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Invariance of the Nayatani metrics for Kleinian manifolds

Katsuhiko Matsuzaki · **Yasuhiro Yabuki**

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Abstract The Nayatani metric g_N is a Riemannian metric on a Kleinian manifold M which is compatible with the standard flat conformal structure. It is known that, for M corresponding to a geometrically finite Kleinian group, g_N has large symmetry: the isometry group of (M, g_N) coincides with the conformal transformation group of M. In this paper, we prove that this holds for a larger class of M. In particular, this class contains such M that correspond to Kleinian groups of divergence type.

Keywords Nayatani metric · Patterson-Sullivan measure · Kleinian group · divergence type

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1 Introduction

Let M be a differentiable manifold of dimension $n \geq 2$ endowed with a conformal structure C. We call the pair (M,C) *conformally flat* if each $g \in C$ has an expression of the form

$$g = \lambda(x) \sum_{i=1}^{n} (\mathrm{d}x^i)^2$$

locally for some local coordinates (x^i) of M and a function $\lambda > 0$ on M. If $n = 2$, it follows from the existence of the isothermal coordinates that (M,C) is always conformally flat. Hence a conformally flat structure can be regarded as a generalization of the conformal structure of a surface. Schoen and Yau [7] showed that there is an extensive class of conformally flat manifolds that can be realized as Kleinian manifolds. In particular, this class contains all compact conformally flat manifolds (M,C) of dimension $n \geq 4$ having the property that $R_g > 0$ for some $g \in C$, where R_g denotes the scalar curvature of g. Here, by a Kleinian manifold, we mean a quotient space Ω/Γ of a Γ-invariant subdomain Ω of the

K. Matsuzaki
Department of Mathematics, Okayama University, Okayama 700-8530, Japan
E-mail: matsuzak@math.okayama-u.ac.jp

Y. Yabuki
Mathematical Institute, Tohoku University, Aoba-ku, Sendai 980-8578, Japan
E-mail: sa3m30@math.tohoku.ac.jp

n-dimensional sphere S^n, by a Kleinian group Γ acting freely and properly discontinuously on Ω. Since the action of Γ preserves the conformal structure of Ω, Ω/Γ has the standard flat conformal structure C_0 induced by the natural projection.

Let Γ be a Kleinian group with the domain of discontinuity $\Omega(\Gamma) \neq \emptyset$ and the critical exponent $\delta(\Gamma) > 0$. Nayatani [4] constructed a Γ-invariant metric g_N on $\Omega(\Gamma)$, which is conformal to the standard sphere metric g_0 on $S^n = \left\{x \in \mathbb{R}^{n+1} \mid |x| = 1\right\}$, by

$$g_N = \left(\int_{S^n} \left(\frac{2}{|x-y|^2}\right)^{\delta(\Gamma)} \mathrm{d}\mu_0(y)\right)^{2/\delta(\Gamma)} g_0, \qquad x \in \Omega(\Gamma),$$

where $\delta(\Gamma)$ denotes the critical exponent of Γ and μ_0 is a Patterson-Sullivan measure based at $0 \in B^{n+1}$. Here a Patterson-Sullivan measure is a $\delta(\Gamma)$-dimensional Γ-invariant conformal measure on S^n having its support on the limit set $\Lambda(\Gamma)$ of Γ. Note that $\delta(\Gamma)$ measures the size of the action of Γ.

A Kleinian group Γ is said to be *convex cocompact* if the convex core $\mathrm{Hull}(\Gamma)/\Gamma$ is compact, where $\mathrm{Hull}(\Gamma)$ denotes the hyperbolic convex hull of $\Lambda(\Gamma)$ in B^{n+1}. Note that a convex cocompact Γ is characterized as a geometrically finite group without parabolic elements. It is also known that if Γ is non-elementary and convex cocompact, there exists a unique Patterson-Sullivan measure, which coincides with, up to a constant multiple, the $\delta(\Gamma)$-dimensional Hausdorff measure on the limit set of Γ. In general, the uniqueness of the Patterson-Sullivan measure holds for a large class of Kleinian groups, which in particular contains those of divergence type (see Theorem 2.4).

Since the metric g_N is Γ-invariant, it projects to a metric on the Kleinian manifold Ω/Γ, denoted by the same symbol g_N, which is compatible with C_0. The metric g_N is called the *Nayatani metric*.

There are remarkable relations between curvatures of the Nayatani metric and the critical exponent $\delta(\Gamma)$ of Γ. For instance, the scalar curvature is positive (resp. zero, negative) if and only if $\delta(\Gamma) <$ (resp. $=,>$) $(n-2)/2$ for $n \geq 3$ (see [4, Theorem 3.3]). These relations enable us to study Kleinian groups by means of Riemannian geometry of Kleinian manifolds and vice versa.

On the other hand, the Nayatani metric has large symmetry in the following sense:

Theorem 1.1 (Nayatani [4]) *Let Ω/Γ be a Kleinian manifold of dimension $n \geq 3$ with $\delta(\Gamma) > 0$ and g_N a Nayatani metric on Ω/Γ. Suppose that*

(a) *Γ has a unique Patterson-Sullivan measure, and*
(b) *g_N is a complete metric on Ω/Γ.*

Then the isometry group of $(\Omega/\Gamma, g_N)$ coincides with the conformal transformation group of $(\Omega/\Gamma, C_0)$.

If Γ is geometrically finite, it satisfies condition (a) in Theorem 1.1, since a geometrically finite Kleinian group is of divergence type. However, condition (b) does not hold in general even for a geometrically finite Kleinian group (see [3]). In [10], the second author showed that, if Γ is geometrically finite, g_N has large symmetry even when g_N is not complete. In this paper, we show that assumption (b) in Theorem 1.1 can be dropped altogether, that is,

Theorem 3.5 *Let Ω/Γ be a Kleinian manifold of dimension $n \geq 3$ with $\delta(\Gamma) > 0$ and g_N a Nayatani metric on Ω/Γ. Suppose that Γ has a unique Patterson-Sullivan measure up to a constant multiple. Then the isometry group of $(\Omega/\Gamma, g_N)$ coincides with the conformal transformation group of $(\Omega/\Gamma, C_0)$.*

2 Preliminaries

In this section, we briefly review several definitions and basic facts on Kleinian groups, Patterson-Sullivan measures and Nayatani metrics. More details can be found in [4] and [6] for example.

2.1 Kleinian groups

Let (B^{n+1},h), $n \geq 2$, denote the Poincaré ball model of the hyperbolic $(n+1)$-space, where $B^{n+1} = \left\{x \in \mathbb{R}^{n+1} \mid |x| < 1\right\}$ and

$$h = \left(\frac{2}{1-|x|^2}\right)^2 \sum_{i=1}^{n+1} \left(\mathrm{d}x^i\right)^2 .$$

Let also $\mathrm{Isom}(B^{n+1},h)$ denote the group of all orientation-preserving isometries of (B^{n+1},h). As is well-known, the action of $\mathrm{Isom}(B^{n+1},h)$ on B^{n+1} extends to the boundary S^n and gives the conformal action on S^n with the standard conformal structure. In this way, we can identify $\mathrm{Isom}(B^{n+1},h)$ with the group $\mathrm{Conf}(S^n)$ of all orientation-preserving conformal transformations of S^n.

A Kleinian group Γ is a discrete subgroup of $\mathrm{Isom}(B^{n+1},h)$. The *limit set* $\Lambda(\Gamma)$ of Γ is defined as the set of all accumulation points of Γ-orbit of any point in B^{n+1}. Since Γ acts properly discontinuous on B^{n+1}, $\Lambda(\Gamma)$ is contained in S^n. The complement of the limit set is denoted by $\Omega(\Gamma)$ and called the *domain of discontinuity* of Γ. This is the largest open subset of S^n on which Γ acts properly discontinuously.

For a Kleinian group Γ, the Poincaré series of dimension s with the base point $z \in B^{n+1}$ and the orbit point $w \in B^{n+1}$ is defined by

$$P_\Gamma(z,w,s) = \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(z,\gamma w)},$$

where d denotes the hyperbolic distance function on B^{n+1}. Using the Poincaré series, the *critical exponent* $\delta(\Gamma)$ of Γ is defined by

$$\delta(\Gamma) = \inf\{s > 0 \mid P_\Gamma(z,w,s) < \infty\}, \qquad z,w \in B^{n+1}.$$

It is easy to see that $\delta(\Gamma)$ is independent of the particular choice of $z,w \in B^{n+1}$. It is known that $0 \leq \delta(\Gamma) \leq n$, and $\delta(\Gamma) > 0$ if Γ is *non-elementary*, that is, $\Lambda(\Gamma)$ contains at least three points.

We say that Γ is *of divergence type* if $P_\Gamma(z,w,\delta(\Gamma)) = \infty$ and *of convergence type* if $P_\Gamma(z,w,\delta(\Gamma)) < \infty$. It is known that if Γ is geometrically finite, then Γ is of divergence type. Here Γ is said to be *geometrically finite* if the ε-neighborhood of the convex core $\mathrm{Hull}(\Gamma)/\Gamma$ has finite volume for every $\varepsilon > 0$, where $\mathrm{Hull}(\Gamma)$ denotes the hyperbolic convex hull of $\Lambda(\Gamma)$. In particular, if Γ is cofinite (namely, the hyperbolic volume of B^{n+1}/Γ is finite), then $\delta(\Gamma) = n$ and Γ is of divergence type. It is known that the hyperbolic manifold B^{n+1}/Γ does not admit the Green function if and only if $\delta(\Gamma) = n$ and Γ is of divergence type.

2.2 Patterson-Sullivan measures

An s-dimensional *conformal measure* on S^n is a family of positive finite Borel measures $\{\mu_z\}_{z\in B^{n+1}}$ such that $\mu_z = |h_z'|^s\mu_0$, where h_z is an element of $\mathrm{Conf}(S^n)$ sending z to $0 \in B^{n+1}$ and $|h_z'|$ is the linear stretch factor of h_z with respect to g_0. Here we identified $\mathrm{Conf}(S^n)$ with $\mathrm{Isom}(B^{n+1},h)$. Using the Poisson kernel

$$k(z,x) = \frac{1-|z|^2}{|z-x|^2}, \qquad z \in B^{n+1},\ x \in S^n,$$

the linear stretch factor of $h_z \in \mathrm{Conf}(S^n)$ is represented as $|h_z'(x)| = k(z,x)$. For a Kleinian group Γ, an s-dimensional conformal measure $\{\mu_z\}_{z\in B^{n+1}}$ is said to be Γ-invariant if $\gamma^*\mu_z = \mu_{\gamma^{-1}z}$ for every $z \in B^{n+1}$ and for every $\gamma \in \Gamma$, where $\gamma^*\mu_z$ is the pull-back of the measure μ_z by γ. A Γ-invariant conformal measure $\{\mu_z\}_{z\in B^{n+1}}$ of dimension $\delta(\Gamma)$ is called a Patterson-Sullivan measure if each μ_z has the support on the limit set, that is,

Definition 2.1 For a Kleinian group Γ, a Patterson-Sullivan measure is a family of positive finite Borel measures $\{\mu_z\}_{z\in B^{n+1}}$ on S^n satisfying the following properties:

(a) $\mu_z = k(z,\cdot)^{\delta(\Gamma)}\mu_0$ for every $z \in B^{n+1}$, where k is the Poisson kernel and $\delta(\Gamma)$ is the critical exponent of Γ.
(b) $\gamma^*\mu_z = \mu_{\gamma^{-1}z}$ for every $z \in B^{n+1}$ and every $\gamma \in \Gamma$.
(c) Each μ_z is supported on the limit set $\Lambda(\Gamma)$.

The Patterson-Sullivan measure was introduced by Patterson [5] when Γ is Fuchsian, that is, for a discrete subgroup of $\mathrm{Isom}(B^2,h)$, and was generalized by Sullivan [8] to general Kleinian groups.

Theorem 2.2 (Patterson, Sullivan) *For any Kleinian group Γ, there exists a Patterson-Sullivan measure.*

We now describe the construction of a Patterson-Sullivan measure by Patterson.

First, suppose that Γ is of divergence type. We define Borel measures on $\overline{B^{n+1}} = B^{n+1} \cup S^n$ by

$$\mu_{z,w,s} = \frac{1}{P_\Gamma(w,w,s)} \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(z,\gamma w)} D_{\gamma w},$$

where $z,w \in B^{n+1}, s > \delta(\Gamma)$, and $D_{\gamma w}$ denotes the Dirac measure supported at γw. Then $\{\mu_{z,w,s}\}_{z\in B^{n+1}}$ is Γ-invariant; for $\tilde\gamma \in \Gamma$, we have

$$\begin{aligned}
\tilde\gamma^*\mu_{z,w,s} &= \frac{1}{P_\Gamma(w,w,s)} \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(z,\gamma w)} \tilde\gamma^* D_{\gamma w} \\
&= \frac{1}{P_\Gamma(w,w,s)} \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(\tilde\gamma^{-1}z,\tilde\gamma^{-1}\gamma w)} D_{\tilde\gamma^{-1}\gamma w} \\
&= \mu_{\tilde\gamma^{-1}z,w,s}.
\end{aligned}$$

We also have

$$\lim_{r\to 0} \frac{\mu_{z,w,s}(B(x,r))}{\mu_{0,w,s}(B(x,r))} = k(z,x)^s, \tag{2.1}$$

where $x \in \Lambda(\Gamma)$ and $B(x,r)$ denotes the intersection of $\overline{B^{n+1}}$ and the ball of Euclidean radius $r > 0$ centered at x. To see this, we recall that

$$\sinh^2(d(z,w)/2) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}, \qquad z,w \in B^{n+1}. \tag{2.2}$$

It follows from (2.2) that

$$\frac{\sinh^2(d(z,w)/2)}{\sinh^2(d(0,w)/2)} = \frac{|z-w|^2}{(1-|z|^2)|w|^2}. \tag{2.3}$$

If w approaches to $x \in S^n$, the right hand side of (2.3) tends to $k(z,x)^{-1}$ and the left hand side is asymptotic to $e^{d(z,w)}/e^{d(0,w)}$. With the aid of the equation

$$\mu_{z,w,s}(B(x,r)) = \frac{1}{P_\Gamma(w,w,s)} \sum_{\gamma\in\Gamma:\gamma w\in B(x,r)} \frac{e^{-sd(z,\gamma w)}}{e^{-sd(0,\gamma w)}} e^{-sd(0,\gamma w)} D_{\gamma w}(B(x,r)),$$

we obtain (2.1).

We now let $\{s_i\}_{i=1}^\infty$ be a sequence of real numbers satisfying $s_i > \delta(\Gamma)$ and $s_i \to \delta(\Gamma)$. By Helly's theorem, renumbering a subsequence if necessary, we see that $\{\mu_{z,w,s_i}\}_{i=1}^\infty$ has a weak limit, denoted by $\mu_{z,w}$, which forms a Γ-invariant conformal measure $\{\mu_{z,w}\}_{z\in B^{n+1}}$ of dimension $\delta(\Gamma)$. The conformality is seen from (2.1). Since Γ is of divergence type, each $\mu_{z,w}$ has the support on $\Lambda(\Gamma)$. In this way, we obtain a Patterson-Sullivan measure $\{\mu_{z,w}\}_{z\in B^{n+1}}$.

In the case that Γ is of convergence type, we need a certain modification of the Poincaré series. See [6].

Lemma 2.3 *Let Γ be a Kleinian group with the critical exponent $\delta(\Gamma)$. Then there exists a continuous and non-decreasing function $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{>0}$ that has the following properties*:

(a) *For any $z,w \in B^{n+1}$,*

$$\sum_{\gamma\in\Gamma} e^{-sd(z,\gamma w)} f\left(e^{d(z,\gamma w)}\right) \begin{cases} = \infty, & s \leq \delta(\Gamma) \\ < \infty, & s > \delta(\Gamma). \end{cases}$$

(b) *For any $\varepsilon > 0$, there exists r_0 such that, if $r \geq r_0$ and $t > 1$, then $f(tr) \leq t^\varepsilon f(r)$.*

We call a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ as in Lemma 2.3 a *Patterson function*. Let Γ be a Kleinian group of convergence type with the critical exponent $\delta(\Gamma)$. Fix a Patterson function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. We define the *modified Poincaré series* of dimension s by

$$\tilde{P}_\Gamma(z,w,s) = \sum_{\gamma\in\Gamma} e^{-sd(z,\gamma w)} f\left(e^{d(z,\gamma w)}\right), \qquad z,w \in B^{n+1}.$$

Fix $w \in B^{n+1}$ and consider Borel measures $\{\mu_{z,w,s}\}_{z\in B^{n+1}}$ on $\overline{B^{n+1}}$ defined by

$$\mu_{z,w,s} = \frac{1}{\tilde{P}_\Gamma(w,w,s)} \sum_{\gamma\in\Gamma} e^{-sd(z,\gamma w)} f\left(e^{d(z,\gamma w)}\right) D_{\gamma w},$$

where $s > \delta(\Gamma)$ and $D_{\gamma w}$ is the Dirac measure supported at γw. By an argument similar to the case when Γ is of divergence type, we can obtain a Patterson-Sullivan measure $\mu_{z,w}$ as a weak limit of $\{\mu_{z,w,s_i}\}_{i=1}^\infty$ for some sequence $\{s_i\}$ with $s_i \downarrow \delta(\Gamma)$.

Concerning the uniqueness of the Patterson-Sullivan measure, it is known that

Theorem 2.4 *Let Γ be a Kleinian group of divergence type with the critical exponent $\delta(\Gamma) > 0$. Then Γ has a unique Patterson-Sullivan measure up to a constant multiple.*

Remark 2.5 There exists a Kleinian group of convergence type that has a unique Patterson-Sullivan measure up to a constant multiple. This can be found in [1] and [9].

2.3 Nayatani metrics

Let Γ be a Kleinian group with the domain of discontinuity $\Omega(\Gamma) \neq \emptyset$ and the critical exponent $\delta(\Gamma) > 0$. Let $\{\mu_z\}_{z\in B^{n+1}}$ be a Patterson-Sullivan measure for Γ. Nayatani [4] constructed a conformally flat metric on $\Omega(\Gamma)$ by deforming the standard sphere metric g_0:

$$g_N = \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|x-y|^2} \right)^{\delta(\Gamma)} \mathrm{d}\mu_0(y) \right)^{2/\delta(\Gamma)} g_0, \qquad x \in \Omega(\Gamma).$$

We call g_N a *Nayatani metric* for Γ. By using the hyperbolic metric h, g_N is naturally extended to a Riemannian metric on B^{n+1} as follows:

$$\begin{aligned} g_N &= \left(\int_{\Lambda(\Gamma)} \left(\frac{1-|z|^2}{|z-y|^2} \right)^{\delta(\Gamma)} \mathrm{d}\mu_0(y) \right)^{2/\delta(\Gamma)} h \\ &= \mu_z(\Lambda(\Gamma))^{2/\delta(\Gamma)} h, \qquad z \in B^{n+1}. \end{aligned} \tag{2.4}$$

Remark 2.6 If we identify B^{n+1} with a hemisphere in S^{n+1} and extend the action of Γ to S^{n+1} naturally, the extension of the Nayatani metric to B^{n+1} is nothing but the restriction of that on $\Omega^{n+1}(\Gamma)$, where $\Omega^{n+1}(\Gamma)$ denotes the domain of discontinuity of Γ on S^{n+1}.

It is easy to see that a Nayatani metric is Γ-invariant on $\Omega(\Gamma)\cup B^{n+1}$. Hence if Γ acts freely and properly discontinuously on $\Omega(\Gamma)\cup B^{n+1}$, g_N projects to a metric, denoted by the same symbol g_N, on the quotient manifold $[\Omega(\Gamma)\cup B^{n+1}]/\Gamma$. If Γ has the quotient B^{n+1}/Γ of finite volume in the hyperbolic sense, then $\Omega(\Gamma) = \emptyset$ and g_N is nothing but the hyperbolic metric on B^{n+1} up to a constant multiple.

3 Proof of Theorem 3.5

Let Γ be a Kleinian group acting on S^n, $n \geq 3$, with $\Omega(\Gamma) \neq \emptyset$. We suppose that Ω is a Γ-invariant subdomain of $\Omega(\Gamma)$ on which Γ acts freely and properly discontinuously. Let C_0 denote the standard flat conformal structure on the quotient manifold Ω/Γ. First we recall a result of Nayatani [4] on the conformal transformation group $\mathrm{Conf}(\Omega/\Gamma, C_0)$ of $(\Omega/\Gamma, C_0)$. Let $N(\Gamma)$ denote the normalizer of Γ in $\mathrm{Conf}(S^n)$. Define a homomorphism $F : N(\Gamma)\cap \mathrm{Conf}(\Omega) \longrightarrow \mathrm{Conf}(\Omega/\Gamma, C_0)$ as follows:

$$F(\alpha)(\Gamma x) = \Gamma \alpha x, \qquad \alpha \in N(\Gamma)\cap \mathrm{Conf}(\Omega),$$

where $x \in \Omega$ and Γx is regarded as a point of Ω/Γ. Then we have that F is surjective and the kernel of F coincides with Γ. Namely, we have

Proposition 3.1 *Let Γ, Ω, C_0 and $N(\Gamma)$ be as above. Then $[N(\Gamma)\cap \mathrm{Conf}(\Omega)]/\Gamma$ is isomorphic to the conformal transformation group $\mathrm{Conf}(\Omega/\Gamma, C_0)$.*

Next we consider the pull-back of a Patterson-Sullivan measure for Γ by an element of the normalizer $N(\Gamma)$.

Lemma 3.2 *Let Γ be a Kleinian group and $\{\mu_z\}_{z\in B^{n+1}}$ a Patterson-Sullivan measure for Γ. Then, for $\alpha \in N(\Gamma)$, $\{\alpha^*\mu_{\alpha z}\}_{z\in B^{n+1}}$ is also a Patterson-Sullivan measure for Γ.*

Proof We recall that

$$k(\eta w, \eta y) = k(\eta^{-1}0, y)^{-1} k(w, y), \qquad \eta \in \mathrm{Conf}(S^n). \tag{3.1}$$

Using (3.1), we compute

$$\begin{aligned}
\alpha^*\mu_{\alpha z} &= k(\alpha z, \alpha\cdot)^{\delta(\Gamma)} \alpha^*\mu_0 \\
&= \left(k(\alpha^{-1}0, \cdot)^{-1} k(z, \cdot)\right)^{\delta(\Gamma)} k(\alpha 0, \alpha\cdot)^{-\delta(\Gamma)} \alpha^*\mu_{\alpha 0} \\
&= \left(k(\alpha^{-1}0, \cdot)^{-1} k(z, \cdot)\right)^{\delta(\Gamma)} \left(k(\alpha^{-1}0, \cdot)^{-1} k(0, \cdot)\right)^{-\delta(\Gamma)} \alpha^*\mu_{\alpha 0} \\
&= k(z, \cdot)^{\delta(\Gamma)} \alpha^*\mu_{\alpha 0}.
\end{aligned}$$

This shows that $\{\alpha^*\mu_{\alpha z}\}_{z\in B^{n+1}}$ is a $\delta(\Gamma)$-dimensional conformal measure. Let $\gamma \in \Gamma$ and set $\tilde{\gamma} = \alpha\gamma\alpha^{-1} \in \Gamma$. Then we have

$$\gamma^*\alpha^*\mu_{\alpha z} = \alpha^*\tilde{\gamma}^*\mu_{\alpha z} = \alpha^*\mu_{\tilde{\gamma}^{-1}\alpha z} = \alpha^*\mu_{\alpha\gamma^{-1}z}.$$

This proves the Γ-invariance of $\{\alpha^*\mu_{\alpha z}\}_{z\in B^{n+1}}$. Since $\alpha \in N(\Gamma)$ leaves $\Lambda(\Gamma)$ invariant, $\{\alpha^*\mu_{\alpha z}\}_{z\in B^{n+1}}$ is a Patterson-Sullivan measure. □

Lemma 3.3 (Nayatani [4]) *Let Γ be a Kleinian group with $\delta(\Gamma) > 0$, and g_N a Nayatani metric on $\Omega(\Gamma)\cup B^{n+1}$. Suppose that Γ has a unique Patterson-Sullivan measure $\{\mu_z\}_{z\in B^{n+1}}$ up to a constant multiple. Then $\alpha \in N(\Gamma)$ is a homothety with respect to g_N, that is,*

$$\alpha^* g_N = c g_N$$

for some constant $c > 0$.

Proof By Remark 2.6, we may assume that g_N is a metric on B^{n+1}. By Lemma 3.2 and the uniqueness of the Patterson-Sullivan measure, there exists a constant $\tilde{c} > 0$ such that $\alpha^*\mu_{\alpha z} = \tilde{c}\mu_z$ for any $z \in B^{n+1}$. It follows from (2.4) and the $N(\Gamma)$-invariance of $\Lambda(\Gamma)$ that

$$(\alpha^* g_N)_z = \mu_{\alpha z}(\Lambda(\Gamma))^{2/\delta(\Gamma)} (\alpha^* h)_z = [\alpha^*\mu_{\alpha z}(\Lambda(\Gamma))]^{2/\delta(\Gamma)} h_z = \tilde{c}^{2/\delta(\Gamma)} (g_N)_z.$$

Putting $c = \tilde{c}^{2/\delta(\Gamma)}$ completes the proof. □

The following theorem is the key to our proof, which asserts that α as in Lemma 3.3 is actually an isometry.

Theorem 3.4 *Let Γ be a Kleinian group with $\delta(\Gamma) > 0$. Suppose that Γ has a unique Patterson-Sullivan measure $\{\mu_z\}_{z\in B^{n+1}}$ up to a constant multiple. Then $\{\mu_z\}_{z\in B^{n+1}}$ is $N(\Gamma)$-invariant, that is,*

$$\alpha^*\mu_z = \mu_{\alpha^{-1}z} \tag{3.2}$$

for $\alpha \in N(\Gamma)$ and $z \in B^{n+1}$.

Proof It follows from Lemma 3.2 and the uniqueness of the Patterson-Sullivan measure that $\alpha^*\mu_z = c\mu_{\alpha^{-1}z}$ for some constant $c > 0$. We now suppose that Γ is of divergence type. It is easy to see that the Poincaré series of dimension $s > \delta(\Gamma)$ has the following properties:

(a) $P_\Gamma(z,w,s) = P_\Gamma(w,z,s)$, and
(b) $P_\Gamma(\alpha z, \alpha w, s) = P_\Gamma(z,w,s)$

for any $z, w \in B^{n+1}$ and any $\alpha \in N(\Gamma)$. Define $\varphi_\mu(z) = \mu_z(S^n)$ for $z \in B^{n+1}$. We then have

$$\frac{\varphi_\mu(z)}{\varphi_\mu(z')} = \lim_{s\to\delta(\Gamma)} \frac{P_\Gamma(z,w,s)}{P_\Gamma(z',w,s)} \tag{3.3}$$

for $z, z', w \in B^{n+1}$. Indeed, by the uniqueness of the Patterson-Sullivan measure, μ_z is obtained as the weak limit of Borel measures

$$\frac{C}{P_\Gamma(w,w,s_i)} \sum_{\gamma\in\Gamma} \mathrm{e}^{-s_i d(z,\gamma w)} D_{\gamma w},$$

for some constant $C > 0$ independent of z and for any sequence $s_i \downarrow \delta(\Gamma)$. Hence

$$\varphi_\mu(z) = C \lim_{s\to\delta(\Gamma)} \frac{P_\Gamma(z,w,s)}{P_\Gamma(w,w,s)},$$

which implies (3.3). In particular, we have

$$c = \frac{\alpha^*\mu_z(S^n)}{\mu_{\alpha^{-1}z}(S^n)} = \frac{\mu_z(S^n)}{\mu_{\alpha^{-1}z}(S^n)} = \frac{\varphi_\mu(z)}{\varphi_\mu(\alpha^{-1}z)} \tag{3.4}$$

for any $z \in B^{n+1}$. Similarly, since $\alpha^*\mu_{\alpha z} = c\mu_z$, we also have

$$\frac{\varphi_\mu(z)}{\varphi_\mu(\alpha z)} = c^{-1}, \qquad z \in B^{n+1}. \tag{3.5}$$

Take $z = w$. Then it follows from (a), (b), (3.3), (3.4), and (3.5) that

$$c = \lim_{s\to\delta(\Gamma)} \frac{P_\Gamma(w,w,s)}{P_\Gamma(\alpha^{-1}w,w,s)} = \lim_{s\to\delta(\Gamma)} \frac{P_\Gamma(w,w,s)}{P_\Gamma(w,\alpha w,s)} = \lim_{s\to\delta(\Gamma)} \frac{P_\Gamma(w,w,s)}{P_\Gamma(\alpha w,w,s)} = c^{-1}.$$

Therefore c must be 1, which implies that (3.2) holds if Γ is of divergence type.

In the case when Γ is of convergence type, properties (a), (b) and (3.3) are satisfied by the modified Poincaré series $\tilde{P}_\Gamma$. Hence (3.2) follows from the same argument as in the case of divergence type. □

As a direct consequence of Theorem 3.4, we have

Theorem 3.5 *Let Ω/Γ be a Kleinian manifold of dimension $n \geq 3$ with $\delta(\Gamma) > 0$ and g_N a Nayatani metric on Ω/Γ. Suppose that Γ has the unique Patterson-Sullivan measure up to a constant multiple. Then the isometry group of $(\Omega/\Gamma, g_N)$ coincides with the conformal transformation group of $(\Omega/\Gamma, C_0)$.*

Remark 3.6 Theorem 3.4 can be also applied to a problem in Kleinian group theory, which asks for us conditions for Kleinian groups to have no proper conjugation. Here we say that a Kleinian group G has proper conjugation if there exists $\alpha \in \mathrm{Isom}(B^{n+1}, h)$ such that the conjugate $\alpha G \alpha^{-1}$ is a proper subgroup of G. Using Theorem 3.4, we can show that Kleinian groups of divergence type have no proper conjugation (see [2]).

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