THE NIELSEN REALIZATION PROBLEM FOR ASYMPTOTIC TEICHMÜLLER MODULAR GROUPS

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Dedicated to Professor Masahiko Taniguchi on the occasion of his 60th birthday

Abstract. Under a certain geometric assumption on a hyperbolic Riemann surface, we prove an asymptotic version of the fixed point theorem for the Teichmüller modular group, which asserts that every finite subgroup of the asymptotic Teichmüller modular group has a common fixed point in the asymptotic Teichmüller space. For its proof, we use a topological characterization of the asymptotically trivial mapping class group, which has been obtained in the authors' previous paper but a simpler argument is given here. As a consequence, every finite subgroup of the asymptotic Teichmüller modular group is realized as a group of quasiconformal automorphisms modulo coincidence near the infinity. Furthermore, every finite subgroup of a certain geometric automorphism group of the asymptotic Teichmüller space is realized as an automorphism group of the Royden boundary of the Riemann surface. These results can be regarded as asymptotic versions of the Nielsen realization theorem.

1. Introduction and statement of results

We consider the group $\mathrm{QC}(R)$ of all quasiconformal automorphisms of a Riemann surface R, the quasiconformal mapping class group $\mathrm{MCG}(R)$ of all homotopy equivalence classes of the elements of $\mathrm{QC}(R)$, and the surjective homomorphism $q : \mathrm{QC}(R) \to \mathrm{MCG}(R)$. The *realization problem* of the mapping class group is asking whether there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for a given subgroup Γ of $\mathrm{MCG}(R)$. In particular, we refer to the realization problem for a finite subgroup of $\mathrm{MCG}(R)$ as the *Nielsen realization problem*.

For an analytically finite Riemann surface R (in other words, for a hyperbolic surface R of finite area), Kerckhoff [16] proved the fixed point theorem for a finite subgroup of the Teichmüller modular group $\mathrm{Mod}(R)$. It asserts that every finite subgroup of $\mathrm{Mod}(R)$ has a common fixed point in the Teichmüller space $T(R)$. This is equivalent to the statement that every finite subgroup of

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$\mathrm{MCG}(R)$ can be realized as a group of conformal automorphisms of the Riemann surface corresponding to the fixed point, and thus gives an affirmative answer to the Nielsen realization problem. The Nielsen realization problem is also true even for an analytically infinite Riemann surface. Indeed, a generalization of the fixed point theorem to analytically infinite Riemann surfaces follows from a result by Markovic [20].

In this paper, we consider an asymptotic version of the fixed point theorem for the asymptotic Teichmüller modular group. The *asymptotic Teichmüller space* $AT(R)$ is defined in a similar manner to the Teichmüller space by replacing conformal equivalence with asymptotically conformal equivalence. Every element of $\mathrm{MCG}(R)$ induces a biholomorphic automorphism of $AT(R)$, which gives a representation ι_{AT} from $\mathrm{MCG}(R)$ to the group $\mathrm{Aut}(AT(R))$ of all biholomorphic automorphisms of $AT(R)$. The *asymptotic Teichmüller modular group* $\mathrm{Mod}_{AT}(R)$ is defined to be the image of ι_{AT}. Our fixed point theorem can be stated as follows.

Theorem 2.5. *Every finite subgroup of* $\mathrm{Mod}_{AT}(R)$ *has a common fixed point in* $AT(R)$ *if* R *satisfies the weak bounded geometry condition.*

The (weak) bounded geometry condition is a condition of hyperbolic geometry on Riemann surfaces. Its definition is given in the next section. We will prove Theorem 2.5 in Section 4 after reviewing a topological characterization of the kernel of ι_{AT} as Theorem 3.2 in Section 3.

As a consequence of Theorem 2.5, we have two asymptotic versions of the Nielsen realization theorem. The first one is for the asymptotic Teichmüller modular group. The end quasiconformal automorphism group $\mathrm{QC}_e(R)$ is the group of all end equivalence classes of quasiconformal automorphisms of R. Here we say that two quasiconformal automorphisms of R are *end equivalent* if they are coincident outside some topologically finite subsurface of finite area in R. The projection is denoted by $e : \mathrm{QC}(R) \to \mathrm{QC}_e(R)$, which induces a surjective homomorphism $q_e : \mathrm{QC}_e(R) \to \mathrm{Mod}_{AT}(R)$. Then the realization problem for the asymptotic Teichmüller modular group is asking whether there exists a local section for q_e. The following theorem says that this problem is true for every finite subgroup of $\mathrm{Mod}_{AT}(R)$ under the same geometric assumption on R, and thus this can be regarded as an asymptotic version of the Nielsen realization theorem.

Theorem 5.2. *For every finite subgroup* $\hat{\Gamma}$ *of* $\mathrm{Mod}_{AT}(R)$*, there exists a homomorphism* $\mathcal{E}_e : \hat{\Gamma} \to \mathrm{QC}_e(R)$ *such that* $q_e \circ \mathcal{E}_e = \mathrm{id}|_{\hat{\Gamma}}$ *under the weak bounded geometry condition of* R*.*

In fact, every finite subgroup $\hat{\Gamma}$ of $\mathrm{Mod}_{AT}(R)$ can be realized as a subgroup of $\mathrm{QC}_e(R)$ whose elements are conformal outside some topologically finite subsurface of finite area in the Riemann surface corresponding to the fixed point. We demonstrate these arguments in Section 5.

Another asymptotic version of the Nielsen realization theorem is for the boundary geometric automorphism group of the asymptotic Teichmüller space. We consider the *Royden boundary* dR of R and its subset d_0R that is obtained by removing all boundary points corresponding to punctures of R. Every quasiconformal automorphism of R extends to a homeomorphism of dR and keeps d_0R invariant. Let $\mathrm{Homeo}^*(d_0R)$ denote the group of all homeomorphic automorphisms of d_0R that extend continuously to quasiconformal homeomorphisms of some neighborhoods of d_0R. This extension of an element of $\mathrm{Homeo}^*(d_0R)$, which is called a supporting map, induces a biholomorphic automorphism of $AT(R)$.

We define the boundary geometric automorphism group $\mathrm{Aut}^*(AT(R))$ as the group of all biholomorphic automorphisms of $AT(R)$ induced by supporting maps. Then we have a surjective homomorphism q_b from $\mathrm{Homeo}^*(d_0R)$ to $\mathrm{Aut}^*(AT(R))$ and the following commutative diagram, where i stands for the inclusion map:

$$\begin{array}{ccccc}
\mathrm{QC}(R) & \xrightarrow{e} & \mathrm{QC}_e(R) & \xrightarrow{\nu} & \mathrm{Homeo}^*(d_0R) \\
\Big\downarrow q & & \Big\downarrow q_e & & \Big\downarrow q_b \\
\mathrm{MCG}(R) \cong \mathrm{Mod}(R) & \xrightarrow{\iota_{AT}} & \mathrm{Mod}_{AT}(R) & \xrightarrow{i} & \mathrm{Aut}^*(AT(R))
\end{array}$$

The realization problem for the boundary geometric automorphism group is asking whether there exists a local section of the surjective homomorphism q_b for a given subgroup of $\mathrm{Aut}^*(AT(R))$. The following theorem states that this problem is true for every finite subgroup of $\mathrm{Aut}^*(AT(R))$ under the same geometric assumption on R as before.

Theorem 6.5. *For every finite subgroup* $\widetilde{\Gamma}$ *of* $\mathrm{Aut}^*(AT(R))$*, there exists a homomorphism* $\mathcal{E}_b : \widetilde{\Gamma} \to \mathrm{Homeo}^*(d_0R)$ *such that* $q_b \circ \mathcal{E}_b = \mathrm{id}|_{\tilde{\Gamma}}$ *under the weak bounded geometry condition of* R*.*

We will see this version of the realization theorem using the Royden boundary in Section 6. Finally in Section 7, we introduce the space of ends of R and deal with the pure mapping class group, which is defined to be a subgroup of $\mathrm{MCG}(R)$ consisting of all elements fixing each non-cuspidal end.

2. Statement of the fixed point theorem

In this section, we first review the fixed point theorem for the Teichmüller modular group and the Nielsen realization theorem for the quasiconformal mapping class group, especially for analytically infinite Riemann surfaces. Then we explain our fixed point theorem for the asymptotic Teichmüller modular group.

Throughout this paper, we assume that a Riemann surface R admits a hyperbolic structure, that is, R is represented as the quotient space $\mathbb{H}/H$ of the hyperbolic plane $\mathbb{H}$ by a torsion-free Fuchsian group H. Let $\mathrm{QC}(R)$ be the group of all quasiconformal automorphisms of R. A quasiconformal mapping class of R is the homotopy equivalence class $[g]$ of quasiconformal automorphisms $g \in \mathrm{QC}(R)$, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes of R. Here the homotopy is considered to be relative to the ideal boundary at infinity $\partial R = (\partial\mathbb{H} - \Lambda(H))/H$, where $\Lambda(H)$ is the limit set of H. The correspondence $g \mapsto [g]$ gives a surjective homomorphism $q : \mathrm{QC}(R) \to \mathrm{MCG}(R)$. The *realization problem* for the quasiconformal mapping class group $\mathrm{MCG}(R)$ is asking whether there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for a given subgroup Γ of $\mathrm{MCG}(R)$.

The quasiconformal mapping class group acts on the Teichmüller space. The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are equivalent if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R. Here the homotopy is again considered to be relative to the ideal boundary at infinity ∂R. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R^-, where R^- is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure. For details, see [17] and [26].

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller-Kobayashi distance. Let $\mathrm{Aut}(T(R))$ denote the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$, and we define the *Teichmüller modular group* by

$$\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R)).$$

We call an element of $\mathrm{Mod}(R)$ a Teichmüller modular transformation. It is proved in [2] that the homomorphism ι_T is injective (faithful) for all Riemann surfaces R of non-exceptional type. See also [6] and [23] for other proofs.

Here we say that a Riemann surface R is of exceptional type if R has finite hyperbolic area and satisfies $2g+n \leq 4$, where g is the genus of R and n is the number of punctures of R. It was a problem to determine the homomorphism ι_T is also surjective, especially for an analytically infinite Riemann surface. By the combination of the results of [1] and [19], this problem has been solved affirmatively, namely, $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. See also [9] for simplifying a part of the proof in a special case.

The *Nielsen realization problem* is the realization problem for a finite subgroup of $\mathrm{MCG}(R)$, namely, it is asking whether there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for a finite subgroup Γ of $\mathrm{MCG}(R)$. For a compact Riemann surface R, Nielsen himself proved that a finite cyclic subgroup Γ can be always realized, and after several partial solutions by Fenchel and Zieschang, Kerckhoff [16] finally proved the following fixed point theorem, which gives an affirmative answer to the Nielsen realization problem.

Theorem 2.1. *Let R be an analytically finite Riemann surface. Then a subgroup of* $\mathrm{Mod}(R)$ *is finite if and only if it has a common fixed point in $T(R)$.*

Since a Teichmüller modular transformation having a fixed point $p = [f] \in T(R)$ is realized as a conformal automorphism of the Riemann surface $R_p = f(R)$ corresponding to p, Theorem 2.1 is equivalent to the statement that, if a subgroup of $\mathrm{MCG}(R)$ is finite, then it can be realized as a group of conformal automorphisms of R_p (and vice versa). Since the groups $\mathrm{QC}(R)$ and $\mathrm{QC}(R_p)$ are quasiconformally conjugate, this implies that there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for every finite group Γ of $\mathrm{MCG}(R)$.

A generalization of Theorem 2.1 to analytically infinite Riemann surfaces follows from the result on uniformly quasisymmetric groups by Markovic [20].

Theorem 2.2. *Let R be a Riemann surface in general. For a subgroup G of* $\mathrm{Mod}(R)$*, the orbit $G(p)$ is bounded for some $p \in T(R)$ if and only if G has a common fixed point in $T(R)$. In particular, a finite subgroup of* $\mathrm{Mod}(R)$ *has a common fixed point in $T(R)$.*

Hence Theorem 2.2 implies the following statement and eventually the Nielsen realization problem is true for all Riemann surfaces.

Corollary 2.3. *Let R be a Riemann surface in general. Every finite subgroup Γ of* $\mathrm{MCG}(R)$ *can be realized as a group of conformal automorphisms of some Riemann surface that is quasiconformally equivalent to R. In particular, there exists a homomorphism $\mathcal{E} : \Gamma \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}|_\Gamma$ for every finite subgroup Γ of* $\mathrm{MCG}(R)$*.*

Note that the realization problem is not true for the whole quasiconformal mapping class group. Indeed, for a compact Riemann surface R of genus $g \geq 2$, there exists no homomorphism $\mathcal{E} : \mathrm{MCG}(R) \to \mathrm{QC}(R)$ such that $q \circ \mathcal{E} = \mathrm{id}$. See [21] and [22].

In what follows, we consider an asymptotic version of the fixed point theorem for the asymptotic Teichmüller modular group. The asymptotic Teichmüller space has been introduced in [15] for the hyperbolic plane and in [3] and [4] for an arbitrary Riemann surface. We say that a quasiconformal homeomorphism f of R is *asymptotically conformal* if, for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 of R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R relative to the ideal boundary at infinity ∂R. The *asymptotic Teichmüller space* $AT(R)$ of R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f of R.

The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Since a conformal homeomorphism is asymptotically conformal, there is a projection $\alpha : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex structure such that α is holomorphic. See also [5] and [14].

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to the asymptotic Teichmüller distance. See [4]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$, and we define the *asymptotic Teichmüller modular group* (the geometric automorphism group of $AT(R)$) by

$$\mathrm{Mod}_{AT}(R) = \iota_{AT}(\mathrm{MCG}(R)).$$

We call an element of $\mathrm{Mod}_{AT}(R)$ an asymptotic Teichmüller modular transformation. It is different from the case of the representation $\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_{AT} is not injective, namely, $\mathrm{Ker}\, \iota_{AT} \neq \{[\mathrm{id}]\}$ unless R is either the unit disc or the once-punctured disc ([2]). We call an element of $\mathrm{Ker}\, \iota_{AT}$ asymptotically trivial and call $\mathrm{Ker}\, \iota_{AT}$ the *asymptotically trivial mapping class group*.

To formulate our results, we need to introduce the following geometric conditions on the hyperbolic structure of an analytically infinite Riemann surface. Hereafter, $\dot{R}$ denotes the non-cuspidal part of R obtained by removing all horocyclic cusp neighborhoods of area one. Also, for a constant $M > 0$, R_M denotes the set of all points x in R satisfying a property that there exists a homotopically non-trivial and non-cuspidal closed curve based at x whose hyperbolic length is less than M.

Definition 2.4. We say that a Riemann surface R satisfies the *bounded geometry condition* if the following three conditions are fulfilled:

(i) *lower bound condition*: there exists a constant $m > 0$ such that, for every point $x \in \dot{R}$, every homotopically non-trivial curve based at x has hyperbolic length greater than or equal to m;

(ii) *upper bound condition*: there exist a constant $M > 0$ and a connected component R_M^0 of R_M such that inclusion map $R_M^0 \to R$ induces a surjective homomorphism $\pi_1(R_M^0) \to \pi_1(R)$.

(iii) R has no ideal boundary at infinity ∂R.

Moreover, we say that a Riemann surface R satisfies the *weak bounded geometry condition* if only two conditions (i) and (ii) above are satisfied, namely, R may have an ideal boundary at infinity.

Every non-universal normal cover of an analytically finite Riemann surface satisfies the bounded geometry condition and every non-universal normal cover of a topologically finite Riemann surface satisfies the weak bounded geometry condition. Moreover, if the convex core of a Riemann surface R admits such pants decomposition that the diameter of the non-cuspidal part of each pair of pants (possibly degenerate) is uniformly bounded, then R satisfies the weak bounded geometry condition. The bounded geometry condition is preserved under quasiconformal homeomorphisms. Thus, this can be regarded as a condition for the Teichmüller space.

In [12], we have proved that every element of $\mathrm{Mod}_{AT}(R)$ of finite order has a fixed point in $AT(R)$ if R satisfies a certain stronger bounded geometry condition. Our main theorem in this paper is an extension of this result to the following asymptotic version of the fixed point theorem.

Theorem 2.5. (Fixed point theorem) *Let R be a Riemann surface satisfying the weak bounded geometry condition. Then every finite subgroup of* $\mathrm{Mod}_{AT}(R)$ *has a common fixed point in $AT(R)$.*

We prove Theorem 2.5 in Section 4 after some preparation in Section 3.

3. Topological characterization of the asymptotically trivial mapping class group

For the proof of Theorem 2.5, a topological characterization of the asymptotically trivial mapping class group plays a central role. To state this result and give a proof for it, we begin with several definitions.

By a subsurface of a Riemann surface R, we mean a surface possibly with boundary in R. We say that a subsurface V of R is topologically finite if its fundamental group is finitely generated and, *in addition*, if each component of the relative boundary $\partial V \subset R$ is homeomorphic to a circle.

For a homotopically non-trivial and non-cuspidal simple closed curve c on a hyperbolic Riemann surface R, let c_* be the unique simple closed geodesic that is freely homotopic to c. For a subsurface V of R whose relative boundary ∂V consists of simple closed curves, let V_* be a subsurface of R each of whose relative boundary components is the simple closed geodesic that is freely homotopic to the corresponding component of ∂V. We call such V_* a *geodesic subsurface*. Remark that, if a relative boundary component of ∂V is homotopically trivial or cuspidal, then we assume that the corresponding component of ∂V_* degenerates.

We consider the following subgroup of the quasiconformal mapping class group.

Definition 3.1. The *stable quasiconformal mapping class group* $G_\infty(R)$ is a subgroup of $\mathrm{MCG}(R)$ consisting of all essentially trivial mapping classes $[g]$ of a Riemann surface R. Here $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* (or trivial near infinity) if there exists a topologically finite subsurface V of finite area in R such that, for each connected component U of $R - V$, the restriction $g|_U : U \to R$ is homotopic to the inclusion map $\mathrm{id}|_U : U \hookrightarrow R$ relative to the ideal boundary at infinity.

It is obvious from definition that $G_\infty(R) \subset \operatorname{Ker} \iota_{AT}$. This inclusion is not necessarily equality, in general. See [10, Remark 4.1] for the difference. However, under the weak bounded geometry condition, we have the equality, which gives the topological characterization of $\operatorname{Ker} \iota_{AT}$.

Theorem 3.2. *Let R be a Riemann surface satisfying the weak bounded geometry condition. Then* $\operatorname{Ker} \iota_{AT} = G_\infty(R)$.

In [12], we have proved this theorem as a consequence of several other related theorems. In this section, we give a rather direct proof for it by summarizing necessary results proved in [12]. Remark that, in the previous paper, we used the upper bound condition in a stronger sense that every point of R has uniformly bounded injectivity radius (and hence there is no ideal boundary at infinity) and imposed this condition on R together with the lower bound condition. This is because of avoiding inessential complexity in the entire arguments. However, only to prove Theorem 3.2, it is enough to assume the upper bound condition introduced in Section 2. Actually, it is easy to verify that Lemma 4.9 in [12], which is used in the proof of Proposition 3.3 below, is still valid under this upper bound condition.

In this paper (with a slight difference from [12]), a *thetaframe* $X = (\vec{c}, \eta)$ in a Riemann surface R is defined by an oriented simple closed geodesic $\vec{c}$ together with a non-degenerate oriented simple geodesic arc η connecting $\vec{c}$ to itself perpendicularly and having no intersection with $\vec{c}$ except for its end points. Furthermore, we say that $X = (\vec{c}, \eta)$ is a D-thetaframe for a constant $D > 0$ if the hyperbolic lengths of $\vec{c}$ and η are not greater than D. For a quasiconformal homeomorphism f of R onto another Riemann surface R' and for a thetaframe $X = (\vec{c}, \eta)$ in R, we denote by $f(X)_*$ the thetaframe in R' that is homotopic to $f(X) = (f(\vec{c}), f(\eta))$. More precisely, the thetaframe $f(X)_* = (f(\vec{c})_*, f(\eta)_*)$ consists of the oriented simple closed geodesic $f(\vec{c})_*$ freely homotopic to $f(\vec{c})$ and the oriented geodesic arc $f(\eta)_*$ defined as follows. Let $f(x_1)$ and $f(x_2)$ be the end points of $f(\eta)$ in $f(\vec{c})$ and let H_t $(0 \leq t \leq 1)$ be a homotopy sending $f(\vec{c})$ to $f(\vec{c})_*$. Then $H_t(f(x_i))$ defines an arc s_i from $f(x_i)$ to a point in $f(\vec{c})_*$ for $i = 1, 2$. The geodesic arc $f(\eta)_*$ connects $f(\vec{c})_*$ to itself in the homotopy class of $s_1^{-1} \cdot f(\eta) \cdot s_2$ with the end points throughout on $f(\vec{c})_*$.

The next proposition, which has been essentially proved in [12], ensures that we can always take a thetaframe that is not fixed by a non-trivial quasiconformal mapping class.

Proposition 3.3. *Let R be a Riemann surface satisfying the weak bounded geometry condition, and U_* a topologically infinite geodesic subsurface of R. Let g be a quasiconformal automorphism of R such that the restriction $g|_{U_*}$ is not homotopic to the inclusion map $U_* \hookrightarrow R$, where the homotopy is not necessarily relative to the ideal boundary at infinity. Then there exists a D-thetaframe X in the B-neighborhood of U_* such that $g(X)_* \neq X$. Here D and B are constants depending only on the constants m and M for the lower and upper bound conditions.*

Proof. We choose a point x in $U_* \cap R_M^0$ sufficiently far away from ∂U_* and a non-trivial, non-cuspidal simple loop c_x based at x whose length is less than M. Clearly c_x is contained in R_M^0. Let c be the simple closed geodesic freely homotopic to c_x. Then the length of c is between m and M, and c is also contained in R_M^0. Moreover, there is a constant $B > 0$ depending only on m and M such that the distance between c_x and c is not greater than B. Since x is sufficiently far away from ∂U_*, we may assume that c is contained in U_*. By [8, Proposition 3.1], giving an orientation to c and choosing a suitable geodesic arc η, we have a D-thetaframe $X = (\vec{c}, \eta)$ for a constant $D > 0$ depending only on m and M. We may also assume that X is contained in U_*.

If $g(X)_* \neq X$, then the thetaframe X is a desired one. If $g(X)_* = X$, then applying [12, Lemma 4.9] and modifying the obtained frame to a thetaframe (see [12, Proposition 4.3]), we can find a $\widetilde{D}$-thetaframe $\widetilde{X}$ in the $\widetilde{B}$-neighborhood of U_* such that $g(\widetilde{X})_* \neq \widetilde{X}$. Here the constants $\widetilde{D}$ and $\widetilde{B}$ again depend only on m and M. □

The following proposition is crucial for our proof of Theorem 3.2. This has been proved in [12, Lemma 7.3].

Proposition 3.4. *Let R be a Riemann surface satisfying the lower bound condition, and $\{c_n\}_{n=1}^{\infty}$ a sequence of mutually disjoint simple closed geodesics on R such that the hyperbolic lengths of c_n are uniformly bounded from above. Let $[g] \in \mathrm{MCG}(R)$ be a quasiconformal mapping class such that $g(c_n)_* \neq c_{n'}$ for any n and n'. Then $[g] \notin \mathrm{Ker}\,\iota_{AT}$.*

Now we are ready to give the proof for our theorem.

Proof of Theorem 3.2. Let R be a Riemann surface satisfying the weak bounded geometry condition for the constants m and M. We may assume that R is topologically infinite. We take a quasiconformal mapping class $[g]$ that does not belong to $G_\infty(R)$, and prove that $[g] \notin \mathrm{Ker}\,\iota_{AT}$. The assumption $[g] \notin G_\infty(R)$ in particular implies that, for each compact subsurface V of R, there exists a topologically infinite connected component U of $R - V$ such that the restriction $g|_{U_*}$ to the geodesic subsurface U_* is not homotopic to the inclusion map $\mathrm{id}|_{U_*} : U_* \hookrightarrow R$ relative to the ideal boundary at infinity.

First, we consider the case where R has no ideal boundary at infinity. Then by Proposition 3.3, there exists a D-thetaframe X in the B-neighborhood of U_* such that $g(X)_* \neq X$. Let $\{R_n\}_{n=1}^{\infty}$ be a regular exhaustion of R. Namely, $\{R_n\}_{n=1}^{\infty}$ is an increasing sequence of compact subsurfaces R_n such that $R = \bigcup_{n=1}^{\infty} R_n$ and each connected component of the complement $R - R_n$ is not relatively compact. Then, for each $n \geq 1$, there exists a D-thetaframe $X_n = (\vec{c}_n, \eta_n)$ in the B-neighborhood of $(U_n)_*$ for some topologically infinite connected component U_n of $R - R_n$ such that $g(X_n)_* \neq X_n$.

We will choose a sequence of simple closed geodesics $\{\tilde{c}_n\}_{n=1}^{\infty}$ of uniformly bounded lengths tending to the infinity of R (that is, escaping from each compact subsurface of R) and satisfying $g(\tilde{c}_n)_* \neq \tilde{c}_n$ as follows. Fix n. For the thetaframe $X_n = (\vec{c}_n, \eta_n)$, if the non-oriented simple closed curve c_n underlying $\vec{c}_n$ satisfies $g(c_n)_* \neq c_n$, then we just set $\tilde{c}_n = c_n$. Hence, hereafter, we only consider the case where $g(c_n)_* = c_n$. The assumption $g(X_n)_* \neq X_n$ implies $g(\eta_n)_* \neq \eta_n$ in this case. The two end points of η_n on c_n divide c_n into two subarcs α_n^1 and α_n^2. Let c_n^i be a closed curve $\alpha_n^i \cdot \eta_n$ for each $i = 1, 2$, which has the length bounded by $2D$. If one of $\{c_n^i\}_{i=1,2}$, say c_n^1, is freely homotopic to a simple closed geodesic $(c_n^1)_*$, then we have $g(c_n^1)_* \neq (c_n^1)_*$ by $g(\eta_n)_* \neq \eta_n$. We set $\tilde{c}_n = (c_n^1)_*$ in this case.

Now we only deal with the case where each of $\{c_n^i\}_{i=1,2}$ is cuspidal, that is, freely homotopic to a simple closed curve around a puncture. Then a connected component of $R - c_n$ must be a twice punctured disk P_n. Consider a half-collar A_n of c_n in $R - P_n$, which is a ring domain within some distance ω from c_n. By the bounded geometry condition for R, the maximal width ω for such A_n is uniformly bounded by a constant $D' > 0$ depending only on m and M. See [8, Proposition 3.1] for this argument. This implies that the length of the shortest geodesic arc η_n' connecting c_n to itself in $R - P_n$ is bounded by

$2D'$. Then we take the other thetaframe $X'_n = (\vec{c}_n, \eta'_n)$ and make the simple closed curves ${c'}^i_n = \alpha^i_n \cdot \eta'_n$ in the same way. Remark that, in this case, at least one of $\{{c'}^i_n\}_{i=1,2}$ must be non-cuspidal, for otherwise, R would be a four-times punctured sphere.

If $g(X'_n)_* = X'_n$, then we choose a simple closed curve of length not greater than $2(D + D')$, which is made of η_n and η'_n together with the subarcs in c_n connecting the end points of η_n and η'_n suitably. We set the geodesic realization of this simple closed curve by $\tilde{c}_n$. Then $g(\tilde{c}_n)_* \neq \tilde{c}_n$. If $g(X'_n)_* \neq X'_n$, then one of $\{{c'}^i_n\}_{i=1,2}$ is freely homotopic to a simple closed geodesic $({c'}^i_n)_*$, which is defined to be $\tilde{c}_n$. Thus we have checked all possibilities.

By passing to a subsequence, we may assume that the sequence $\{\tilde{c}_n\}_{n=1}^{\infty}$ of simple closed geodesics are mutually disjoint as well as each $g(\tilde{c}_n)_*$ is disjoint from $\tilde{c}_{n'}$ for every $n' \neq n$. Also by $g(\tilde{c}_n)_* \neq \tilde{c}_n$, we see that $g(\tilde{c}_n)_* \neq \tilde{c}_{n'}$ for any n and n'. Then we apply Proposition 3.4 to conclude that $[g] \notin \operatorname{Ker} \iota_{AT}$.

Next, we consider the case where R has ideal boundary at infinity ∂R. If there exists a D-thetaframe X in the B-neighborhood of U_* such that $g(X)_* \neq X$, then the proof can be carried out in the same way as above. If there exists no such thetaframe, then Proposition 3.3 implies that $g|_{U_*}$ is homotopic to the inclusion map $\mathrm{id}|_{U_*} : U_* \hookrightarrow R$. However, since $g|_{U_*}$ is not homotopic to the inclusion map relative to the ideal boundary at infinity, g is not the identity on ∂R. Then we apply the following lemma to complete the proof. $\square$

Lemma 3.5. *Let R be a Riemann surface with ideal boundary at infinity ∂R. Then a mapping class $[g] \in \mathrm{MCG}(R)$ induced by a quasiconformal automorphism $g \in \mathrm{QC}(R)$ that is not the identity on ∂R is not asymptotically trivial.*

Proof. Suppose to the contrary that $[g] \in \operatorname{Ker} \iota_{AT}$. Since $g|_{\partial R} \neq \mathrm{id}$, there is some $x_0 \in \partial R$ such that $g(x_0) \neq x_0$. Choose an open interval $I \subset \partial R$ containing x_0 such that $g(I) \cap I = \emptyset$. By the correspondence under a universal covering map of R, we may identify I with an interval $(-1, 1)$ in $\mathbb{R}$ and x_0 with 0. Consider a quasiconformal homeomorphism $f : R \to R$ such that f is the identity on $\partial R - I$ and $f : I \to I$ is defined by $f(x) = \frac{3}{2}x + \frac{1}{2}$ on $(-1, 0]$ and $f(x) = \frac{1}{2}x + \frac{1}{2}$ on $[0, 1)$.

The symmetric ratio

$$m_t(f, x_0) = \frac{f(t) - f(0)}{f(0) - f(-t)}$$

for f locally defined at $x_0 \in I$ satisfies $\lim_{t \to 0} m_t(f, x_0) = \frac{1}{3}$. On the other hand, since $[g]$ is assumed to be asymptotically trivial, both g and fgf^{-1} are homotopic to asymptotically conformal automorphisms of R relative to ∂R. Then, by [7] (see also [15]), the symmetric ratios for g and fgf^{-1} tend to 1 everywhere on ∂R as $t \to 0$. In particular,

$$\lambda = \lim_{t\to 0} m_t(fgf^{-1}, fg^{-1}(x_0)) = 1.$$

Since $g^{-1}(x_0) \notin I$ and f is the identity outside I, we see that λ is actually equal to $\lim_{t\to 0} m_t(fg, g^{-1}(x_0))$. However, by using $\lim_{t\to 0} m_t(g, g^{-1}(x_0)) = 1$ as well as the piecewise linearity of f, we obtain a contradicting value $\lambda = 1/3$. This completes the proof. □

4. Proof of the fixed point theorem

In this section, we prove the fixed point theorem on the asymptotic Teichmüller space.

Proof of Theorem 2.5. We take an arbitrary finite subgroup $\hat{\Gamma}$ of $\mathrm{Mod}_{AT}(R)$ and number all its elements by $\{\hat{\gamma}_i\}_{i=1}^n$, where $\hat{\gamma}_1 = [\mathrm{id}]_{**}$. For each i, we choose a quasiconformal mapping class $[g_i] \in \mathrm{MCG}(R)$ ($[g_1] = [\mathrm{id}]$) such that $\iota_{AT}([g_i]) = \hat{\gamma}_i$ and fix it. For any i and j, there exists a unique $k = k(i,j)$ such that $\hat{\gamma}_i\hat{\gamma}_j = \hat{\gamma}_k$. Then an asymptotically trivial mapping class $[h_{ij}] \in \mathrm{Ker}\,\iota_{AT}$ is so determined that $[g_i][g_j] = [h_{ij}][g_k]$.

Since R satisfies the weak bounded geometry condition, we have $\mathrm{Ker}\,\iota_{AT} = G_\infty(R)$ by Theorem 3.2. Then $[h_{ij}]$ for any i and j is an essentially trivial mapping class. Thus there exists a topologically finite subsurface V_{ij} of finite area in R such that the restriction $h_{ij}|_{R-V_{ij}} : R - V_{ij} \to R$ is homotopic to the inclusion map $\mathrm{id}|_{R-V_{ij}} : R - V_{ij} \hookrightarrow R$ relative to the ideal boundary at infinity. We take a topologically finite subsurface V_0 of finite area so that all V_{ij} ($1 \le i, j \le n$) are contained in V_0. Then, the restriction $h_{ij}|_{R-V_0} : R - V_0 \to R$ is homotopic to the inclusion map $\mathrm{id}|_{R-V_0} : R - V_0 \hookrightarrow R$ relative to the ideal boundary at infinity for any i and j.

Take a representative g_i of each $[g_i]$ and choose a topologically finite subsurface V of finite area that contains all $g_ig_j(V_0)$ ($1 \le i, j \le n$). In particular, V satisfies $g_i(V) \cap V \neq \emptyset$ for every representative g_i.

Take the union

$$\widetilde{V} = g_1(V)_* \cup \cdots \cup g_n(V)_*$$

and consider the topologically finite geodesic subsurface $\widetilde{V}_*$. We see that this is homotopically g_i-invariant for each i, in other words, $g_i(\widetilde{V})_* = \widetilde{V}_*$. This can be verified as follows. Since the restriction of h_{ij} to each connected component of $R - V_0$ is homotopic to the inclusion map for any i and j, and since both $g_ig_j(V)$ and $g_k(V)$ for $k = k(i,j)$ contain V_0, we have $g_ig_j(V)_* = g_k(V)_*$. Hence $\{g_ig_j(V)_*\}_{j=1}^n = \{g_k(V)_*\}_{k=1}^n$ for each i. Note that $\widetilde{V}_*$ can be characterized as the minimal geodesic subsurface that contains all $g_1(V)_*, \ldots, g_n(V)_*$. Then this is ready to show $g_i(\widetilde{V})_* = \widetilde{V}_*$. By replacing the representative g_i of $[g_i]$, we may assume that $g_i(\widetilde{V}_*) = \widetilde{V}_*$.

Let $\{W_*^{(l)}\}_{l=1}^m$ be the family of all connected components of $R - \widetilde{V}_*$. We make the double $\widehat{W}^{(l)}$ of each $W_*^{(l)}$ with respect to the geodesic boundary $\partial W_*^{(l)}$. Extend $g_i|_{R-\widetilde{V}_*}$ to a quasiconformal automorphism $\hat{g}_i$ of the union $S = \bigcup_{l=1}^m \widehat{W}^{(l)}$ of the Riemann surfaces by reflection. Then $\hat{g}_i\hat{g}_j$ is homotopic to $\hat{g}_k$ for $k = k(i,j)$ on each component $\widehat{W}^{(l)}$ of S relative to the ideal boundary at infinity. We consider the mapping classes $[\hat{g}_i]$ on S and define a map $\varepsilon : \hat{\Gamma} \to \mathrm{MCG}(S)$ by the correspondence $\hat{\gamma}_i \mapsto [\hat{g}_i]$. Since ε is a homomorphism by the definition of $k(i,j)$, we see that the image $G = \{[\hat{g}_i]\}_{i=1}^n$ constitutes a finite subgroup of $\mathrm{MCG}(S)$.

The quasiconformal mapping class group $\mathrm{MCG}(S)$ acts on the product Teichmüller space $T(S) = \prod_{l=1}^m T(\widehat{W}^{(l)})$ in the same manner as usual. Then G has a common fixed point $(p_1, \ldots, p_m)$ in $T(S)$. Indeed, for some $l = 1, \ldots, m$, we consider the stabilizer subgroup G_l of G preserving $\widehat{W}^{(l)}$ and apply Theorem 2.2 to obtain a fixed point p_l of G_l in $T(\widehat{W}^{(l)})$. Then we pick the images of p_l as the fixed points for the conjugates of G_l in G, which are the stabilizer subgroups for the images of $\widehat{W}^{(l)}$. We repeat this process for all possible components to assign the fixed point to $T(\widehat{W}^{(l)})$ for every l. Giving each $\widehat{W}^{(l)}$ the complex structure corresponding to p_l, we can realize the mapping class group G as a conformal automorphism group of the corresponding Riemann surfaces $S_{(p_1,\ldots,p_m)}$.

The statement of the theorem follows directly from this consequence. Indeed, by restricting the complex structure p_l to $W_*^{(l)}$ for each l, we have a point $\hat{p}$ of the asymptotic Teichmüller space $AT(R)$. Since the mapping class $[g_i]$ has a representative that is conformal outside $\widetilde{V}_*$ with respect to this complex structure, we see that each $\hat{\gamma}_i = \iota_{AT}([g_i])$ fixes the point $\hat{p}$. □

In light of Theorem 2.2, we will explore a problem asking whether we can extend Theorem 2.5 to the statement that there exists a common fixed point in $AT(R)$ if the orbit of a subgroup of $\mathrm{Mod}_{AT}(R)$ is bounded.

In [24], we constructed a Riemann surface R not satisfying the upper bound condition so that $\mathrm{Mod}_{AT}(R)$ has a common fixed point in $AT(R)$. Moreover in [25], we gave an example where $\mathrm{Mod}_{AT}(R)$ acts on $AT(R)$ trivially, namely, all points in $AT(R)$ are common fixed points of $\mathrm{Mod}_{AT}(R)$. On the other hand, it was proved in [11, Theorem 3.3] that $\mathrm{Mod}_{AT}(R)$ does not have a common fixed point in $AT(R)$ if R satisfies the upper bound condition. This means that the asymptotic version of the fixed point theorem is not always true for the whole asymptotic Teichmüller modular group.

5. Realization in the end quasiconformal automorphism group

Theorem 2.5 induces two asymptotic Nielsen realization theorems. In this section, we give one of them, which is for the asymptotic Teichmüller modular group. For its statement, we introduce the following concept.

Definition 5.1. We say that two quasiconformal automorphisms of a Riemann surface R are *end equivalent* if they coincide outside some topologically finite subsurface of finite area in R. The *end quasiconformal automorphism group* $\mathrm{QC}_e(R)$ is the group of all end equivalence classes $(g)_e$ of quasiconformal automorphisms g of R. Furthermore, the end conformal automorphism group $\mathrm{Conf}_e(R)$ is the subgroup of $\mathrm{QC}_e(R)$ consisting of all end equivalence classes that have representatives conformal outside some topologically finite subsurface of finite area in R.

Let $e : \mathrm{QC}(R) \to \mathrm{QC}_e(R)$ be the homomorphism given by the projection. It is clear that e splits the surjective homomorphism $\iota_{AT} \circ q : \mathrm{QC}(R) \to \mathrm{Mod}_{AT}(R)$ into a surjective homomorphism $q_e : \mathrm{QC}_e(R) \to \mathrm{Mod}_{AT}(R)$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{QC}(R) & \xrightarrow{\ e\ } & \mathrm{QC}_e(R) \\ \big\downarrow q & & \big\downarrow q_e \\ \mathrm{MCG}(R) \cong \mathrm{Mod}(R) & \xrightarrow{\ \iota_{AT}\ } & \mathrm{Mod}_{AT}(R) \end{array}$$

The Nielsen realization theorem for the asymptotic Teichmüller modular group can be stated as follows.

Theorem 5.2. *Let R be a Riemann surface satisfying the weak bounded geometry condition. Then every finite subgroup $\hat{\Gamma}$ of $\mathrm{Mod}_{AT}(R)$ can be realized as the end conformal automorphism group $\mathrm{Conf}_e(R_p)$ of some Riemann surface R_p quasiconformally equivalent to R. In particular, there exists a homomorphism $\mathcal{E}_e : \hat{\Gamma} \to \mathrm{QC}_e(R)$ such that $q_e \circ \mathcal{E}_e = \mathrm{id}|_{\hat{\Gamma}}$.*

Proof. By Theorem 2.5, every finite subgroup $\hat{\Gamma}$ of $\mathrm{Mod}_{AT}(R)$ has a common fixed point $\hat{p}$ in $AT(R)$. Let R_p be the Riemann surface corresponding to $p \in T(R)$ with $\alpha(p) = \hat{p}$. Then, for each element $\hat{\gamma} \in \hat{\Gamma}$, we have an asymptotically conformal automorphism g of R_p as a representative in the mapping class $[g] \in \mathrm{MCG}(R)$ satisfying $\iota_{AT}([g]) = \hat{\gamma}$. Moreover, by the proof of Theorem 2.5, these asymptotically conformal automorphisms g are actually conformal outside some topologically finite subsurface of finite area in R_p, namely, it determines an element $(g)_e$ of $\mathrm{Conf}_e(R_p)$. We note that such an element $(g)_e \in \mathrm{Conf}_e(R_p)$ is uniquely determined for each $\hat{\gamma} \in \hat{\Gamma}$. Indeed, if there exist two such elements $(g_1)_e$ and $(g_2)_e$ in $\mathrm{Conf}_e(R_p)$, then $g_1 \circ g_2^{-1}$ is homotopic to the identity relative to the ideal boundary at infinity and is conformal outside some topologically finite subsurface V of finite area in $g_2(R_p)$. We may assume that each component of the complement of V is topologically infinite. By [18], we see that $g_1 \circ g_2^{-1}$ is the identity outside V. This shows that $(g_1)_e = (g_2)_e$. Since the groups $\mathrm{QC}(R)$ and $\mathrm{QC}(R_p)$ are quasiconformally conjugate, the correspondence $\hat{\gamma} \mapsto g$ induces the homomorphism $\mathcal{E}_e : \hat{\Gamma} \to \mathrm{QC}_e(R)$ such that $q_e \circ \mathcal{E}_e = \mathrm{id}|_{\hat{\Gamma}}$. □

6. Realization in the automorphism group of the Royden boundary

In this section, we give another asymptotic Nielsen realization theorem for the boundary geometric automorphism group of the asymptotic Teichmüller space. The asymptotic Teichmüller modular group is determined by the action of the group of quasiconformal automorphisms near the infinity of the Riemann surface. We extend this action to a certain ideal boundary of the Riemann surface and investigate the realization problem with respect to a homomorphism from the group of such boundary automorphisms to the biholomorphic automorphism group of the asymptotic Teichmüller space.

For a Riemann surface R, the *Royden algebra* $M(R)$ is a complex Banach algebra consisting of all bounded continuous functions f on R that are differentiable in the distribution sense with the Dirichlet integral $D(f) = \int_R df \wedge * \overline{df}$ finite. The norm is defined by $\|f\| = \sup |f| + D(f)^{1/2}$. There exists a compact Hausdorff space R^* containing R as an open and dense subset such that each function $f \in M(R)$ can be extended to a continuous function on R^* and every pair of points in R^* is separable by a function in $M(R)$. Then R^* is uniquely determined up to homeomorphisms that are the identity on R. This is called the Royden compactification of R and the boundary $dR = R^* - R$ is called the *Royden boundary* of R. An important property of the Royden compactification is that every quasiconformal homeomorphism f of R onto another Riemann surface R' extends continuously to a homeomorphism $\bar{f} : dR \to dR'$. See [28, Chapter III] for details.

We deal with a smaller subset d_0R of the Royden boundary dR as in [29], which is obtained by removing all boundary points corresponding to punctures of R. More precisely, d_0R is defined as follows. There is a canonical continuous map π from the Royden boundary dR onto the space of all topological ends of R, which extends continuously to the identity of R. The inverse image $\pi^{-1}(a)$ of a topological end a corresponding to a puncture constitutes a connected component of dR. We remove all such connected components $\pi^{-1}(a)$ from dR and define the remaining compact subset as d_0R, which we call the non-cuspidal Royden boundary. Note that a neighborhood of dR in R is given by the complement of a compact subsurface V of R whereas a neighborhood of

d_0R in R is given by the union of the non-cuspidal connected components of $R-V$, which is the complement of some topologically finite subsurface of finite area in R. Since a quasiconformal homeomorphism f of R preserves punctures, its extension $\bar{f}$ to dR preserves d_0R.

The converse property for the above extendability of quasiconformal homeomorphisms to the Royden boundary is also true in a certain sense, and we know that, if a homeomorphism $\bar{f}: dR \to dR'$ is the restriction of a homeomorphism of R^* onto R'^* that is orientation preserving on R, then $\bar{f}$ extends continuously to a quasiconformal homeomorphism $f: R \to R'$. This was proved in [27]. In fact, local extendability is essential for its proof, and we can formulate this claim for homeomorphic automorphisms of the non-cuspidal Royden boundary as in [29, Proposition 3].

Proposition 6.1. *Let $\bar{g}$ be a homeomorphic automorphism of d_0R extending continuously to an orientation preserving homeomorphism g of some neighborhood of d_0R in R into R. Then there exists a topologically finite subsurface V of finite area in R such that g is quasiconformal on the complement $R-V$.*

The homeomorphism g as in Proposition 6.1 is called a *supporting map* for $\bar{g}$. Note that the image of the neighborhood of d_0R by g is also a neighborhood of d_0R in R. We say that a homeomorphic automorphism $\bar{g}$ of d_0R is *extendable* if there is a supporting map for $\bar{g}$. We denote by $\mathrm{Homeo}^*(d_0R)$ the group of all extendable homeomorphic automorphisms of d_0R. Proposition 6.1 says that, for every element of $\mathrm{Homeo}^*(d_0R)$, its supporting map is always quasiconformal on $R-V$ for some topologically finite subsurface V of finite area in R. Since $g(R-V)$ is a neighborhood of d_0R in R, we see that $R-g(R-V)$ is also topologically finite.

Let $\nu: \mathrm{QC}_e(R) \to \mathrm{Homeo}^*(d_0R)$ be a homomorphism defined by extending a quasiconformal automorphism of R to d_0R. Since an element of $\mathrm{QC}_e(R)$ has ambiguity on topologically finite subsurfaces of finite area, it cannot determine an automorphism of the entire boundary dR uniquely, but the restriction to d_0R kills such ambiguity to define the unique element of $\mathrm{Homeo}^*(d_0R)$.

The homomorphism ν is injective but not surjective. The reason why ν is not surjective is that a supporting map does not necessarily extend to the entire surface R. An example has been given in [29, Theorem 4]. In contrast, the injectivity of ν can be shown by a usual argument as follows.

Proposition 6.2. *Let g be a supporting map for $\bar{g} \in \mathrm{Homeo}^*(d_0R)$. If $\bar{g}$ is the identity, then there exists a topologically finite subsurface V of finite area in R such that g is the identity on $R-V$.*

Proof. Assume that g is not the identity outside any topologically finite subsurface V of finite area in R. Then we can choose a discrete sequence of points $\{z_n\}_{n=1}^{\infty}$ in R tending to the infinity of R but not to any puncture such that

$Z = \{z_n\}$ and $g(Z) = \{g(z_n)\}$ are disjoint. Then it is easy to construct a function φ in the Royden algebra $M(R)$ such that $\varphi(z_n) = 1$ and $\varphi(g(z_n)) = 0$ for every n. Indeed, choose a disk neighborhood $U_n \subset R - V$ of z_n such that $U_n \cap (Z \cup g(Z))$ consists only of z_n for each n. Take a Riemann map ζ_n of U_n onto the unit disk Δ such that $\zeta_n(z_n) = 0$. We consider a function

$$h_n(\zeta) = \min\left\{-\frac{\log|\zeta|}{n^2}, 1\right\} \geq 0 \qquad (h_n(0) = 1)$$

on Δ. An easy calculation shows that the Dirichlet integral of h_n is $D(h_n) = 2\pi/n^2$. Define the function $\varphi(z)$ on R so that $\varphi(z) = h_n(\zeta_n(z))$ if $z \in U_n$ for some n and otherwise $\varphi(z) = 0$. Then φ is continuous and bounded with the Dirichlet integral $D(\varphi) = \pi^3/3$, and hence belongs to $M(R)$. By construction, we see that $\varphi(z_n) = 1$ and $\varphi(g(z_n)) = 0$.

Since every infinite set in a compact Hausdorff space has an accumulation point, Z has an accumulation point z_∞ in d_0R. Then $\bar{g}(z_\infty)$ is an accumulation point of $g(Z)$. If the extension $\bar{g}$ of g is the identity on d_0R, then $\bar{g}(z_\infty) = z_\infty$. This implies that Z and $g(Z)$ have the common accumulation point z_∞. On the other hand, since $\varphi \in M(R)$ extends to a continuous function on R^*, the sets $X_1 = \{z \in R^* \mid \varphi(z) = 1\}$ and $X_0 = \{z \in R^* \mid \varphi(z) = 0\}$ are both closed. Since $Z \subset X_1$ and $g(Z) \subset X_0$, the common accumulation point z_∞ belongs to both X_1 and X_0. This contradiction proves the statement. □

Next, we consider a representation of $\mathrm{Homeo}^*(d_0R)$ in the biholomorphic automorphism group $\mathrm{Aut}(AT(R))$ of the asymptotic Teichmüller space $AT(R)$. This has been also introduced in [29]. A quasiconformal supporting map $g : R - V \to R$ for $\bar{g} \in \mathrm{Homeo}^*(d_0R)$ induces a biholomorphic automorphism of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$ for each element $[[f]]$ in $AT(R)$. Although the quasiconformal homeomorphism $f \circ g^{-1}$ is defined only on $g(R - V)$, it determines the asymptotic Teichmüller class $[[f \circ g^{-1}]]$ because the value of $f \circ g^{-1}$ on the topologically finite subsurface $R - g(R - V)$ of finite area is negligible and hence it can be arbitrarily given. In this manner, we have a homomorphism $q_b : \mathrm{Homeo}^*(d_0R) \to \mathrm{Aut}(AT(R))$.

Definition 6.3. The *boundary geometric automorphism group* is the subgroup of $\mathrm{Aut}(AT(R))$ consisting of all elements induced by supporting maps for $\mathrm{Homeo}^*(d_0R)$ and is denoted by $\mathrm{Aut}^*(AT(R))$:

$$\mathrm{Aut}^*(AT(R)) = q_b(\mathrm{Homeo}^*(d_0R)).$$

It is clear that the asymptotic Teichmüller modular group $\mathrm{Mod}_{AT}(R)$ is contained in $\mathrm{Aut}^*(AT(R))$, namely, there is an inclusion map $i : \mathrm{Mod}_{AT}(R) \to \mathrm{Aut}^*(AT(R))$.

Remark 6.4. We conjecture that the boundary geometric automorphism group $\mathrm{Aut}^*(AT(R))$ actually coincides with the whole biholomorphic automorphism group $\mathrm{Aut}(AT(R))$. This claim is corresponding to the fact for Teichmüller spaces that the Teichmüller modular group $\mathrm{Mod}(R)$ coincides with $\mathrm{Aut}(T(R))$. We also expect $\mathrm{Mod}_{AT}(R)$ to be a proper subgroup of $\mathrm{Aut}^*(AT(R))$. Actually, it is proved in [12] that $\mathrm{Mod}_{AT}(R)$ is geometrically isomorphic to the biholomorphic automorphism group $\mathrm{Aut}(IT(R))$ of the intermediate Teichmüller space $IT(R)$ under the bounded geometry condition.

Now having the following commutative diagram

$$\begin{array}{ccc} \mathrm{QC}_e(R) & \xrightarrow{\nu} & \mathrm{Homeo}^*(d_0R) \\ \downarrow q_e & & \downarrow q_b \\ \mathrm{Mod}_{AT}(R) & \xrightarrow{i} & \mathrm{Aut}^*(AT(R)) \end{array}$$

we formulate the Nielsen realization theorem for the surjective homomorphism $q_b : \mathrm{Homeo}^*(d_0R) \to \mathrm{Aut}^*(AT(R))$ and give a proof for it.

Theorem 6.5. *Let R be a Riemann surface satisfying the weak bounded geometry condition. Then, for every finite subgroup $\widetilde{\Gamma}$ of $\mathrm{Aut}^*(AT(R))$, there exists a homomorphism $\mathcal{E}_b : \widetilde{\Gamma} \to \mathrm{Homeo}^*(d_0R)$ such that $q_b \circ \mathcal{E}_b = \mathrm{id}|_{\widetilde{\Gamma}}$.*

As is shown below, the essential part of this theorem lies in the following lemma. Then Theorem 6.5 will be just an interpretation of Theorem 5.2, the realization for $q_e : \mathrm{QC}_e(R) \to \mathrm{Mod}_{AT}(R)$.

Lemma 6.6. *Let R be a Riemann surface satisfying the weak bounded geometry condition. Then every element $\tilde{\gamma}$ of $\mathrm{Aut}^*(AT(R))$ of finite order belongs to $\mathrm{Mod}_{AT}(R)$.*

Proof. We take a quasiconformal supporting map g inducing $\tilde{\gamma}$, which maps $R - V_0$ into R for some topologically finite subsurface V_0 of finite area in R. If $\tilde{\gamma}$ is of order n, then g^n maps $R - V_0$ into R and induces the trivial action on $AT(R)$. By the same reason as the assertion that an asymptotically trivial mapping class is essentially trivial (Theorem 3.2), we see that there exists a topologically finite subsurface V of finite area containing V_0 such that $g^n|_{R-V}$ is homotopic to the inclusion map $\mathrm{id}|_{R-V} : R - V \hookrightarrow R$ relative to the ideal boundary at infinity.

Similarly to the proof of Theorem 2.5, we consider $\bigcap_{i=0}^{n-1} g^i(R - V)_*$, whose complement in R is defined to be $\widetilde{V}$. Since $g^n(R - V)_* = (R - V)_*$, we see that $g(R - \widetilde{V})_* = (R - \widetilde{V})_*$. By replacing g, we may assume that g preserves $(R - \widetilde{V})_*$. Then we can extend g to the topologically finite geodesic subsurface $\widetilde{V}_*$ to construct a quasiconformal automorphism $\tilde{g}$ of R. Hence $\tilde{\gamma}$ is induced by $\tilde{g}$, which means that $\tilde{\gamma} \in \mathrm{Mod}_{AT}(R)$. □

We are ready to prove our theorem.

Proof of Theorem 6.5. Let $\widetilde{\Gamma}$ be a finite subgroup of $\mathrm{Aut}^*(AT(R))$. By Lemma 6.6, each element of $\widetilde{\Gamma}$ actually belongs to $\mathrm{Mod}_{AT}(R)$, and hence $\widetilde{\Gamma} \subset \mathrm{Mod}_{AT}(R)$. We define a homomorphism $\mathcal{E}_b$ of $\widetilde{\Gamma}$ to be the composition of $\mathcal{E}_e : \widetilde{\Gamma} \to \mathrm{QC}_e(R)$ obtained in Theorem 5.2 and $\nu : \mathrm{QC}_e(R) \to \mathrm{Homeo}^*(d_0R)$. From $q_e \circ \mathcal{E}_e = \mathrm{id}|_{\widetilde{\Gamma}}$, it follows that

$$q_b \circ \mathcal{E}_b = q_b \circ \nu \circ \mathcal{E}_e = q_e \circ \mathcal{E}_e = \mathrm{id}|_{\widetilde{\Gamma}}.$$

This proves the statement. □

7. The automorphism group of the space of the ends

We conclude this paper by giving a comment that the sequence of homomorphisms

$$\mathrm{QC}(R) \to \mathrm{QC}_e(R) \to \mathrm{Homeo}^*(d_0R)$$

for raw mappings and the sequence of homomorphisms

$$\mathrm{Mod}(R) \to \mathrm{Mod}_{AT}(R) \to \mathrm{Aut}^*(AT(R))$$

for mapping classes meet into the group of homeomorphic automorphisms of the space of the ends of R. We use a slightly finer concept of ends as follows, which has been introduced in [13].

The *end compactification* $\overline{R}^{\delta}$ of a Riemann surface R is given by adding all the ends of R and by providing the canonical topology. Here an end means a topological end if R has no ideal boundary at infinity. However, if R has the ideal boundary at infinity ∂R, we first consider the double $\hat{R}$ of R with respect to ∂R and then take the closure of R in the end compactification of $\hat{R}$, which we define to be $\overline{R}^{\delta}$. The boundary $\overline{R}^{\delta} - R$ is denoted by δR. As before, we remove all ends corresponding to punctures from δR and denote the space of all non-cuspidal ends by $\delta_0 R$. A supporting map g for $\bar{g} \in \mathrm{Homeo}^*(d_0R)$ extends to a homeomorphic automorphism of $\delta_0 R$, which defines a homomorphism $\kappa : \mathrm{Homeo}^*(d_0R) \to \mathrm{Homeo}(\delta_0 R)$. Note that there is a continuous surjective map $\hat{\pi} : dR \to \delta R$ from the Royden boundary to the space of the ends that extends to the identity of R. The restriction of $\hat{\pi}$ to d_0R is compatible with the above homomorphism κ.

On the other hand, since each mapping class $[g] \in \mathrm{MCG}(R)$ induces a homeomorphic automorphism of $\delta_0 R$, there is a well-defined homomorphism $\iota_{\delta_0 R} : \mathrm{MCG}(R) \to \mathrm{Homeo}(\delta_0 R)$. The kernel of $\iota_{\delta_0 R}$ is defined to the *pure mapping class group*. In [10], it has been shown that $\mathrm{Ker}\,\iota_{AT} \subset \mathrm{Ker}\,\iota_{\delta_0 R}$, that is, the asymptotically trivial mapping class group is contained in the pure mapping class group, though the definition of $\delta_0 R$ used for this group is slightly coarser than the above one. The inclusion $\mathrm{Ker}\,\iota_{AT} \subset \mathrm{Ker}\,\iota_{\delta_0 R}$ would yield a homomorphism of $\mathrm{Mod}_{AT}(R)$ to $\mathrm{Homeo}(\delta_0 R)$.

In what follows, we will show the existence of a homomorphism from the larger group $\mathrm{Aut}^*(AT(R))$ to $\mathrm{Homeo}(\delta_0 R)$. For this purpose, we prove the following inclusion relation under the new definition of the end compactification.

Lemma 7.1. $\mathrm{Ker}\,q_b \subset \mathrm{Ker}\,\kappa$.

Proof. Let $\bar g \in \mathrm{Homeo}^*(d_0 R)$ not belong to $\mathrm{Ker}\,\kappa$. We take a supporting map g for $\bar g$ defined on $R - V$, where V is a topologically finite subsurface of finite area in R. Then there is an end $e \in \delta_0 R$ such that $g(e) \neq e$. If R has no ideal boundary at infinity, then there exists a neighborhood of e whose intersection with R is a topologically infinite subsurface W in $R - V$ such that $g(W) \cap W = \emptyset$. In this case, g induces a non-trivial automorphism of $AT(R)$ by the same argument as that for the inclusion $\mathrm{Ker}\,\iota_{AT} \subset \mathrm{Ker}\,\iota_{\delta_0 R}$ proved in [10, Theorem 1.5].

Next we assume that R has ideal boundary at infinity ∂R. If g extends to ∂R as a non-identical map, then, by the same argument as Lemma 3.5, we see that g induces a non-trivial automorphism of $AT(R)$. If g is the identity on ∂R, then its extension to the boundary δR of the end compactification is the identity on the closure $\overline{\partial R}^{\delta}$ of ∂R. Since $g(e) \neq e$ for some $e \in \delta_0 R$, this means that e is not in $\overline{\partial R}^{\delta}$. Then again we can find a topologically infinite subsurface W of R such that $g(W) \cap W = \emptyset$ and hence g induces a non-trivial automorphism of $AT(R)$ by [10]. In all cases, we have shown that $\bar g$ does not belong to $\mathrm{Ker}\,q_b$. □

Thus we have the following diagram by factorizing κ into q_b and κ':

$$\begin{array}{ccc} \mathrm{Homeo}^*(d_0 R) & \xrightarrow{\kappa} & \mathrm{Homeo}(\delta_0 R) \\ \big\downarrow{\scriptstyle q_b} & & \big\| \\ \mathrm{Aut}^*(AT(R)) & \xrightarrow{\kappa'} & \mathrm{Homeo}(\delta_0 R) \end{array}$$

The existence of the homomorphism $\kappa' : \mathrm{Aut}^*(AT(R)) \to \mathrm{Homeo}(\delta_0 R)$ in particular verifies the following fact we have mentioned above.

Corollary 7.2. *The pure mapping class group* $\mathrm{Ker}\,\iota_{\delta_0 R}$ *contains the asymptotically trivial mapping class group* $\mathrm{Ker}\,\iota_{AT}$.

This is the *end* of our supplementary comments.

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