NON-DIVERGENT INFINITELY DISCRETE TEICHMÜLLER MODULAR TRANSFORMATION

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Abstract. We consider a classification of Teichmüller modular transformations of an analytically infinite Riemann surface. If the orbit of a Teichmüller modular transformation diverges to the point at infinity (divergent type), then it has no accumulation points in the Teichmüller space (infinitely discrete type). In this note, we show that there is a Teichmüller modular transformation for some Riemann surface that is of infinitely discrete type but not of divergent type.

We consider the Teichmüller space $T(R)$ of an analytically infinite Riemann surface R and the Teichmüller modular group $\mathrm{Mod}(R)$ acting on $T(R)$, where $T(R)$ is not finite dimensional and $\mathrm{Mod}(R)$ is not finitely generated.

The Teichmüller space $T(R)$ is the set of all Teichmüller equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that $f_1 : R \to R_1$ and $f_2 : R \to R_2$ are Teichmüller equivalent if there exists a conformal homeomorphism $h : R_1 \to R_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to h relative to the ideal boundary at infinity of R_1. It is known that $T(R)$ is a complex Banach manifold. Also, it has a metric structure such that the distance between $p_1 = [f_1]$ and $p_2 = [f_2]$ is given by $d_T(p_1, p_2) = \log K(f)$, where $K(f)$ is the maximal dilatation of an extremal quasiconformal homeomorphism f in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$, which is called the Teichmüller distance. It is known that the Teichmüller distance on $T(R)$ is the same as its Kobayashi distance. See [3] and [4] for fundamental facts on Teichmüller spaces.

A quasiconformal mapping class is a homotopy class $[g]$ of quasiconformal automorphisms $g : R \to R$ relative to the ideal boundary at infinity of R. The quasiconformal mapping class group $\mathrm{MCG}(R)$ is the group of all quasiconformal mapping classes. Each $\gamma = [g] \in \mathrm{MCG}(R)$ acts on $T(R)$ from the left in such a way that $\gamma_* : [f] \mapsto [f \circ g^{-1}]$. It is evident from the definition that $\mathrm{MCG}(R)$ acts on $T(R)$ isometrically with respect to the Teichmüller distance d_T. Also, it acts biholomorphically on $T(R)$.

2000 *Mathematics Subject Classification.* Primary 30F60, Secondary 37F30.

Key words and phrases. Riemann surface, hyperbolic geometry, Teichmüller space, quasiconformal mapping class group.

Let $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ be the homomorphism defined by $\gamma \mapsto \gamma_*$, where $\mathrm{Aut}(T(R))$ denotes the group of all isometric biholomorphic automorphisms of $T(R)$. The image $\mathrm{Im}\,\iota \subset \mathrm{Aut}(T(R))$ is called the Teichmüller modular group and is denoted by $\mathrm{Mod}(R)$. Each element $\gamma_* \in \mathrm{Mod}(R)$ is called a Teichmüller modular transformation. Dynamics of the action of $\mathrm{Mod}(R)$ on $T(R)$ when R is analytically infinite has been studied since [2].

We classify the Teichmüller modular transformations into several types according to the behavior of their orbits.

Definition. A Teichmüller modular transformation $\gamma_* \in \mathrm{Mod}(R) - \{\mathrm{id}\}$ is of

(i) *divergent type* if the orbit $\langle \gamma_* \rangle(p) = \{\gamma_*^n(p)\}_{n \in \mathbb{Z}}$ diverges to the point at infinity of $T(R)$ as $n \to \pm\infty$;

(ii) *infinitely discrete type* if it is of infinite order and if the orbit $\langle \gamma_* \rangle(p)$ of each point $p \in T(R)$ has no accumulation points in $T(R)$;

(iii) *unbounded type* if the orbit $\langle \gamma_* \rangle(p)$ is unbounded in $T(R)$.

Conditions (i) and (iii) are independent of the choice of $p \in T(R)$.

Note that the type of a Teichmüller modular transformation $\gamma_* \in \mathrm{Mod}(R) - \{\mathrm{id}\}$ is consistent in $\langle \gamma_* \rangle - \{\mathrm{id}\}$ ([5, Theorem 1]). Namely, γ_* is of infinitely discrete (divergent, unbounded, respectively) type if and only if so is γ_*^m for some and every $m \in \mathbb{Z} - \{0\}$.

It is evident from the definition that, if $\gamma_* \in \mathrm{Mod}(R)$ is of divergent type, then it is of infinitely discrete type. Furthermore it was proved in [5, Theorem 3] that if $\gamma_* \in \mathrm{Mod}(R)$ is of infinitely discrete type, then it is of unbounded type. Thus we have the following inclusion relations for the classes of Teichmüller modular transformations:

$$\{\text{divergent type}\} \subset \{\text{infinitely discrete type}\} \subset \{\text{unbounded type}\}.$$

In [5, Theorem 4], it was proved that the second inclusion is proper. In this note, we will prove that the first inclusion is also proper. This was mentioned in [5], and an example was given in [6, Example 2.10] without proof.

Theorem 1. *There exists a Teichmüller modular transformation $\gamma_* \in \mathrm{Mod}(R)$ for some Riemann surface R that is of infinitely discrete type but not of divergent type.*

The construction of the Riemann surface R as in this theorem is given below. Let S be a closed Riemann surface of genus 3 and we take 3 mutually disjoint non-dividing simple closed geodesics a, b and c. We cut S along a and b to make a totally geodesic surface S' of genus 1 with 4 boundary components and give a pants decomposition for S' having a, b and c as boundary geodesics. We prepare infinitely many copies of S' and paste them to make an abelian covering surface R_0 of S with the covering transformation group isomorphic to $\mathbb{Z}^2$. Then we index all the lifts of c to R_0 in such a way that c_{ij} is the image of some fixed lift c_{00} under the covering transformation corresponding to $(i,j) \in \mathbb{Z}^2$. We extend the pants decomposition of S' to R_0 so that the action of the covering transformation group $\mathbb{Z}^2$ preserves this decomposition.

By assigning the geodesic lengths $\ell(c_{ij})$ to each c_{ij} and keeping the lengths of the other boundary geodesics of the pants decomposition invariant, we construct our Riemann surface R. This is performed by a locally quasiconformal deformation but it is not necessarily globally quasiconformal. In our purpose, we define

$$\ell(c_{ij}) = \exp\left\{-2^{|i|+1}\, h(2^{-|i|(|i|+1)/2} j)\right\},$$

where h is a periodic function of period one defined on $\mathbb{R}$ such that $h(x) = x$ for $0 \leq x \leq 1/2$ and $h(x) = 1 - x$ for $1/2 \leq x \leq 1$.

Take a mapping class $[g] \in \mathrm{MCG}(R)$ corresponding to the element of the covering transformation $j \mapsto j+1$ in $\mathbb{Z}^2$. Then, we will see that $[g]$ gives a Teichmüller modular transformation $\gamma_* \in \mathrm{Mod(R)}$ such that $\langle \gamma_* \rangle$ acts discontinuously on $T(R)$. We will also see that there is a subsequence $\{n_k\}$ such that $\{\gamma_*^{n_k}(o)\}$ is bounded for the basepoint $o = [\mathrm{id}] \in T(R)$.

We use the following elementary fact on a bilateral sequence.

Proposition 2. *Let $\{a_j\}_{j\in\mathbb{Z}}$ be a bilateral sequence of real numbers and T a positive integer that is a multiple of 4. For a positive constant $L > 0$, assume that a_j satisfies $a_j \geq L + a_0$ if $T/4 \leq j \leq 3T/4 \pmod T$. Then, for every integer $m \neq 0 \pmod T$, there exists some integer k such that*

$$|a_{k+m} - a_k| \geq 4L/T$$

holds.

Proof. Suppose to the contrary that there is some $m \neq 0 \pmod T$ such that

$$|a_{k+m} - a_k| < 4L/T$$

for every integer k. Let $t \geq 1$ be the smallest positive integer satisfying

$$T/4 \leq tm \leq 3T/4 \pmod T.$$

Then $t \leq T/4$ always holds. For $k = 0, m, \ldots, (t-1)m$, we apply the above inequality, which yields

$$|a_{tm} - a_0| < t \cdot 4L/T \leq L.$$

However, this contradicts the assumption on $\{a_j\}_{j\in\mathbb{Z}}$. □

First, we show that γ_* is of infinitely discrete type.

Lemma 3. *For the cyclic group $\langle \gamma_* \rangle$ generated by $\gamma_* \in \mathrm{Mod(R)}$, the orbit $\langle \gamma_* \rangle(p)$ of each point $p \in T(R)$ is a discrete set in $T(R)$.*

Proof. Consider an arbitrary point $p \in T(R)$ and set $d_T(o,p) = \log K$. Let $i_0(p)$ be the smallest integer $i_0 \geq 1$ such that $\log K \leq 2^{i_0-2}$. We will show that $d_T(\gamma_*^n(p),p)$ is uniformly bounded away from zero for all $n \in \mathbb{Z} - \{0\}$. Since $d_T(\gamma_*^{-n}(p),p) = d_T(\gamma_*^n(p),p)$, we have only to show this for $n \geq 1$. Represent each integer $n \geq 1$ uniquely by

$$n = \sum_{i=0}^{\infty} a_i \cdot 2^{i(i+1)/2} \qquad (0 \leq a_i \leq 2^{i+1}-1).$$

Here, the summation is actually a finite sum. Let $i(n)$ be the smallest integer $i \geq 1$ such that $a_i \neq 0$.

Set $l_p(i,j) = -\log \ell_p(c_{ij})$ for $p \in T(R)$ and for $(i,j) \in \mathbb{Z}^2$. Consider the i-th row for $i > i_0(p)$. As a function of $j \in \mathbb{Z}$, $l_o(i,j)$ is periodic with prime period $\theta(i) = 2^{i(i+1)/2}$. Then

$$\begin{aligned} &l_o(i,j) = 0 \quad \text{for} \quad j = 0 \ (\mathrm{mod}\ \theta(i)); \\ &l_o(i,j) \geq 2^i/2 \quad \text{for} \quad \theta(i)/4 \leq j \leq 3\theta(i)/4 \ (\mathrm{mod}\ \theta(i)). \end{aligned}$$

Since

$$|l_p(i,j) - l_o(i,j)| \leq \log K \leq 2^{i_0(p)-2},$$

by Sorvali [7] and Wolpert [8], we have

$$\begin{aligned} &l_p(i,j) \leq 2^{i_0(p)-2} \quad \text{for} \quad j = 0 \ (\mathrm{mod}\ \theta(i)); \\ &l_p(i,j) \geq 2^i/2 - 2^{i_0(p)-2} \quad \text{for} \quad \theta(i)/4 \leq j \leq 3\theta(i)/4 \ (\mathrm{mod}\ \theta(i)). \end{aligned}$$

Then we define the difference of these values by

$$v(i,p) = 2^i/2 - 2^{i_0(p)-2} - 2^{i_0(p)-2} = 2^{i_0(p)-1}(2^{i-i_0(p)} - 1)$$

for each integer $i > i_0(p)$.

Case 1: $i(n) \leq i_0(p)$. Consider the i-th row for $i = i_0(p)+1$. We apply Proposition 2 for $a_j = l_p(i,j)$, $T = \theta(i)$ and $L = v(i,p)$. Then there is some k such that

$$|l_p(i,k+m) - l_p(i,k)| \geq 4L/T$$

for all $m \neq 0$ (mod T). Since $n \neq 0$ (mod T) by $i(n) < i$, we can apply this estimate for $m = n$. Then

$$\begin{aligned} d_T(\gamma^n(p),p) &\geq |l_p(i,k+n) - l_p(i,k)| \\ &\geq 4 \cdot 2^{i_0(p)-1}/2^{(i_0(p)+1)(i_0(p)+2)/2} \\ &= 2^{-i_0(p)(i_0(p)+1)/2}. \end{aligned}$$

Case 2: $i(n) \geq i_0(p)+1$. Consider the i-th row for $i = i(n)+1$. We apply Proposition 2 for $a_{j'} = l_p(i,\theta(i(n))j')$, $T = \theta(i)/\theta(i(n))$ and $L = v(i,p)$. Then there is some $k = \theta(i(n))k'$ such that

$$|l_p(i,\theta(i(n))(k'+m')) - l_p(i,\theta(i(n))k')| \geq 4L/T$$

for all $m' \neq 0 \pmod T$. Since $n' \neq 0 \pmod T$ for $n = \theta(i(n))n'$ due to the definition of $i(n)$, we can apply this estimate for $m' = n'$. Then

$$\begin{aligned} d_T(\gamma_*^n(p), p) &\geq |l_p(i, k+n) - l_p(i,k)| \\ &\geq 4 \cdot 2^{i_0(p)-1} \cdot 2^{i(n)-i_0(p)+1}/2^{i(n)+1} \\ &= 2. \end{aligned}$$

By dividing the arguments into the above two cases, we have seen that $d_T(\gamma_*^n(p), p)$ is uniformly bounded away from zero for all $n \in \mathbb{Z} - \{0\}$. By isometric group invariance, this implies that the orbit $\langle \gamma_* \rangle(p)$ is a discrete set in $T(R)$. □

Next, we show that γ_* is not of divergent type.

Lemma 4. *The orbit $\langle \gamma_* \rangle(o)$ of the basepoint $o \in T(R)$ does not diverge to the infinity. More precisely, $\{\gamma_*^n(o) \mid n = 2^{m(m+1)/2},\ m \geq 1\}$ is a bounded set in $T(R)$.*

Proof. By its construction, the Riemann surface R has a pants decomposition that is invariant under the action of the mapping class $[g] \in \mathrm{MCG}(R)$. We consider a pair of pants P_{ij} for each $(i,j) \in \mathbb{Z}^2$ that has c_{ij} as a geodesic boundary component. (There are two such pairs of pants for each (i,j).) The lengths of the other geodesic boundary components c' and c'' of P_{ij} are mutually the same for all (i,j).

The mapping class $[g^n]$ for each $n \in \mathbb{Z}$ is realized by sending each pair of pants in the invariant pants decomposition to the corresponding pair of pants quasiconformally in such a way that the boundary geodesics are mapped linearly with respect their length parameters and without twists. Note that the lengths of geodesic boundaries of the pants decomposition are uniformly bounded and can be taken small enough if necessary. Then, by Bishop [1], the maximal dilatation of the quasiconformal homeomorphism of each pair of pants can be estimated above in terms of the ratio of the geodesic length $\ell(c_{ij})$ of c_{ij} to that of the image of c_{ij}. Therefore, $d_T(\gamma_*^n(o), o)$ is estimated above by the supremum of the difference of $\log \ell(c_{ij})$ and $\log \ell(g^n(c_{ij}))$ taken over all $(i,j) \in \mathbb{Z}^2$.

Set $l(i,j) = -\log \ell(c_{ij})$ as before and consider $|l(i,j+n) - l(i,j)|$ for $n = 2^{m(m+1)/2}$ with $m \geq 1$. If $|i| \leq m$, then $l(i,j+n) = l(i,j)$. Regarding $l(i,j) = l(-i,j)$, we have only to consider the case for $i > m$. Then an elementary observation using the period $\theta(i) = 2^{i(i+1)/2}$ on the i-th row shows that

$$\begin{aligned} |l(i,j+n) - l(i,j)| &\leq 2^{i+1} n/\theta(i) \\ &\leq 2^{i+1} 2^{m(m+1)/2}/2^{i(i+1)/2} \\ &\leq 2 \end{aligned}$$

for all integer j. This implies that $d_T(\gamma_*^n(o), o)$ is uniformly bounded by some constant for all $n = 2^{m(m+1)/2}$. □

Proof of Theorem 1. We have already done. □

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