LARGE AND SMALL COVERS OF A HYPERBOLIC MANIFOLD

PETRA BONFERT-TAYLOR, KATSUHIKO MATSUZAKI, AND EDWARD C. TAYLOR

Abstract. The exponent of convergence of a non-elementary discrete group of hyperbolic isometries measures the Hausdorff dimension of the conical limit set. In passing to a non-trivial regular cover the resulting limit sets are pointwise equal though the exponent of convergence of the cover uniformization may be strictly less than the exponent of convergence of the base. We show in this paper that, for closed hyperbolic surfaces, the previously established lower bound of one half on the exponent of convergence of "small" regular covers is sharp but is not attained. We also consider "large" (non-regular) covers. Here large and small are descriptive of the size of the exponent of convergence. We show that a Kleinian group that uniformizes a manifold homeomorphic to a surface fibering over a circle contains a Schottky subgroup whose exponent of convergence is arbitrarily close to two.

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1. Introduction

It is well known by now that the size of the exponent of convergence $\delta(\Gamma)$ of the Poincaré series of a discrete group Γ acting on hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ contains important geometric, analytic and topological information. For instance, via work of Canary [8], Sullivan [38], Tukia [41], and the solution of the tameness conjecture ([2], [12]) it is now known that if Γ is a finitely generated Kleinian group with non-empty regular set acting on $\mathbb{H}^3$ then Γ is geometrically finite if and only if $\delta(\Gamma) < 2$. The geometric and topological aspect of the Poincaré series that will concern us most in this note issues from the following theorem of Bishop and Jones [4].

Theorem 1.1. *Let Γ be a non-elementary discrete group of isometries of $\mathbb{H}^{n+1}$. Then $\delta(\Gamma)$ is the Hausdorff dimension of the conical limit set of Γ.*

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The exponent of convergence records the size of the conical limit set of a non-elementary discrete group of isometries. We are interested in passing to a subgroup $\hat{\Gamma}$ of Γ: it is an elementary observation that $\delta(\hat{\Gamma}) \leq \delta(\Gamma)$. In particular, if $\hat{\Gamma}$ is a non-trivial normal subgroup of Γ then the limit sets $L(\Gamma)$ and $L(\hat{\Gamma})$ on the sphere at infinity $\partial_\infty(\mathbb{H}^{n+1}) = \mathbb{S}^n$ are equal, however it may still be the case that $\delta(\hat{\Gamma})$ is *strictly less* than $\delta(\Gamma)$. An example of such phenomena was first given by Patterson [33]. As another example, let $S = \mathbb{H}^2/\Gamma$ be a closed hyperbolic surface, then the Schottky group defined by a retrosection of S is non-amenable, and so by a result of Brooks [7] the regular set of this Schottky group, regarded as a hyperbolic surface, has a uniformization $\hat{\Gamma}$ whose exponent of convergence is strictly less than $1 = \delta(\Gamma)$. The geometric point here is that in passing to a non-elementary subgroup a conical limit point may no longer be conical with respect to the subgroup.

But the size of the set of conical limit points that remain in passing to regular non-universal cover can only be made so small. Based on Theorem 3.1 which will be stated later, we now know the following result given by Falk and Stratmann [17].

Theorem 1.2. *Let $\hat{\Gamma}$ be a non-trivial normal subgroup of a non-elementary discrete group Γ of isometries of $\mathbb{H}^{n+1}$. Then $\delta(\hat{\Gamma}) \geq \dfrac{\delta(\Gamma)}{2}$.*

We remark that without the assumption of normality the result is false. In fact, for each $\epsilon > 0$ one can find a Schottky subgroup $\hat{G}$ of an arbitrary non-elementary group G so that $\delta(\hat{G}) < \epsilon$. (For an explicit construction of an example of a Schottky group Γ where $\delta(\Gamma) < \epsilon$ but $L(\Gamma) = \mathbb{S}^n$ see [34].)

Our first result shows that for $n = 1$ the bound in Theorem 1.2 can not be improved for any Fuchsian group Γ uniformizing a closed hyperbolic surface.

Theorem 1.3. *Let Γ be a Fuchsian group uniformizing a closed hyperbolic surface. Then there exists a sequence $\{\Gamma_i\}$ of normal subgroups of Γ so that $\delta(\Gamma_i) \to \dfrac{1}{2}$.*

Next we show that for convex co-compact groups the bound in Theorem 1.2 is strict:

Theorem 1.4. *Let Γ be a non-elementary convex co-compact discrete isometry group acting on $\mathbb{H}^{n+1}$, and let $\hat{\Gamma}$ be a non-trivial normal subgroup of Γ. Then*

$$\delta(\hat{\Gamma}) > \frac{\delta(\Gamma)}{2}.$$

The proof is carried out by using the ergodicity of the geodesic flow on $\mathbb{H}^{n+1}/\Gamma$ combined with a result on escaping geodesics due to Lundh [22].

We note that the conclusion of Theorem 1.4 leads to the following observation: If Γ uniformizes a closed hyperbolic manifold then every non-trivial normal subgroup of Γ uniformizes a hyperbolic manifold admitting a non-trivial L^2 spectral theory (see [39]). Hyperbolic manifolds exhibit an intricate and illuminating intertwining of their ergodic, conformal and geometric properties (see [32], [37], [38]). For instance, the Hausdorff dimension of the limit set can be seen as a measure of the geometric and topological complexity of a hyperbolic manifold (see e.g. [11], [10] and [27]).

On the other hand, we can show that there are large (non-regular) covers of a certain class of closed hyperbolic 3-manifolds.

Theorem 1.5. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold that fibers over a circle. Then, for every $\epsilon > 0$, there exists a Schottky subgroup G of Γ such that the exponent of convergence of G is greater than $2 - \epsilon$.*

We note that this theorem implies the existence of non-classical Schottky subgroups in groups fibering over circles (see the next section for details). Unlike classical Schottky groups, non-classical Schottky groups are relatively hard to both make explicit and to find (see [42]). See the below in Section 2 for a further discussion.

2. Large Schottky Subgroups

In this section we prove Theorem 1.5. Recall that a *Schottky group* G is a free, finitely generated, and purely loxodromic Kleinian group having non-empty regular set $\Omega(G) \subset \mathbb{S}^2$. (For the basics of Kleinian group theory see [25] and [30].) It is well known that Schottky groups are convex co-compact and as such the exponent of convergence of a Schottky group is the Hausdorff dimension of its limit set. It is also not hard to show that, since the Hausdorff dimension function is continuous on Schottky space $\mathcal{S}_m$ ($m \geq 2$ is the number of generators), then for any constant $k \in (0, 2)$ there exists a Schottky group $G \in \mathcal{S}_m$ so that $\delta(G) = k$. One says that a Schottky group $G \in \mathcal{S}_m$ is a *classical*

Schottky group if there exists $2m$ round circles $\{C_1, C_1', \dots, C_m, C_m'\}$ on $\mathbb{S}^2$ and a generating set $\{g_1, \dots, g_m\}$ so that the circles bound a domain D in $\mathbb{S}^2$ so that $g_j(D) \cap D = \emptyset$ and so that $g_j(C_j) = C_j'$ for all $1 \leq j \leq m$. If a Schottky group is not classical it is called a *non-classical Schottky group* (see [24] and [42]).

In higher dimension, Schottky and classical Schottky groups are also defined similarly. Doyle [15] has shown that, in terms of the exponent of convergence, a classical Schottky group can be only so large: *For each dimension $n \geq 2$ there exists a universal constant $C(n) < n$ so that for any classical Schottky group G acting on $\mathbb{H}^{n+1}$ the exponent of convergence $\delta(G)$ of G satisfies $\delta(G) < C(n)$.*

In light of Doyle's theorem, we have the following immediate corollary of Theorem 1.5:

Corollary 2.1. *If Γ uniformizes a closed hyperbolic 3-manifold that fibers over the circle then Γ contains a non-classical Schottky subgroup.*

We commence with the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold that fibers over a circle. Then M is represented by a mapping torus so that $\pi_1(M) \cong \pi_1(S) \rtimes \pi_1(\mathbb{S}^1)$, where S is a closed surface of genus ≥ 2. "Unwrapping" the $\mathbb{S}^1$ factor we pass to a regular cover $\hat{M} = \mathbb{H}^3/\hat{\Gamma}$ so that $\hat{\Gamma} \cong \pi_1(S)$; $\hat{\Gamma}$ is a doubly degenerate group. Since $\hat{\Gamma}$ is normal in Γ we have that the limit set $L(\Gamma) = \mathbb{S}^2 = L(\hat{\Gamma})$, and because $\hat{\Gamma}$ is topologically tame (see Bonahon [5]) and geometrically infinite we have by Canary [8] that $\delta(\hat{\Gamma}) = 2$. To obtain this, we may alternatively use a result for a cyclic cover of a closed manifold by Rees [36] (for an amenable cover of a manifold of finite topological type in general by Brooks [7]).

We now find the desired covers of M via finding covers of $\hat{M}$; we will work group-theoretically to do so. As $\hat{\Gamma}$ is isomorphic to the fundamental group of the closed surface S we work S as a model. On S we can find a simple, homotopically non-trivial and non-separating curve β. We cut S open along β to form a surface $\tilde{S}_0$ with two boundary components β^+ and β^- and form an infinite regular cyclic cover $\tilde{S}$ of S by gluing β^+ to β^-. Then $\tilde{S}_0$ also serves as the closure of a fundamental domain for the action of the covering group on $\tilde{S}$ which is isomorphic to $\pi_1(S)/\pi_1(\tilde{S}) \cong \mathbb{Z}$.

We take certain subgroups of $\pi_1(\tilde{S})$ as follows. Note that $\pi_1(\tilde{S})$ is a free group (see Propositions 3.5 and 3.6 below). Let f be the cyclic generator of the covering group $\pi_1(S)/\pi_1(\tilde{S})$, and let $\tilde{S}_n = \bigcup_{i=-n}^{n} f^i(\tilde{S}_0)$. Fixing base points of the fundamental groups, we assume $\pi_1(\tilde{S}_n) \subset$

$\pi_1(\tilde{S}) \subset \pi_1(S)$. Recall that $\pi_1(S) \cong \hat{\Gamma}$ and so we can find a sequence $\{\hat{\Gamma}_n\}$ of free subgroups of $\hat{\Gamma}$ so that $\pi_1(\tilde{S}_n) \cong \hat{\Gamma}_n \subset \pi_1(\tilde{S})$ for each index n. Note that, since $\tilde{S}_n \subset \tilde{S}_{n+1}$ we have that $\hat{\Gamma}_n \subset \hat{\Gamma}_{n+1}$, and so let $\hat{\Gamma}_\infty = \bigcup_{n=1}^{\infty} \hat{\Gamma}_n$, which is isomorphic to $\pi_1(\tilde{S})$. Hence $\hat{\Gamma}_\infty$ is a normal subgroup of $\hat{\Gamma}$ such that $\hat{\Gamma}/\hat{\Gamma}_\infty \cong \mathbb{Z}$.

We now need to verify two claims.

Claim 1: *$\hat{\Gamma}_n$ is a Schottky group for each index n.*

Proof of Claim 1: This is an application of the Canary Covering Theorem. In particular we will show that any free purely loxodromic and finitely generated subgroup of a doubly degenerate group is Schottky. By construction we have that, for each index n, $\hat{\Gamma}_n$ is purely loxodromic and finitely generated, and we have observed that it is a free group.

To derive a contradiction, we assume that $\hat{\Gamma}_n$ is not Schottky; thus it is a free, purely loxodromic, and finitely generated subgroup of $\hat{\Gamma}$ that must necessarily have empty regular set (Theorem X.H.5 [25]). Let $\hat{M}_n = \mathbb{H}^3/\hat{\Gamma}_n$; thus $\hat{M}_n$ is a topologically tame cover of the doubly degenerate manifold $\hat{M} = \mathbb{H}^3/\hat{\Gamma}$ (Agol [2], Calegari and Gabai [12]). Since $\hat{\Gamma}_n$ is finitely generated, free and purely loxodromic, $\hat{M}_n$ is homeomorphic to the interior of a handlebody and thus $\hat{M}_n$ has only one geometrically infinite end. By Canary's Covering Theorem [9] we have that the geometrically infinite end of $\hat{M}_n$ covers an end of $\hat{M}$ in a finite to one fashion. This would imply that $\hat{\Gamma}$, which is isomorphic to the group $\pi_1(S)$ where S is a closed surface, has a finite index free subgroup. This is the desired contradiction: each $\hat{\Gamma}_n$ is thus Schottky.

Claim 2: *The group $\hat{\Gamma}_\infty$ is the geometric limit of the sequence $\{\hat{\Gamma}_n\}$, and* $\lim_{n\to\infty} \delta(\hat{\Gamma}_n) = \delta(\hat{\Gamma}_\infty)$=2.

Proof of Claim 2: Let $\hat{\Gamma}_G$ be any geometric limit of a subsequence of $\{\hat{\Gamma}_n\}$. Since $\{\hat{\Gamma}_n\}$ is an increasing nested sequence we have that $\hat{\Gamma}_n \subset \hat{\Gamma}_G$ for each index n and so we can observe that $\hat{\Gamma}_\infty \subseteq \hat{\Gamma}_G$. We need to show $\hat{\Gamma}_G \subseteq \hat{\Gamma}_\infty$; this too is an easy observation from the discreteness of $\hat{\Gamma}_G$, and from the fact that each element of $\hat{\Gamma}_G$ is an accumulation point of a sequence $\{\gamma_n \in \hat{\Gamma}_n\}$.

Since $\hat{\Gamma}_\infty$ is the geometric limit of $\{\hat{\Gamma}_n\}$ we have that $\liminf \delta(\hat{\Gamma}_n) \geq \delta(\hat{\Gamma}_\infty)$ ([37], see also [26] and [40]). The inequality in the other direction comes from the fact that $\hat{\Gamma}_n \subset \hat{\Gamma}_\infty$ for all indices n.

Finally, we claim that $\delta(\hat{\Gamma}_\infty) = 2$. Since $\hat{\Gamma}_\infty$ is a normal subgroup of $\hat{\Gamma}$ such that $\hat{\Gamma}/\hat{\Gamma}_\infty \cong \mathbb{Z}$, the result by Brooks (Theorem 1 [7]) yields that $\delta(\hat{\Gamma}_\infty) = \delta(\hat{\Gamma}) = 2$ (also by using well-known Theorem 3.4 below).

The result now follows immediately from claim 1 and claim 2. We have found a nested sequence $\{\hat{\Gamma}_n\}$ of Schottky subgroups of $\hat{\Gamma}_\infty$ so that $\{\hat{\Gamma}_n\}$ converges geometrically to $\hat{\Gamma}_\infty$. The resulting sequence $\{\delta(\hat{\Gamma}_n)\}$ of exponents of convergence has the property that

$$\lim_{n\to\infty} \delta(\hat{\Gamma}_n) = \delta(\hat{\Gamma}_\infty) = 2.$$

This completes the proof of Theorem 1.5. □

Remark 2.2. If Thurston's virtual fiber question has a positive answer, then the proof of Theorem 1.5 shows that all closed hyperbolic orbifolds (by passing to a finite index torsion-free cover) contain a non-classical Schottky group whose exponent is arbitrarily close to 2. Thurston's virtual fiber question does have a positive answer in many cases; see [1] for an overview.

3. Small Regular Covers

This section is devoted to the discussion and proof of Theorem 1.3. Recall that the *conical* limit set of a discrete group Γ of isometries of $\mathbb{H}^{n+1}$ is a set of limit points on $\mathbb{S}^n$, each of which has the property that there exists a sequence $\{\gamma_i \in \Gamma\}$ and a cone region in $\mathbb{H}^{n+1}$ based at the limit point, so that the orbit $\{\gamma_i(0)\}$ limits to this point within the fixed cone region. Similarly, the (big) *horospherical* limit set of Γ is a set of limit points, each of which has the property that there exists a sequence $\{\gamma_i \in \Gamma\}$ and a horoball in $\mathbb{H}^{n+1}$ based at the limit point, so that the orbit $\{\gamma_i(0)\}$ limits to this point within the fixed horoball. Note that the conical limit set is contained in the horospherical limit set and the conical limit set of any non-elementary discrete group is non-empty. Denote the conical limit set of Γ by $L_c(\Gamma)$ and the horospherical limit set by $L_h(\Gamma)$. In the following the Hausdorff dimension of a limit set A is represented by $\dim A$. It is shown in [17] and [29] that

$$\delta(\Gamma) \geq \frac{\dim L_h(\Gamma)}{2}.$$

(A generalization of this fact is given in Proposition 4.2 later.)

Now we specialize further: let S be a fixed closed hyperbolic surface, and let F be a Fuchsian group so that $S = \mathbb{H}^2/F$. It is well known that $L(F) = L_c(F) = \mathbb{S}^1$ and $\delta(F) = 1$. Let $\hat{S}$ be any non-universal regular cover of S, and let $\hat{F}$ be a normal subgroup of F such that $\hat{S} = \mathbb{H}^2/\hat{F}$. It is also well known that $L(\hat{F}) = L(F)$, however recall from the introduction that it may be the case that $L_c(\hat{F})$ is properly contained in $L(F) = \mathbb{S}^1$. The geometric degeneration of the conical limit set of F in passing to a regular cover is described by

Theorem 3.1 ([28] Theorem 6)**.** *Let $\hat{\Gamma}$ be a non-trivial normal subgroup of a discrete isometry group Γ acting on $\mathbb{H}^{n+1}$. Then $L_c(\Gamma) \subseteq L_h(\hat{\Gamma})$.*

It follows from this theorem that any non-trivial normal subgroup $\hat{\Gamma}$ of a discrete isometry group Γ whose conical limit set $L_c(\Gamma)$ is of full Patterson-Sullivan measure μ for Γ has the property that the horospherical limit set $L_h(\hat{\Gamma})$ is of full measure with respect to μ ([28] Corollary 7) and so in particular we have the following result. Note that this is a special case of Theorem 1.2 in the introduction ([17]).

Corollary 3.2. *If $\hat{F}$ is any non-trivial normal subgroup of a Fuchsian group F uniformizing a closed surface, then $\delta(\hat{F}) \geq \dfrac{1}{2}$.*

And so this is our definition of a *small non-trivial regular cover* of a closed hyperbolic surface: such a cover has the property that its uniformizing group has exponent of convergence (equivalently the Hausdorff dimension of the conical limit set) that is close to $\frac{1}{2}$. We show in this section that –for any closed surface– such small regular covers exist.

Let λ_0 denote the bottom of the discrete spectrum of the Laplacian operator on a Riemannian manifold. Let $p : Y \to X$ be a regular covering. We say that the covering (p, Y, X) is *amenable* by definition if there is a linear functional $m : L^\infty(\pi_1(X)/\pi_1(Y)) \to \mathbb{R}$, invariant under the left action of $\pi_1(X)$, so that $\inf(f) \leq m(f) \leq \sup(f)$ for all $f \in L^\infty(\pi_1(X)/\pi_1(Y))$. A result of Brooks states:

Theorem 3.3 ([6], [7])**.** *If Y is a non-amenable regular cover of a smooth closed Riemannian manifold X, then $\lambda_0(Y) > \lambda_0(X) = 0$.*

There exists a fundamental relationship between $\lambda_0(\mathbb{H}^{n+1}/\Gamma)$ and $\delta(\Gamma)$: in fact the magnitude of $\delta(\Gamma)$ determines $\lambda_0(\mathbb{H}^{n+1}/\Gamma)$, which is due to Elstrodt, Patterson and Sullivan.

Theorem 3.4 ([39])**.** *For a hyperbolic manifold $M = \mathbb{H}^{n+1}/\Gamma$, $\lambda_0(M) = \frac{n^2}{4}$ if $\delta(\Gamma) \leq \frac{n}{2}$ and $\lambda_0(M) = \delta(\Gamma)(n - \delta(\Gamma))$ if $\delta(\Gamma) \geq \frac{n}{2}$.*

The following two facts give the basic algebraic characteristics of a non-amenable and non-universal regular cover. Recall that a finite index regular covering of a closed surface is amenable, and thus any non-amenable cover must necessarily be of infinite index.

Proposition 3.5. *Let S be a closed hyperbolic surface, and suppose $\hat{S} \neq \mathbb{H}^2$ is an infinite index regular covering. Then $\pi_1(\hat{S})$ is infinitely generated.*

Proof. Suppose not. Then if $\hat{S} = \mathbb{H}^2/\hat{F}$ we assume that $\hat{F}$ is a finitely generated Fuchsian group. Since $\hat{S}$ is a regular cover of the closed surface S we have that $L(\hat{F}) = L(F) = \mathbb{S}^1$, where $S = \mathbb{H}^2/F$. A finitely generated Fuchsian group whose limit set is all of $\mathbb{S}^1$ has co-finite area. In particular, this implies that the index of $\hat{F}$ in F is finite, and so $\hat{S}$ is thus a finite cover of S. This is a contradiction. □

More is true: using the fact below we see that the fundamental group of an infinite index regular covering of a closed hyperbolic surface is free.

Proposition 3.6 ([35] Theorem 4)**.** *If F is an infinitely generated torsion-free Fuchsian group then it is a free group.*

See also [20] p.137 and [19] p.210.

Assume that $p : \hat{S} \to S$ is a non-amenable regular covering of a closed hyperbolic surface. Using Corollary 3.2 and Theorems 3.3 and 3.4 we have:

Theorem 3.7. *Let $p : \hat{S} \to S$ be a non-universal non-amenable regular covering. As before, let $S = \mathbb{H}^2/F$ and let $\hat{S} = \mathbb{H}^2/\hat{F}$. Then*

$$\frac{1}{2} \leq \delta(\hat{F}) < \delta(F) = 1.$$

This motivates the following definition. Let M be a hyperbolic manifold. To simplify the notation, let $\delta(M)$ be the exponent of convergence $\delta(\Gamma)$, where $M = \mathbb{H}^{n+1}/\Gamma$. For S a closed hyperbolic surface, let

$$b(S) = \inf\{\delta(\hat{S}) : \ \hat{S} \text{ is a non-universal regular cover}\}.$$

We note that, by the Retrosection Theorem (see e.g. [3]) and Theorem 3.7, we have that $b(S) < 1$. Similarly, we define

$$e(S) = \sup\{\delta(\hat{S}) : \ \hat{S} \text{ is a non-amenable regular cover}\}.$$

Our main result in this section is the following theorem, which is equivalent to Theorem 1.3 in the introduction.

Theorem 3.8. *Let S be a closed hyperbolic surface. Then there exists a sequence of regular covers $\hat{S}_i$ of S so that*

$$\delta(\hat{S}_i) \to \frac{1}{2}.$$

In particular, $b(S) = \frac{1}{2}$.

Remark 3.9. We defined $e(S)$ to ask the following question: *Analogous to $b(S)$, is $e(S)$ is equal to one*?

The remainder of this section is devoted to a proof of Theorem 3.8. We start by providing an explicit construction of the surfaces on which we will demonstrate the proof this theorem.

Lemma 3.10. *Let S be a closed hyperbolic surface. For an arbitrary positive integer i, there exists a finite index regular cover S_i of S such that the lengths of all closed geodesics in S_i are greater than i.*

Proof. As S is closed it is well known that the length spectrum of S is discrete, and thus for any number $i \in \mathbb{Z}^+$ there exist only finitely many closed geodesics of length less than or equal to i. Let $S = \mathbb{H}^2/F$, and let $\{a_1, \ldots, a_{n(i)}\}$ be (hyperbolic) representatives in F that uniformize this collection of short closed geodesics. Because F is residually finite [23] there exists a finite index subgroup $\tilde{F}_i$ so that $a_j \notin \tilde{F}_i$ for all $1 \leq j \leq n(i)$. Let F_i be the finite index normal subgroup of F formed by intersecting all conjugates in F of $\tilde{F}_i$. We have thus found a subgroup F_i of F with the property that all of its non-trivial elements have translation length greater than i, and the surface $S_i = \mathbb{H}^2/F_i$ is the desired cover. □

Fix a simple closed geodesic c in S and let γ be an element of F representing c. For each i, since the index $[F : F_i]$ is finite, there exists an integer $m_i \geq 1$ dividing $[F : F_i]\,!$ such that γ^{m_i} belongs to F_i. We then consider the normal closure $\hat{F}_i$ of γ^{m_i} in F; recall that this is the smallest normal subgroup of F that contains γ^{m_i}. Since F_i is normal in F and also contains γ^{m_i}, we have that $\hat{F}_i \subset F_i$. This implies that the surface $\hat{S}_i = \mathbb{H}^2/\hat{F}_i$ covers $S_i = \mathbb{H}^2/F_i$ and hence that the lengths of all closed geodesics in $\hat{S}_i$ are also greater than i. Applying Proposition X.A.3 of Maskit [25], we see that $\hat{S}_i$ is planar, that is, all non-trivial simple closed curves in $\hat{S}_i$ are dividing.

The sequence $\{\hat{S}_i\}$ is our candidate sequence of regular covers for showing $\frac{1}{2}$ is the best possible constant. That is, we will show that $\delta(\hat{S}_i) \to \frac{1}{2}$. To this end, we again use the relationship between the bottom of the spectrum $\lambda_0(\hat{S}_i)$ of the hyperbolic Laplacian and the critical exponent $\delta(\hat{S}_i)$ as detailed in Theorem 3.4. In particular we see from this theorem that $0 \leq \lambda_0(\hat{S}_i) \leq \frac{1}{4}$ and if $\lambda_0(\hat{S}_i) \nearrow \frac{1}{4}$ then $\delta(\hat{S}_i) \searrow \frac{1}{2}$ as $i \to \infty$.

For a hyperbolic surface S, the *isoperimetric constant* (sometimes called the *Cheeger constant*) is defined by

$$h(S) = \sup_W \frac{A(W)}{\ell(\partial W)},$$

where the supremum is taken over all compact subsurfaces $W \subset S$ with smooth boundary. Here $A(W)$ is the hyperbolic area of W and $\ell(\partial W)$ is the hyperbolic length of the boundary ∂W. Note that the isoperimetric constant always satisfies $h(S) \geq 1$. An estimate of $\lambda_0(\hat{S}_i)$ from below is given by the following result due to Cheeger [14] (see also [13]).

Theorem 3.11. *The bottom of the spectrum of the Laplacian and the isoperimetric constant for a hyperbolic surface S satisfy the inequality*

$$(1/4 \geq)\ \lambda_0(S) \geq \frac{1}{4h(S)^2}.$$

In the definition of the isoperimetric constant, if a subsurface W has a boundary curve that bounds a topological disk in S, then by filling the disk, the area becomes larger but the boundary length becomes smaller. Hence W can be assumed to have no trivial boundary curves. For each boundary curve c of W, there is a unique simple closed geodesic c^* freely homotopic to c. Let W^* denote a compact subsurface of S where each boundary curve c of W is replaced by the simple geodesic c^* in the homotopy class of c. When two boundary curves c_1 and c_2 of W are freely homotopic, we assume that W^* has two geodesic boundaries corresponding to c_1 and c_2 that are the same simple closed geodesic in W. In general, a compact subsurface is called a geodesic subsurface if its boundary consists of a finite number of simple closed geodesics. By restricting W to geodesic subsurfaces W^* in the definition of the isoperimetric constant, we have another isoperimetric constant

$$h^*(S) = \sup_{W^*} \frac{A(W^*)}{\ell(\partial W^*)},$$

which was introduced by Fernández and Rodríguez [18]. The inequality $h^*(S) \leq h(S)$ is clearly true. Conversely, as the following proposition shows, $h(S)$ is almost bounded from above by $h^*(S)$.

Theorem 3.12 ([29])**.** *Let W be a compact subsurface with smooth boundary in a hyperbolic surface S without cusp and assume that its boundary curves are non-trivial. Then the geodesic subsurface W^* homotopically equivalent to W in S satisfies*

$$\frac{A(W)}{\ell(\partial W)} \leq \frac{A(W^*)}{\ell(\partial W^*)} + 1.$$

In particular $(1 \leq)\ h(S) \leq h^*(S) + 1$.

We are now in a position to finish the proof of Theorem 3.8.

Proof of Theorem 3.8. Consider an arbitrary geodesic subsurface W^* in $\hat{S}_i$. Since $\hat{S}_i$ is planar, W^* is an n-ply connected domain for some $n \geq 3$. The area of W^* is $2\pi(n-2)$ by the Gauss-Bonnet formula. Since lengths of simple closed geodesics on $\hat{S}_i$ are all greater than i, the total length of the boundary geodesics of W^* is greater than ni. Hence

$$\frac{A(W^*)}{\ell(\partial W^*)} < \frac{2\pi(n-2)}{ni} < \frac{2\pi}{i},$$

and taking the supremum over all W^* we have $h^*(\hat{S}_i) \leq 2\pi/i$. This implies that, as $i \to \infty$, $h^*(\hat{S}_i) \to 0$ and so $h(\hat{S}_i) \to 1$ by Theorem 3.12. Using the Theorem 3.11 we see that $\lambda_0(\hat{S}_i) \to \frac{1}{4}$, and so by Theorem 3.4 we have $\delta(\hat{S}_i) \to \frac{1}{2}$, as desired. □

4. $b(S)$ Is Not Realized

In this section we will show that $b(S) = \frac{1}{2}$ is not realized by any regular cover of a closed surface S. This follows immediately from the more general Theorem 1.4. See also the problem in Section 5 of [29].

Corollary 4.1. *Let Γ be a Fuchsian group uniformizing a closed hyperbolic surface, and let $\hat{\Gamma}$ be a non-trivial normal subgroup of Γ. Then*

$$\delta(\hat{\Gamma}) > \frac{1}{2}.$$

We will give a proof of Theorem 1.4 in this section. The idea of the proof is as follows. By Theorem 3.1, we see that every limit point of $\hat{\Gamma}$ is a horospherical limit point, from which we see $\delta(\hat{\Gamma}) \geq \frac{\delta(\Gamma)}{2}$. In order to obtain the strict inequality, we will show that almost all limit points of $\hat{\Gamma}$ are in a certain class of limit points which are accumulated by the orbits of better (i.e., closer to conical) approach than the horospherical one.

First we introduce a continuous family of limit sets of a discrete group by the approaching order of its orbits (Nicholls [31]). Fix $k > 0$ and $\alpha \in (0, 1]$. For a point $z \in \mathbb{H}^{n+1}$, an (k, α)-shadow is a disk in $\mathbb{S}^n$ defined by

$$I(z : k, \alpha) = \left\{ \xi \in \mathbb{S}^n : \left| \xi - \frac{z}{|z|} \right| < k(1 - |z|)^\alpha \right\}.$$

For a discrete group Γ of isometries of $\mathbb{H}^{n+1}$, consider the orbit $\Gamma(z) = \{\gamma(z)\}_{\gamma \in \Gamma}$ of $z \in \mathbb{H}^{n+1}$ and define $\mathcal{L}_{k,\alpha}(\Gamma)$ to be a set of all points

$\xi \in \mathbb{S}^n$ such that ξ belongs to infinitely many $I(\gamma(z) : k, \alpha)$. This set $\mathcal{L}_{k,\alpha}(\Gamma) \subseteq L(\Gamma)$ depends on the choice of z, but the union

$$\mathcal{L}_\alpha(\Gamma) := \bigcup_{k>0} \mathcal{L}_{k,\alpha}(\Gamma),$$

taken over all $k > 0$, is defined independently of z. When $\alpha = 1$ the set $\mathcal{L}_\alpha(\Gamma)$ is (if nonempty) nothing more than the conical limit set $L_c(\Gamma)$, and when $\alpha = \frac{1}{2}$, $\mathcal{L}_\alpha(\Gamma)$ is coincident with the (big) horospherical limit set $L_h(\Gamma)$. By moving α between $\frac{1}{2}$ and 1, we are interpolating between the horospherical limit set and the conical limit set.

The Hausdorff dimension of the conical limit set $L_c(\Gamma) = \mathcal{L}_1(\Gamma)$ is bounded from above by the exponent of convergence $\delta(\Gamma)$ (Γ could be elementary). This fact can be generalized to a bound on the Hausdorff dimension $\dim \mathcal{L}_\alpha(\Gamma)$ as follows. For $\alpha = \frac{1}{2}$, this fact has been stated at the beginning of Section 3. Note that essentially the same inequality has appeared in p.575 of Falk and Stratmann [17]. Here we give a proof for the readers' convenience.

Proposition 4.2. *Any discrete group Γ of isometries of $\mathbb{H}^{n+1}$ satisfies*

$$\delta(\Gamma) \geq \alpha \dim \mathcal{L}_\alpha(\Gamma).$$

Proof. We can show that there is a constant $A > 0$ independent of $\gamma \in \Gamma$ such that the Euclidian radius of $I(\gamma(z) : k, \alpha)$ is bounded by $A(1 - |\gamma(z)|)^\alpha$ (e.g., see p.24 in Nicholls [31]). Since the Poincaré series $\sum_{\gamma\in\Gamma}(1 - |\gamma(z)|)^s$ converges at the exponent s for every $s > \delta(\Gamma)$, a standard argument (e.g., pp.76–77 in [31]) yields that (s/α)-dimensional Hausdorff measure of $\mathcal{L}_{k,\alpha}(\Gamma)$ is zero. Since $\mathcal{L}_\alpha(\Gamma)$ can be written as a countable union of such $\mathcal{L}_{k,\alpha}(\Gamma)$, (s/α)-dimensional Hausdorff measure of $\mathcal{L}_\alpha(\Gamma)$ is also zero. This implies that $\dim \mathcal{L}_\alpha(\Gamma) \leq \frac{s}{\alpha}$, and hence, by taking $s \to \delta(\Gamma)$, $\dim \mathcal{L}_\alpha(\Gamma) \leq \frac{\delta(\Gamma)}{\alpha}$ follows. □

The limit set $\mathcal{L}_\alpha(\Gamma)$ can be interpreted geometrically via another classification of limit points. We introduce certain subsets of limit points according to Lundh [22], but one may notice that Falk and Stratmann [17] also intended some arguments of a similar flavor. Let $\tilde{g}_{\xi,z}(t)$ be a geodesic ray of unit speed starting from a given point $z \in \mathbb{H}^{n+1}$ towards $\xi \in \mathbb{S}^n$. The projection of $\tilde{g}_{\xi,z}(t)$ in $M = \mathbb{H}^{n+1}/\Gamma$ is denoted by $g_{\xi,z}(t)$ and we set

$$\varphi_\xi(t) := d(g_{\xi,z}(t), g_{\xi,z}(0)),$$

which is the hyperbolic distance in the quotient manifold M between $g_{\xi,z}(t)$ and the initial point $g_{\xi,z}(0)$. Alternatively, it is defined as the distance of the orbit $\Gamma(z)$ from $\tilde{g}_{\xi,z}(t)$ in $\mathbb{H}^{n+1}$. It is clear that $\varphi_\xi(t) \leq t$.

The ratio $\varphi_\xi(t)/t$ measures how rapidly or slowly the geodesic ray $g_{\xi,z}(t)$ escapes to infinity as $t \to \infty$. For instance, $g_{\xi,z}(t)$ is called a *linearly escaping geodesic* if there exists a positive constant $c > 0$ such that $\varphi_\xi(t)/t \geq c$ for all t. However, what we investigate here are geodesic rays that are escaping slowly or are non-escaping. For each $r \in [0,1]$ we define the following set of end points of geodesic rays:

$$\Lambda_r(\Gamma) = \{\xi \in \mathbb{S}^n \mid \liminf_{t\to\infty} \frac{\varphi_\xi(t)}{t} \leq r\}.$$

The conical limit points correspond to non-escaping geodesic rays and hence $L_c(\Gamma)$ is contained in $\Lambda_0(\Gamma)$. When $r = 1$, $\Lambda_1(\Gamma)$ is coincident with the entire sphere $\mathbb{S}^n$ since $\varphi_\xi(t)/t \leq 1$ for all $\xi \in \mathbb{S}^n$ and all $t > 0$.

Concerning the relationship between $\mathcal{L}_\alpha(\Gamma)$ and $\Lambda_r(\Gamma)$, Lundh [22] obtained the following result.

Theorem 4.3. *Let $\frac{1}{2} < \alpha < 1$. If $r = \frac{1-\alpha}{\alpha}$ then $\mathcal{L}_\alpha(\Gamma) \subset \Lambda_r(\Gamma)$. On the other hand, if $r < \frac{1-\alpha}{\alpha}$, then $\Lambda_r(\Gamma) \subset \mathcal{L}_\alpha(\Gamma)$.*

An essential step towards the proof of Theorem 1.4 is the following claim, which is the core of our arguments.

Lemma 4.4. *Let Γ be a non-elementary convex co-compact discrete isometry group acting on $\mathbb{H}^{n+1}$ and let $\hat{\Gamma}$ be a non-trivial normal subgroup of Γ. Then, for some positive constant $\rho > 0$, the Hausdorff dimension of $\Lambda_{1-\rho}(\hat{\Gamma})$ is $\delta(\Gamma)$.*

Assuming for the moment the proof of this lemma we assemble the proof of Theorem 1.4.

Proof of Theorem 1.4. By Lemma 4.4, $\dim\Lambda_{1-\rho}(\hat{\Gamma}) = \delta(\Gamma)$ for some $\rho > 0$. Choose $\alpha > \frac{1}{2}$ such that $\frac{1-\alpha}{\alpha} > 1-\rho$. Then $\mathcal{L}_\alpha(\hat{\Gamma})$ contains $\Lambda_{1-\rho}(\hat{\Gamma})$ by Theorem 4.3, and hence $\dim\mathcal{L}_\alpha(\hat{\Gamma}) \geq \dim\Lambda_{1-\rho}(\hat{\Gamma}) = \delta(\Gamma)$. Applying Proposition 4.2, we have $\delta(\hat{\Gamma}) \geq \alpha \dim\mathcal{L}_\alpha(\hat{\Gamma}) \geq \alpha\,\delta(\Gamma)$. Since $\alpha > \frac{1}{2}$, this yields the assertion. ☐

The proof of Lemma 4.4 relies on the ergodicity of the geodesic flow on the unit tangent bundle of $M = \mathbb{H}^{n+1}/\Gamma$ with respect to its Patterson-Sullivan measure. Note that Falk [16] investigated the Myrberg limit points in a similar way. See Nicholls [31] for a general reference of the facts explained below.

Let μ be a Γ-invariant conformal measure of dimension $\delta(\Gamma)$ supported on the limit set $L(\Gamma)$, which is the so called *Patterson-Sullivan measure* for Γ. If Γ is convex co-compact this measure is coincident up to multiplication by a positive constant with the $\delta(\Gamma)$-dimensional Hausdorff measure supported on $L(\Gamma)$. The unit tangent space $T^1_z\mathbb{H}^{n+1}$

at $z \in \mathbb{H}^{n+1}$ can be identified with $\mathbb{S}^n$. A measure $\tilde{m}$ on the unit tangent bundle $T^1\mathbb{H}^{n+1}$ that is invariant under the geodesic flow is induced by μ (Sullivan [37]). The quotient unit tangent bundle T^1M is nothing but the quotient of $T^1\mathbb{H}^{n+1}$ by the canonical action of Γ. Since μ is invariant under Γ, so is $\tilde{m}$ and hence it descends to a measure m on T^1M. When Γ is geometrically finite, it is known that the total volume $m(T^1M)$ is finite.

For a unit tangent vector v at $p \in M$ ($(v,p) \in T^1(M)$), let $g_{v,p}(t)$ denote the geodesic line such that $g_{v,p}(0) = p$ and $g'_{v,p}(0) = v$. The *geodesic flow* $\phi_t : T^1M \to T^1M$ is a map sending (v,p) to $(g'_{v,p}(t), g_{v,p}(t))$ for each $t \in \mathbb{R}$. If Γ is geometrically finite, then the geodesic flow on T^1M is ergodic with respect to the measure m. This means that if E is a measurable subset of T^1M that is invariant under ϕ_t for all t then $m(E) = 0$ or $m(T^1M \setminus E) = 0$. From this ergodicity, it follows that the time mean coincides with the space mean, that is, for every measurable subset A of T^1M,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_A(\phi_t(v,p))dt = \frac{m(A)}{m(T^1M)}$$

for almost every $(v,p) \in T^1M$ with respect to m.

Let β be a geodesic segment in $\mathbb{H}^{n+1}$ with end points b_- and b_+. For $\varepsilon > 0$, take n-dimensional hyperbolic closed disks $\tilde{D}_-$ and $\tilde{D}_+$ of radius ε centered at b_- and b_+ respectively that are perpendicular to β. Then the convex hull $\tilde{C}_\varepsilon(\beta)$ of $\tilde{D}_-$ and $\tilde{D}_+$ is the closed convex cylindrical region in $\mathbb{H}^{n+1}$ that is the union of all geodesic segments β' connecting $\tilde{D}_-$ and $\tilde{D}_+$. We define a *flow tube* $\tilde{N}_\varepsilon(\beta)$ in the unit tangent bundle $T^1\mathbb{H}^{n+1}$ by

$$\tilde{N}_\varepsilon(\beta) = \{(v,z) \in T^1\mathbb{H}^{n+1} \mid z \in \tilde{C}_\varepsilon(\beta),\ v = \frac{d\beta'}{dt} \in T^1_z\mathbb{H}^{n+1}\},$$

where β' is some oriented geodesic segment starting from a point in $\tilde{D}_-$ and ending at a point in $\tilde{D}_+$. We now have the objects required to prove Lemma 4.4.

Proof of Lemma 4.4. Let $\pi : \mathbb{H}^{n+1} \to M = \mathbb{H}^{n+1}/\Gamma$ and $\hat{\pi} : \mathbb{H}^{n+1} \to \hat{M} = \mathbb{H}^{n+1}/\hat{\Gamma}$ denote the covering projections. We may also use the same notations π and $\hat{\pi}$ for the projections of the unit tangent bundles $T^1\mathbb{H}^{n+1} \to T^1M$ and $T^1\mathbb{H}^{n+1} \to T^1\hat{M}$ since the unit tangent vectors v are naturally identified under these projections.

Since $\hat{\Gamma}$ is a non-trivial normal subgroup of Γ, it is non-elementary and hence it contains a loxodromic element γ. Consider the axis of γ in $\mathbb{H}^{n+1}$ and take a geodesic segment β on the axis that is a fundamental set for $\langle\gamma\rangle$; β has no equivalent pair under $\langle\gamma\rangle$ except its end points. Then the projection $\hat{\pi}(\beta)$ on $\hat{M}$ turns to be a closed geodesic. Let ℓ be the length of β and choose $\varepsilon > 0$ so small that $\varepsilon \leq \frac{\ell}{6}$. For these β and ε, we take the flow tube $\tilde{N}_\varepsilon(\beta)$ in $T^1\mathbb{H}^{n+1}$ as above and denote its projection $\pi(\tilde{N}_\varepsilon(\beta))$ on T^1M by $N_\varepsilon(\beta)$. For the canonical measure m on T^1M, set $2\rho = m(N_\varepsilon(\beta))/m(T^1M)$. Note that, since $N_\varepsilon(\beta)$ has non-empty interior in T^1M and non-empty intersection with the support of m, the measure of $N_\varepsilon(\beta)$ is positive and hence $\rho > 0$. By the ergodicity of the geodesic flow, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{N_\varepsilon(\beta)}(\phi_t(v,p))dt = 2\rho$$

for almost every $(v,p) \in T^1M$. In particular, there is some $p \in M$ such that almost every $v \in T^1_pM$ satisfies this property. We may assume that $p \notin \pi(\tilde{C}_\varepsilon(\beta))$.

For each $v \in T^1_pM$, let t'_1 be the first time when $\phi_t(v,p)$ starts the first round in $N_\varepsilon(\beta)$ by departing from a tangent vector on $D = \pi(\tilde{D}_-)$, and let t_1 be the first time when $\phi_t(v,p)$ completes the first round in $N_\varepsilon(\beta)$ by returning to a tangent vector on $D = \pi(\tilde{D}_+)$. Similarly we define $t'_2, t'_3, \ldots$ and $t_2, t_3, \ldots$ one after another when $\phi_t(v,p)$ begins and finishes each round in $N_\varepsilon(\beta)$ respectively. Choose $\hat{p} \in \hat{M}$ lying above the $p \in M$ against the projection $\hat{M} \to M$ and consider the geodesic ray $\hat{g}_{v,\hat{p}}(t)$ starting from $\hat{p}$ that is a lift of $g_{v,p}(t)$ to $\hat{M}$.

We look at a subarc $\hat{\beta}_n$ of $\hat{g}_{v,\hat{p}}(t)$ between t'_n and t_n for each integer $n \geq 1$. The length of $\hat{\beta}_n$ is greater than or equal to ℓ and the distance $d(\hat{g}_{v,\hat{p}}(t'_n), \hat{g}_{v,\hat{p}}(t_n))$ between the initial and the terminal points is less than or equal to 2ε. Hence if $\hat{g}_{v,\hat{p}}(t)$ avoided this detour along $\hat{\beta}_n$, it would have a short cut which reduces its itinerary at least $\ell - 2\varepsilon$. Let $\hat{\varphi}_\xi(t)$ be the distance $d(\hat{g}_{v,\hat{p}}(t), \hat{g}_{v,\hat{p}}(0))$ in $\hat{M}$, where $\xi \in \mathbb{S}^n$ is corresponding to the tangent vector v. By the consideration above, we see that

$$\hat{\varphi}_\xi(t_n) \leq t_n - n(\ell - 2\varepsilon)$$

for all integers $n \geq 1$. On the other hand, we have seen

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} 1_{N_\varepsilon(\beta)}(\phi_t(v,p))dt = 2\rho,$$

and this integral represents the total time spent in $N_\varepsilon(\beta)$ and thus estimated from above by $n(\ell + 2\varepsilon)$. Therefore

$$\limsup_{n\to\infty} \frac{n(\ell + 2\varepsilon)}{t_n} \geq 2\rho.$$

Using these estimates, we obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{\hat{\varphi}_\xi(t)}{t} &\leq \liminf_{n\to\infty}\frac{\hat{\varphi}_\xi(t_n)}{t_n}\\ &\leq \liminf_{n\to\infty}\frac{t_n-n(\ell-2\varepsilon)}{t_n}\\ &= 1-\limsup_{n\to\infty}\frac{n(\ell+2\varepsilon)}{t_n}\cdot\frac{\ell-2\varepsilon}{\ell+2\varepsilon}\\ &\leq 1-2\rho\cdot\frac{1}{2}=1-\rho.\end{aligned}$$

This implies that almost every $\xi\in\mathbb{S}^n$ belongs to $\Lambda_{1-\rho}(\hat{\Gamma})$ with respect to the Patterson-Sullivan measure μ for Γ. Since $\Lambda_{1-\rho}(\hat{\Gamma})\subseteq L(\hat{\Gamma})=L(\Gamma)$ and μ is essentially the same as the $\delta(\Gamma)$-dimensional Hausdorff measure restricted to $L(\Gamma)$, we see that the $\delta(\Gamma)$-dimensional Hausdorff measure of $\Lambda_{1-\rho}(\hat{\Gamma})$ is positive and finite, and in particular $\dim\Lambda_{1-\rho}(\hat{\Gamma})=\delta(\Gamma)$. □

We can say a bit more about any sequence $\{S_i\}$ of regular covers of a closed surface S having the property that $\delta(S_i)\to\frac{1}{2}$, which has been obtained in Theorem 3.8. We first need an elementary fact that we include for completeness.

Proposition 4.5. *A geometric limit of a sequence $\{\Gamma_i\}$ of normal subgroups of a discrete group Γ of isometries of $\mathbb{H}^{n+1}$ is itself a normal subgroup of Γ.*

Proof. Let Γ_∞ be a geometric limit of $\{\Gamma_i\}$, and fix an element $\gamma_\infty\in\Gamma_\infty$. By the definition of geometric limit there exists a sequence of elements $\{\gamma_i\in\Gamma_i\}$ so that $\lim\gamma_i=\gamma_\infty$. But $\{\gamma_i\}\subset\Gamma$, and Γ is discrete, so $\gamma_\infty\in\Gamma$. Thus Γ_∞ is a subgroup of Γ.

Once again fix an element $\gamma_\infty\in\Gamma_\infty$, and let $f\in\Gamma$. We wish to show that $f\gamma_\infty f^{-1}\in\Gamma_\infty$. Using the definition of geometric convergence we again observe that there exists a sequence $\{\gamma_i\in\Gamma_i\}$ converging to γ_∞ and we observe as well that $\{f\gamma_i f^{-1}\}$ converges to $f\gamma_\infty f^{-1}$. Let $g_i=f\gamma_i f^{-1}$; since Γ_i is normal in Γ then $g_i\in\Gamma_i$. Using the definition of geometric convergence we again observe that $f\gamma_\infty f^{-1}$ is in Γ_∞, since Γ_∞ contains all of the accumulation points of $\{\Gamma_i\}$. □

Thus in particular any geometric limit of a sequence of regular covers of a hyperbolic manifold is a regular cover of the manifold.

As an immediate corollary of Theorem 1.4 we have the following. Note that a normal subgroup of a non-elementary discrete group of isometries of $\mathbb{H}^{n+1}$ is either non-elementary or trivial.

Corollary 4.6. *Let $\{M_i\}$ be a sequence of regular covers of an $(n+1)$-dimensional closed hyperbolic manifold M so that $\delta(M_i) \to \frac{n}{2}$. Then $\{M_i\}$ converges geometrically to $\mathbb{H}^{n+1}$.*

Proof. Let M_∞ be a geometric limit of a subsequence of $\{M_i\}$. By the above proposition we know that M_∞ is a regular cover of M. Now we recall that the exponent of convergence is lower semicontinuous under geometric convergence. Thus we have that $\delta(M_\infty) \leq \frac{n}{2}$. If $M_\infty \neq \mathbb{H}^{n+1}$ then this contradicts the conclusion of Theorem 1.4, and thus we conclude that $M_\infty = \mathbb{H}^{n+1}$. □

This in particular shows that our sequence $\{S_i\}$ of regular covers of a closed surface S having the property that $\delta(S_i) \to \frac{1}{2}$ necessarily converges geometrically to $\mathbb{H}^2$.

Remark 4.7. By converging geometrically to $\mathbb{H}^2$ it is necessary that the geometry of the regular covers S_i is becoming unbounded. In particular, for any constant $d > 0$ there is a number i_0 so that for all $i > i_0$ there exists an embedded hyperbolic ball of radius greater than d in S_i. We ask whether a stronger result is true: *let $d(S_i)$ be the infimum of the injectivity radius function on S_i, then does $d(S_i) \to \infty$ as $\delta(S_i) \to \frac{1}{2}$?*

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Wesleyan University, Middletown, CT 06459
E-mail address: pbonfert@wesleyan.edu

Wesleyan University, Middletown, CT 06459
E-mail address: kmatsuzaki@wesleyan.edu

Wesleyan. University, Middletown, CT 06459
E-mail address: ectaylor@wesleyan.edu