AN EXAMPLE OF SELF-COVERING OF RIEMANN SURFACE

KATSUHIKO MATSUZAKI AND YASUHIRO YABUKI

Abstract. In this paper, we first review our result on self-covering of Riemann surfaces, or equivalently, proper conjugation of Fuchsian groups. Then we consider a problem asking whether a Fuchsian group has proper conjugation or not whenever it has a unique Patterson measure supported on its limit set.

Assume that a Riemann surface R has a holomorphic (unbranched and unlimited) cover $f : R \to R$ onto itself. If f is not injective (that is, f is not an automorphism), then it is called a *self-cover* of R. Note that a topologically finite Riemann surface R whose fundamental group is not abelian does not admit self-covering by the Riemann-Hurwitz formula.

Example 1. Let $\tilde{R}$ be an infinite cyclic cover of a closed hyperbolic Riemann surface and g an automorphism of infinite order that generates the covering transformation group. We take a subsurface $R' \subset \tilde{R}$ of compact border such that $g(R') \subset R'$ and the homomorphism of the fundamental group $g_{\#} : \pi_1(R') \to \pi_1(R')$ induced by the inclusion map $g|_{R'}$ is not surjective. Consider the Nielsen extension of the bordered surface R' as a complete hyperbolic surface and denote it by R. Then R has self-covering.

Systematic construction of self-covering has been given by Jørgensen, Marden and Pommerenke [3]. Also, self-covering naturally appears in complex dynamics. For instance, let f be a rational map of degree greater than one and D its invariant Fatou component. Assume that the grand orbit $O(f)$ of the critical points of f is discrete in D. Then, for a planar surface $R = D - O(f)$, $f|_R : R \to R$ is a self-cover.

As a necessary condition for a Riemann surface to have self-covering, Heins [2] gave the existence of the Green function. We have obtained a certain generalization of this theorem in [5] in terms of the divergence of the Poincaré series.

Represent a hyperbolic Riemann surface $R = \mathbb{B}/G$ by a Fuchsian group G acting on the unit disk $\mathbb{B}$ properly discontinuously and freely.

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The Poincaré series of dimension $s \geq 0$ for G is defined by

$$P_G^s := \sum_{g \in G} e^{-sd(g(0),0)},$$

where d denotes the hyperbolic distance and 0 the origin of $\mathbb{B}$. The critical exponent of convergence is defined by

$$\delta(G) := \inf\{s \geq 0 \mid P_G^s < \infty\} \ (\leq 1).$$

A Fuchsian group G is of *divergence type* if $P_G^{\delta(G)} = \infty$. It is known that every finitely generated Fuchsian groups is of divergence type.

As a special case of the result in [5], we have the following. We will give a sketch of the proof later.

Theorem 2. *If a Fuchsian group G is of divergence type, then the Riemann surface $R = \mathbb{B}/G$ does not admit self-covering.*

A Riemann surface $R = \mathbb{B}/G$ has no Green function if and only if $\delta(G) = 1$ and G is of divergence type. See [N]. Hence Theorem 2 generalizes the Heins theorem. We have proved that the assertion of Theorem 2 is also valid for higher dimensional hyperbolic manifolds (for higher dimensional Kleinian groups).

A Riemann surface $R = \mathbb{B}/G$ has a self-cover if and only if the Fuchsian group G has *proper conjugation*, that is, there exists a conformal automorphism α of $\mathbb{B}$ such that $\Gamma := \alpha G \alpha^{-1}$ is properly contained in G.

In this case, define $\Gamma_n := \alpha^{-n}\Gamma\alpha^n$ ($\Gamma_0 = \Gamma, \ \Gamma_1 = G$). Then we have a strictly increasing sequence

$$\Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \Gamma_2 \subsetneqq \cdots .$$

Set $\Gamma_\infty := \bigcup_{n \geq 0} \Gamma_n$, which is the geometric limit of the sequence. Then Γ_∞ is discrete and satisfies

$$\alpha \Gamma_\infty \alpha^{-1} = \Gamma_\infty.$$

Also $\delta(\Gamma_\infty) = \lim_{n\to\infty} \delta(\Gamma_n)$, and Γ_∞ is of divergence type if so is G. The conformal automorphism α defines a conformal automorphism f_∞ of $R_\infty = \mathbb{B}/\Gamma_\infty$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{\alpha} & \mathbb{B} \\
\downarrow & & \downarrow \\
R & \xrightarrow{f} & R \\
\downarrow & & \downarrow \\
R_\infty & \xrightarrow{f_\infty} & R_\infty
\end{array}
$$

A conformal density $\{\mu_z\}_{z\in\mathbb{B}}$ of dimension $s \geq 0$ is a family of positive finite Borel measures on $\partial\mathbb{B}$ such that they satisfy

$$\frac{d\mu_z(\xi)}{d\mu_0(\xi)} = k(z,\xi)^s$$

for any $z \in \mathbb{B}$ and for any $\xi \in \partial\mathbb{B}$. Here, $k(z,\xi) = (1-|z|^2)/|z-x|^2$ is the Poisson kernel. For a Fuchsian group Γ, a conformal density $\{\mu_z\}_{z\in\mathbb{B}}$ of dimension $\delta(\Gamma)$ that is invariant under Γ (meaning that the pull-back $\gamma^*\mu_{\gamma(z)}$ is equal to μ_z for every $\gamma \in \Gamma$) and that has support on the limit set $\Lambda(\Gamma)$ is called a *Patterson measure*. The existence (construction) of a Patterson measure for any Fuchsian group was first given in [7]. The patterson measure is unique up to positive constant multiples for a Fuchsian group of divergence type. See [8] and [10].

Our arguments are depending on the following result, which also has its own interest.

Lemma 3. *If a Fuchsian group Γ of divergence type satisfies $\alpha\Gamma\alpha^{-1} = \Gamma$ for some conformal automorphism α of $\mathbb{B}$, then the Patterson measure $\{\mu_z\}$ for Γ is invariant under α, that is, $\alpha^*\mu_{\alpha(z)} = \mu_z$. In particular, if a Fuchsian group G contains a Fuchsian group Γ of divergence type as a normal subgroup, then $\delta(G) = \delta(\Gamma)$, the Patterson measures for G and Γ are coincident, and G is also of divergence type.*

Indeed, $\alpha^*\mu_{\alpha(z)}$ is also a Patterson measure for Γ and its uniqueness gives a constant c such that

$$\alpha^*\mu_{\alpha(z)} = c\mu_z.$$

To show that $c = 1$, we trace back the construction of the Patterson measure, which will be sketched in the proof of Theorem 2 below. In our arguments, we consider the Poincaré series

$$P_G^s(x,z) = \sum_{g\in G} e^{-sd(g(x),z)}$$

defined for x and z in $\mathbb{B}$ and utilize its elementary properties such as

$$\begin{aligned} P^s_G(z,x) &= P^s_G(x,z); \\ P^s_G(\alpha(x),\alpha(z)) &= P^s_G(x,z). \end{aligned}$$

Outline of the proof of Theorem 2 : The Patterson measure $\{\mu_z\}$ for $\Gamma_\infty = \bigcup_{n\geq 0} \Gamma_n$ is also for G since $\delta(\Gamma_\infty) = \delta(G)$. Being of divergence type, G has the unique Patterson measure μ_z obtained by a weak limit of a sequence of weighted Dirac measures on $\overline{\mathbb{B}}$:

$$\begin{aligned} \mu_z &= \text{w-}\lim_{s\downarrow\delta} \frac{1}{P^s_G} \sum_{g\in G} e^{-sd(g(0),z)} D_{g(0)} \\ &= \text{w-}\lim_{s\downarrow\delta} \frac{1}{P^s_G} \left(\sum_{\gamma\in\Gamma} e^{-sd(\gamma(0),z)} D_{\gamma(0)} + \sum_{g\in G-\Gamma} e^{-sd(g(0),z)} D_{g(0)} \right). \end{aligned}$$

Then, by using $\Gamma = \alpha G \alpha^{-1}$, we calculate the first summation as

$$\begin{aligned} & \frac{1}{P^s_G} \sum_{\gamma\in\Gamma} e^{-sd(\gamma(0),z)} D_{\gamma(0)} \\ = \; & \frac{1}{P^s_G} \sum_{g\in G} e^{-sd(\alpha g\alpha^{-1}(0),z)} D_{\alpha g \alpha^{-1}(0)} \\ = \; & \frac{P^s_G(g\alpha^{-1}(0),\alpha^{-1}z)}{P^s_G} \\ \times \; & \frac{1}{P^s_G(g\alpha^{-1}(0),\alpha^{-1}z)} \sum_{g\in G} e^{-sd(g\alpha^{-1}(0),\alpha^{-1}z)} (\alpha^{-1})^* D_{g\alpha^{-1}(0)}. \end{aligned}$$

The second factor in the last term converges to $(\alpha^{-1})^*\mu_{\alpha^{-1}(z)}$ as $s \to \delta$. Also, by using properties of the Poincaré series, we see that the first factor in the last term converges to one. By the invariance of μ_z under α proved in Lemma 3, we conclude that the above equation converges to μ_z as $s \to \delta$. However, this is achieved only by the summation over Γ. The summation over G also yields the same μ_z. And it is relatively easy to see that the summation over $G - \Gamma$ does not vanish. This is possible only when $G = \Gamma$. □

Lemma 3 is still valid under a weaker assumption that a Fuchsian group Γ has a unique Patterson measure (since the proof uses only its uniqueness). This is given in [4]. Accordingly, Theorem 2 can be slightly generalized as follows.

Theorem 4. *Let G be a Fuchsian group whose Patterson measure is unique up to constant multiples. If the conjugate $\Gamma = \alpha G \alpha^{-1}$ is contained in G for some conformal automorphism α of $\mathbb{B}$ and the limit sets $\Lambda(\Gamma)$ and $\Lambda(G)$ are coincident, then $\Gamma = G$.*

The assumption $\Lambda(\Gamma) = \Lambda(G)$ is necessary for Theorem 4. (Lemma 3 is a statement for normal subgroups and this condition is automatically satisfied.) Without this assumption, Theorem 4 is not valid any more. A counterexample is given by the Riemann surface illustrated in Example 1.

Theorem 5. *There exists a Fuchsian group G whose Patterson measure is unique up to constant multiples but that admits proper conjugation.*

Proof. We take an infinite cyclic cover $\tilde{R}$ of a closed hyperbolic Riemann surface, which has two topological ends. It is well-known that $\tilde{R}$ does not admit the Green function. This implies that, on any end neighborhood $U \subset \tilde{R}$ of compact border ∂U, a positive harmonic function vanishing on ∂U is unique up to positive constant multiples (see [9]). The subsurface $R' \subset \tilde{R}$ as in Example 1 is an end neighborhood and its Nielsen extension R satisfies the same property for positive harmonic functions. This R admits self-covering.

We represent R by a Fuchsian group G. It is also known that $\delta(G) = 1$ (for example, the isoperimetric constant for R gives this conclusion). However, since $\Omega(G) \subset \partial\mathbb{B}$ is non empty, G is of convergence type. The uniqueness of positive harmonic functions on R as above is equivalent to the uniqueness of G-invariant positive harmonic functions on $\mathbb{B}$ whose boundary values vanish on $\Omega(G)$. And this is equivalent to saying that the Patterson measure for G of dimension 1 is unique up to positive constant multiples (see [1] and [11]). Indeed, the total mass $f(z)$ of a conformal density $\{\mu_z\}_{z \in \mathbb{B}}$ of dimension 1 is a harmonic function on $\mathbb{B}$ since

$$f(z) = \int_{\partial\mathbb{B}} k(z, \xi) d\mu_0(\xi)$$

and the Poisson kernel $k(z, \xi)$ satisfies $\Delta_z k(z, \xi) = 0$. Conversely, every positive harmonic function $f(z)$ is represented by the Poisson integral of some positive finite Borel measure μ on $\partial\mathbb{B}$. If $f(z)$ is G-invariant then so is μ, and if $f(z)$ vanishes on $\Omega(G)$ then so does μ. By defining a conformal density $\{\mu_z\}_{z \in \mathbb{B}}$ of dimension 1 so that $d\mu_z(\xi) = k(z, \xi) d\mu(\xi)$, we have a Patterson measure for G. □

We choose a conformal automorphism α of $\mathbb{B}$ corresponding to the conformal automorphism g of $\tilde{R}$ in Example 1 and consider the conjugate $\Gamma = \alpha G \alpha^{-1}$ corresponding to $\pi_1(g(R'))$, which is properly contained in G. Then we see that $\Lambda(\Gamma) \subsetneqq \Lambda(G)$, which does not satisfy the requirement for Theorem 4.

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(Katsuhiko Matsuzaki) Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459-0128, USA

E-mail address: kmatsuzaki@wesleyan.edu

(Yasuhiro Yabuki) Mathematical Institute, Tohoku University, Aoba-ku, Sendai 980-8578, Japan

E-mail address: sa3m30math.tohoku.ac.jp