$ for all $i \in I$ then $\bigcap_{i \in I} A_i$ is G-invariant. Then the compactness of $\mathcal{A}(X)$ implies that $\mathcal{A}_G(X)$ is inductive and Zorn's Lemma ensures the existence of a minimal element $A_0 \in \mathcal{A}_G(X)$ with respect to the inclusion relation.

We will prove that A_0 consists of a single point $a \in X$. This shows that a is a fixed point of G. Suppose to the contrary that A_0 is not a single point set. Then $\mathrm{diam}(A_0) > 0$ and the normal structure of $\mathcal{A}(X)$ implies that $\mathrm{rad}(A_0) < \mathrm{diam}(A_0)$. Choose a constant r with $\mathrm{rad}(A_0) < r < \mathrm{diam}(A_0)$ and set

$$C = \{x \in A_0 \mid A_0 \subset B(x, r)\}.$$

This is not empty since $\mathrm{rad}(A_0) < r$. Then we have

$$C = \left(\bigcap_{y \in A_0} B(y, r) \right) \cap A_0,$$

which can be verified as follows. Take $x \in C \subset A_0$ arbitrarily. Since $A_0 \subset B(x, r)$ by the definition of C, every $y \in A_0$ satisfies $d(x, y) \leq r$. Hence x belongs to $\bigcap_{y \in A_0} B(y, r)$.

Conversely, take $x \in (\bigcap_{y \in A_0} B(y,r)) \cap A_0$ arbitrarily. Then every $y \in A_0$ satisfies $d(x,y) \leq r$. This implies $A_0 \subset B(x,r)$ and the definition of C says that x belongs to C.

The above representation of C in particular implies that $C \in \mathcal{A}(X)$. Moreover, we will prove that $C \in \mathcal{A}_G(X)$, that is, $g(C) = C$ for every $g \in G$. It is enough to show that $g(C) \subset C$ for every $g \in G$ because this includes $g^{-1}(C) \subset C$ and hence $C \subset g(C)$. Take an arbitrary $x \in C$, which satisfies $d(x,y) \leq r$ for every $y \in A_0$. It follows that $d(g(x), g(y)) \leq r$ for every $g \in G$. This implies that $g(y) \in B(g(x), r)$ for every $y \in A_0$, that is, $A_0 = g(A_0) \subset B(g(x), r)$. On the other hand, we know that $g(x) \in A_0$ from $x \in C \subset A_0$. Therefore $g(x)$ belongs to C. This means that $g(C) \subset C$.

However, we see that $\mathrm{diam}(C) \leq r$. Indeed, for any x and y in C, it holds that $d(x,y) \leq r$ because $x \in A_0 \subset B(y,r)$. Since $\mathrm{diam}(C) \leq r < \mathrm{diam}(A_0)$, we have $C \subsetneqq A_0$. This contradicts the minimality of A_0 in $\mathcal{A}_G(X)$. Thus we have proved that $A_0 = \{a\}$, which is fixed by G. □

If (X,d) is complete and $\mathcal{A}(X)$ has uniformly normal structure then $\mathcal{A}(X)$ is compact, which was proved in [1] and [5] for a weaker condition of the compactness (countable compactness) and completed by the work of [9] (see Section 5). We may also consult [7, Theorem 5.4]. Consequently, we obtain the following result as a corollary to Theorem 2. The fact is that an argument in [5] can directly show this result without using the compactness.

Corollary 3. *If the admissible family $\mathcal{A}(X)$ of a complete metric space (X,d) has uniformly normal structure, then (X,d) has the fixed point property.*

Also, Theorem 1 and Corollary 3 yield the following result.

Corollary 4. *If a complete metric space (X,d) is uniformly convex, then (X,d) has the fixed point property.*

Note that this result is already known and can be proved directly. Actually, every bounded subset A in a uniformly convex complete metric space (X,d) has the unique *circumcenter*, which is the center of a closed metric ball containing A with the minimum radius attained. See below in the next section for precise definition. This fact will be also proved later in Theorem 8 under a weaker assumption. If we take A as the bounded orbit of $G \subset \mathrm{Aut}(X,d)$, then its unique circumcenter is a fixed point of G. This result is called the Bruhat-Tits theorem [4]. The presentation using circumcenter can be found in [3, Section VI.4]. In the case of p in general, see [10, Lemma 2.3].

4. Uniform pseudo-convexity

We extend the concept of uniform convexity of a complete metric space so that it still holds the fixed point property.

Definition. A metric space (X,d) is *uniformly k-pseudo-convex* for $k \in [0,1)$ if there are some constants $p \geq 1$ and $c > 0$ such that for any $x, y \in X$ there is some $m \in X$ such that every $z \in X$ satisfies

$$d(z,m)^p \le \frac{1+k^p c}{2}\{d(z,x)^p + d(z,y)^p\} - c\,d(x,y)^p.$$

If there is such k, then we simply say that (X,d) is *uniformly pseudo-convex*.

We define the following points close to circumcenter for each bounded subset A of a metric space (X,d). For $A \subset X$ and $x \in X$, set $r_x(A) = \sup_{a\in A} d(x,a)$. Let us define the *circumradius* of A by

$$r_X(A) = \inf_{x\in X} r_x(A) = \inf\{r > 0 \mid A \subset B(x,r),\ x \in X\},$$

which is not greater than the Chebyshev radius $\mathrm{rad}(A)$. In general, we see that

$$r_X(A) \le \mathrm{rad}(A) \le \mathrm{diam}(A) \le 2r_X(A).$$

For every $\varepsilon \ge 0$, we say that $x \in X$ is an *ε-circumcenter* of a bounded subset $A \subset X$ if it satisfies

$$r_x(A) \le r_X(A) + \varepsilon.$$

When $\varepsilon = 0$, this is nothing but a circumcenter of A. Clearly an ε-circumcenter always exists for every bounded subset A and for every positive $\varepsilon > 0$.

Lemma 5. *Let A be a bounded subset with $r_X(A) > 0$ (or $\mathrm{diam}(A) > 0$) in a uniformly k-pseudo-convex metric space (X,d). Then, for any $\tilde{k} \in (k,1)$, there is some $\varepsilon > 0$ such that any ε-circumcenters $x, y \in X$ of A satisfy $d(x,y) \le \tilde{k} r_X(A)$.*

Proof. For an arbitrary $\varepsilon > 0$, consider any ε-circumcenters $x, y \in X$ of A. For these x and y, we choose some $m \in X$ that satisfies the inequality of uniform k-pseudo-convexity for some $p \ge 1$ and $c > 0$. We also take $z \in A$ such that

$$d(z,m) \ge r_m(A) - \varepsilon.$$

By definition, $d(z,x) \le r_x(A)$ and $d(z,y) \le r_y(A)$. Substituting these three estimates to the uniformly pseudo-convex inequality, we have

$$(r_m(A) - \varepsilon)^p \le \frac{1+k^p c}{2}\{r_x(A)^p + r_y(A)^p\} - c\,d(x,y)^p.$$

Moreover, since $r_x(A) \le r_X(A) + \varepsilon$ and $r_y(A) \le r_X(A) + \varepsilon$, it follows that

$$(r_X(A) - \varepsilon)^p \le (r_X(A)+\varepsilon)^p + k^p c (r_X(A)+\varepsilon)^p - c\,d(x,y)^p.$$

By using $(r_X(A)+\varepsilon)^p - (r_X(A)-\varepsilon)^p \le 2p\varepsilon(r_X(A)+\varepsilon)^{p-1}$ for $p \ge 1$, we see that

$$\begin{aligned} d(x,y)^p &\le \frac{2p\varepsilon}{c}(r_X(A)+\varepsilon)^{p-1} + k^p (r_X(A)+\varepsilon)^p \\ &= \left\{\frac{2p\varepsilon}{c(r_X(A)+\varepsilon)} + k^p\right\}\left\{1 + \frac{\varepsilon}{r_X(A)}\right\}^p r_X(A)^p. \end{aligned}$$

Then, for any $\tilde{k} \in (k,1)$, we can make the last term bounded by $\tilde{k}^p r_X(A)^p$ if we choose a sufficiently small $\varepsilon > 0$. □

Applying this lemma, we obtain the required result.

Theorem 6. *If a complete metric space (X, d) is uniformly pseudo-convex, then it has the fixed point property.*

Proof. Suppose that (X, d) is uniformly k-pseudo-convex for $k \in [0, 1)$ and that $G \subset \mathrm{Aut}(X, d)$ has a bounded orbit $G(x_0)$ for $x_0 \in X$. Choose any $\tilde{k} \in (k, 1)$ and fix it. First we apply Lemma 5 to $A_0 = G(x_0)$. We may assume that $\mathrm{diam}\, A_0 > 0$ for otherwise we obtain a fixed point x_0 of G. Then there is some $\varepsilon_1 \in (0, \mathrm{diam}\, A_0)$ such that any ε_1-circumcenters x_1 and y_1 of A_0 satisfy $d(x_1, y_1) \le \tilde{k} r_X(A_0)$. Note that every point of the orbit $G(x_1)$ is an ε_1-circumcenter of A_0. Hence, for $A_1 = G(x_1)$, the above inequality implies that

$$\mathrm{diam}\, A_1 \le \tilde{k} r_X(A_0) \le \tilde{k}\, \mathrm{diam}\, A_0.$$

Next we apply Lemma 5 to $A_1 = G(x_1)$, for which we may assume that $\mathrm{diam}\, A_1 > 0$. Then there is some $\varepsilon_2 \in (0, \mathrm{diam}\, A_1)$ such that any ε_2-circumcenters x_2 and y_2 of A_1 satisfy $d(x_2, y_2) \le \tilde{k} r_X(A_1)$. For $A_2 = G(x_2)$, we have

$$\mathrm{diam}\, A_2 \le \tilde{k} r_X(A_1) \le \tilde{k}\, \mathrm{diam}\, A_1.$$

Repeating this process, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that each x_n is an ε_n-circumcenter of the orbit $A_{n-1} = G(x_{n-1})$ for some $\varepsilon_n \in (0, \mathrm{diam}\, A_{n-1})$ and that the orbits satisfy

$$\mathrm{diam}\, A_{n-1} \le \tilde{k}\, \mathrm{diam}\, A_n$$

for every $n \in \mathbb{N}$. Since $\tilde{k} < 1$, this implies that $\mathrm{diam}\, A_n \to 0$ as $n \to \infty$.

On the other hand, we see that $\{x_n\}$ is a Cauchy sequence. Indeed, since x_n is an ε_n-circumcenter of $A_{n-1} = G(x_{n-1})$,

$$d(x_n, x_{n-1}) \le r_X(A_{n-1}) + \varepsilon_n \le 2\, \mathrm{diam}\, A_{n-1}.$$

Then

$$\sum_{n=1}^{\infty} d(x_n, x_{n-1}) \le 2(\mathrm{diam}\, A_0) \sum_{n=0}^{\infty} \tilde{k}^n < \infty,$$

which shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, the limit x_∞ of $\{x_n\}$ exists in X. For every $g \in G$, we see that

$$d(g(x_\infty), x_\infty) = \lim_{n\to\infty} d(g(x_n), x_n) \le \lim_{n\to\infty} \mathrm{diam}\, A_n = 0.$$

Thus x_∞ is a fixed point of G. □

Uniform pseudo-convexity is preserved under a bi-Lipschitz homeomorphism with a small Lipschitz constant. We say that a (surjective) homeomorphism $f : X \to X'$ between metric spaces (X, d) and (X', d') is λ-bi-Lipschitz for $\lambda \ge 1$ if

$$\frac{1}{\lambda} d(x, y) \le d'(f(x), f(y)) \le \lambda d(x, y)$$

is satisfied for any $x, y \in X$.

Theorem 7. *Let (X,d) be a (p,c)-uniformly convex metric space. If $f : X \to X'$ is a λ-bi-Lipschitz homeomorphism onto another metric space (X',d') with Lipschitz constant $\lambda < (1+c)^{1/(2p)}$. Then (X',d') is uniformly pseudo-convex.*

Proof. Take any $x', y' \in X'$. For $x, y \in X$ with $f(x) = x'$ and $f(y) = y'$, we choose $m \in X$ that satisfies the inequality for (p,c)-uniform convexity of (X,d). Then set $m' = f(m) \in X'$. For every $z' \in X'$, we have

$$\begin{aligned} d'(z',m')^p + cd'(x',y')^p &\le \lambda^p\{d(z,m)^p + cd(x,y)^p\} \\ &\le \frac{\lambda^p}{2}\{d(z,x)^p + d(z,y)^p\} \le \frac{\lambda^{2p}}{2}\{d'(z',x')^p + d'(z',y')^p\}, \end{aligned}$$

where $z \in X$ is taken as $f(z) = z'$. Here, since $1 \le \lambda^{2p} < 1+c$, there is some $k \in [0,1)$ such that $\lambda^{2p} = 1 + k^p c$. This shows that (X',d') is uniformly pseudo-convex. ☐

Similarly, we can prove that if (X,d) is uniformly pseudo-convex and $f : X \to X'$ is a λ-bi-Lipschitz homeomorphism onto another metric space (X',d') with $\lambda \ge 1$ sufficiently close to 1, then (X',d') is also uniformly pseudo-convex.

If there is a bi-Lipschitz homeomorphism $f : (X,d) \to (X',d')$, then the conjugate $G' = fGf^{-1}$ for an isometry group $G \subset \mathrm{Aut}(X,d)$ acts on (X',d') as uniformly bi-Lipschitz homeomorphisms, meaning that the Lipschitz constants λ are uniformly bounded for all elements of G'. In this situation, the existence of a fixed point of G is equivalent to that of G'. In [6, Theorem 3.1], a certain fixed point property of a uniformly Lipschitz map is investigated.

Example. Let $ds(x,y) = \sqrt{dx^2 + dy^2}$ be the Euclidean metric on $\mathbb{R}^2$. Define a new metric $d\tilde{s}(x,y)$ on $\mathbb{R}^2$ by $d\tilde{s}(x,y) = ds(x,y)$ if $(x,y) \in \mathbb{R}^2 - \mathbb{D}$ and $d\tilde{s}(x,y) = \lambda ds(x,y)$ if $(x,y) \in \mathbb{D}$, where $\mathbb{D}$ is the unit disk and λ is a constant with $1 < \lambda < (5/4)^{1/4}$. Then the identity map is a λ-bi-Lipschitz homeomorphism between $(\mathbb{R}^2, ds)$ and $(\mathbb{R}^2, d\tilde{s})$. Since $(\mathbb{R}^2, ds)$ is $(2, 1/4)$-uniformly convex, Theorem 7 shows that $(\mathbb{R}^2, d\tilde{s})$ is uniformly pseudo-convex. On the other hand, the family of admissible subsets of $(\mathbb{R}^2, d\tilde{s})$ does not have normal structure. Indeed, by taking two closed balls with large radii and centers far away from the origin, which are put in a symmetric position with respect to the origin, we can make an admissible subset A as the intersection of these balls that consists of exactly two points. Actually, the distance between $(R+1, 0)$ and the origin with respect to $d\tilde{s}$ is $R+\lambda$ whereas the distance between $(R+1, 0)$ and $(0,1)$ is less than

$$(R+1)\cos\theta - \sin\theta + 2\theta \quad (\tan\theta = (R+1)^{-1}).$$

This verifies that $\mathrm{rad}(A) = \mathrm{diam}(A)$, which implies that the admissible family of $(\mathbb{R}^2, d\tilde{s})$ does not have normal structure. In fact, although $(\mathbb{R}^2, d\tilde{s})$ is a geodesic metric space, the distance function is not convex on geodesic segments. This example shows that neither uniform convexity nor normal structure of the admissible family are invariant under bi-Lipschitz homeomorphisms.

5. Uniform convexity in the wider sense

This section is added in revision. We will see here that our fashion of defining uniform convexity also works for more general uniform convexity, which is the generalization of uniform convexity of Banach spaces to metric spaces. See [6] for a recent account of such usual definition.

Definition. A metric space X with distance d is *uniformly convex in the wider sense* if there is a function $\delta(r,t) : [t/2,\infty) \times [0,\infty) \to [0,1]$, which is called the modulus of convexity, such that

(1) $\delta(r,t) = 0$ if and only if $t = 0$;
(2) for each fixed r, $\delta(r,t)$ is increasing with respect to t;
(3) for each fixed t, $\delta(r,t)$ is decreasing with respect to r,

and if for any $x, y \in X$ there is some $m \in X$ such that

$$d(z,m) \leq \max\{d(z,x), d(z,y)\}(1 - \delta(\max\{d(z,x), d(z,y)\}, d(x,y)))$$

is satisfied for every $z \in X$.

It is easy to see that if X is (p,c)-uniformly convex, then it is uniformly convex in the wider sense for modulus of convexity $\delta(r,t) = ct^p/(pr^p)$.

We will state two theorems which are closely related to the fixed point property of a complete metric space uniformly convex in the wider sense. The first one is concerning the existence and uniqueness of a circumcenter of every bounded subset. This property implies the fixed point property. Indeed, the circumcenter of a bounded orbit of the isometric action of G is fixed by G. This is well-known as the Bruhat-Tits theorem, which was mentioned at Corollary 4.

Theorem 8. *Let (X,d) be a complete metric space that is uniformly convex in the wider sense. Then every bounded subset $A \subset X$ has the unique circumcenter.*

Proof. For any $x, y \in X$, there is some $m \in X$ that satisfies the inequality for the definition of uniform convexity in the wider sense. We consider $r_m(A)$ for this m. For every $\varepsilon > 0$, there is $z_\varepsilon \in A$ such that $d(z_\varepsilon, m) \geq r_m(A) - \varepsilon$. Then we have

$$r_X(A) - \varepsilon \leq d(z_\varepsilon, m) \leq \max\{d(z_\varepsilon,x), d(z_\varepsilon,y)\}(1 - \delta(\max\{d(z_\varepsilon,x), d(z_\varepsilon,y)\}, d(x,y))).$$

Using property (3) of δ for $d(z_\varepsilon, x) \leq r_x(A)$, $r_y(A)$, and taking $\varepsilon \to 0$, we obtain an inequality

$$(*) \qquad r_X(A) \leq \max\{r_x(A), r_y(A)\}(1 - \delta(\max\{r_x(A), r_y(A)\}, d(x,y)))$$

for any $x, y \in X$.

The existence of a circumcenter of A is proved as follows. We may assume that $r_X(A) > 0$. Take a sequence $\{x_n\} \subset X$ such that $r_{x_n}(A) \to r_X(A)$ as $n \to \infty$. For every $k \in \mathbb{N}$, there is n_k such that $n \geq n_k$ implies $r_{x_n}(A) < r_X(A) + 1/k$. We apply inequality $(*)$ for

$x = x_n$ and $y = x_{n'}$ with $n,\ n' \geq n_k$. It turns out that

$$\begin{aligned} r_X(A) &\leq \max\{r_{x_n}(A), r_{x_{n'}}(A)\}(1 - \delta(\max\{r_{x_n}(A), r_{x_{n'}}(A)\}, d(x_n, x_{n'}))) \\ &\leq (r_X(A) + 1/k)(1 - \delta(r_X(A) + 1/k, d(x_n, x_{n'}))) \\ &\leq r_X(A) + \frac{1}{k} - r_X(A)\,\delta(r_X(A) + 1, d(x_n, x_{n'})). \end{aligned}$$

Here, the latter two estimates come again from property (3) of δ. This implies that

$$\delta(r_X(A) + 1, d(x_n, x_{n'})) \leq \frac{1}{k\, r_X(A)}$$

for any $n,\ n' \geq n_k$. Letting $k \to \infty$ and using properties (1) and (2) of δ, we see that $d(x_n, x_{n'}) \to 0$ as $n,\ n' \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is the limit $x_0 = \lim_{n\to\infty} x_n$ in X. By $r_{x_0}(A) = \lim_{n\to\infty} r_{x_n}(A) = r_X(A)$, we find that x_0 is a circumcenter of A.

The uniqueness is already seen from the above argument. Or, if x and y are circumcenters of A, then the substitution of $r_x(A) = r_X(A)$ and $r_y(A) = r_X(A)$ to $(*)$ gives $\delta(r_X(A), d(x, y)) = 0$. This is possible only when $x = y$. □

The second one is concerning normal structure and compactness of the family of admissible subsets. For the normal structure, we have only to modify the proof of Theorem 1. For the (countable) compactness, we refer to [6, Theorem 2.2] in the case of geodesic metric spaces.

Theorem 9. *If a metric space (X, d) is uniformly convex in the wider sense, then it has normal structure. If (X, d) is complete in addition, then the family $\mathcal{A}(X)$ of admissible subsets is compact.*

Proof. Take an arbitrary $A \in \mathcal{A}(X)$ with $d = \mathrm{diam}(A) > 0$. Then there are $x, y \in A$ such that $d(x, y) \geq d - \epsilon$ for an arbitrary $\epsilon > 0$. For these x and y, there is $m \in X$ that satisfies the inequality for uniform convexity in the wider sense. We will check that m belongs to A. Suppose that A is the intersection of closed metric balls $B(z_i, r_i)$ for all indices $i \in I$. Since $x, y \in A \subset B(z_i, r_i)$, we have $d(z_i, x) \leq r_i$ and $d(z_i, y) \leq r_i$ for each $i \in I$. It follows from the inequality that

$$d(z_i, m) \leq \max\{d(z_i, x), d(z_i, y)\} \leq r_i.$$

This implies that $m \in B(z_i, r_i)$ and hence $m \in A$.

Consider an arbitrary $z \in A$. Then $d(z, x) \leq d$ and $d(z, y) \leq d$. Substituting these bounds and $d(x, y) \geq d - \epsilon$ to the inequality and using the properties of δ, we obtain

$$d(z, m) \leq d(1 - \delta(d, d - \epsilon)) < d.$$

Thus we have $\mathrm{rad}(A) < \mathrm{diam}(A)$.

Now we assume that X is complete and consider a decreasing sequence of admissible subsets $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(X)$. For a fixed point $z \in X$, the distances $d(z, A_n)$ from z to A_n are bounded and increasing, so we have $R = \lim_{n\to\infty} d(z, A_n) < \infty$. Also, we can choose a point $x_n \in A_n$ for each $n \in \mathbb{N}$ such that $\lim_{n\to\infty} d(z, x_n) = R$. We will show that $\{x_n\}$ is a Cauchy sequence. Then there is the limit point $x_\infty \in X$ of $\{x_n\}$ since X is complete. Each A_n contains x_∞ because $x_{n'} \in A_n$ for every $n' \geq n$. Thus $x_\infty \in \bigcap_{n\in\mathbb{N}} A_n$, which shows that the intersection is not empty.

Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there is some $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there are $n_1, n_2 \geq n$ with $d(x_{n_1}, x_{n_2}) \geq \varepsilon$. We apply the inequality of uniform convexity in the wider sense to x_{n_1} and x_{n_2}; there is some $m_n \in X$ such that

$$d(z, m_n) \leq \max\{d(z, x_{n_1}), d(z, x_{n_2})\}(1 - \delta(\max\{d(z, x_{n_1}), d(z, x_{n_2})\}, d(x_{n_1}, x_{n_2})))$$

for every $z \in X$. By choosing the fixed z in particular for this inequality, we have

$$d(z, m_n) \leq R(1 - \delta(R, \varepsilon)) < R.$$

Here m_n is contained in A_n by the same reason as in the first paragraph. Hence $d(z, A_n) \leq d(z, m_n)$. Taking the limit as $n \to \infty$, we have a contradiction. This proves that $\{x_n\}$ is a Cauchy sequence.

We have shown that any decreasing sequence of admissible subsets $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(X)$ has non-empty intersection. This property is called countably compact. It was proved in [9] (see also [7, Theorem 5.5]) that if $\mathcal{A}(X)$ has normal structure then compactness and countable compactness of $\mathcal{A}(X)$ are equivalent. □

This theorem combined with Theorem 2 also implies that a complete metric space that is uniformly convex in the wider sense has the fixed point property.

Remark. We can modify the definition of uniformly convexity in the wider sense by changing the modulus of convexity δ to an increasing function

$$\hat{\delta} : [0, 2] \to [0, 1] \quad (\hat{\delta}(s) = 0 \Leftrightarrow s = 0)$$

and the inequality to

$$d(z, m) \leq \max\{d(z, x), d(z, y)\} \left(1 - \hat{\delta}\left(\frac{d(x, y)}{\max\{d(z, x), d(z, y)\}}\right)\right).$$

This condition is stronger than the previous one because $\delta(r, t) = \hat{\delta}(t/r)$ gives the implication. The above proof of Theorem 9 shows that if X is uniformly convex in this sense then $\mathcal{A}(X)$ has uniformly normal structure with the property $\mathrm{rad}(A) \leq (1 - \alpha)\,\mathrm{diam}(A)$ for $\alpha = \lim_{s\nearrow 1} \hat{\delta}(s)$. Note that if (X, d) is an ultrametric space where

$$d(x, y) \leq \max\{d(z, x), d(z, y)\}$$

is always satisfied, then the modulus of convexity is uniformly bounded by $\hat{\delta}(1)$.

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