THE PROJECTION OF LIMIT SETS OF MODULAR GROUPS ON ASYMPTOTIC TEICHMÜLLER SPACES

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ABSTRACT. For a Riemann surface of analytically infinite type, the action of the Teichmüller modular group is not discontinuous in general and the action of the asymptotic Teichmüller modular group on the asymptotic Teichmüller space is also not discontinuous. In this paper, we study the dynamics of these actions and prove that every point in the limit set of the Teichmüller modular group is projected to the limit set on the asymptotic Teichmüller modular group.

1. INTRODUCTION

1.1. **Teichmüller and asymptotic Teichmüller spaces.** Throughout this paper, we assume that a Riemann surface R admits a hyperbolic structure. The *Teichmüller space* $T(R)$ of R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The homotopy is considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$ which is called the Teichmüller distance. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R', where R' is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure. For details, see [18] and [23].

The asymptotic Teichmüller space has been introduced in [17] when R is the hyperbolic plane and in [3], [4] and [16] when R is an arbitrary hyperbolic Riemann surface. We say that a quasiconformal homeomorphism f of R is *asymptotically conformal* if, for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 of R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The *asymptotic Teichmüller space* $AT(R)$ of a Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. In fact, it is infinite

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dimensional. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic. See also [3] and [5].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-E})$, where the infimum is taken over all compact subsets E of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(f')$, where the infimum is taken over all elements $f' \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_{AT}([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_{AT} is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

1.2. **Teichmüller and asymptotic Teichmüller modular groups.** A *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes of R. Here the homotopy is considered to be relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to d_T. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$, and we define the *Teichmüller modular group* of R by

$$\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R)).$$

It is proved in [2], [6] and [20] that the homomorphism ι_T is injective (faithful) for all Riemann surfaces R of non-exceptional type. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g+n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι_T is also surjective for every Riemann surface R of non-exceptional type. In this case, $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. The proof is a combination of the results of [1] and [19]. See [9] for a survey of the proof.

The action of $\mathrm{MCG}(R)$ preserves the fibers of the projection $\pi : T(R) \to AT(R)$. Then every element $[g] \in \mathrm{MCG}(R)$ also induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_{AT}. See [4]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$, and we define the *asymptotic Teichmüller modular group* (the geometric automorphism group) of R by

$$\mathrm{Mod}_{AT}(R) = \iota_{AT}(\mathrm{MCG}(R)).$$

A surjective homomorphism $A : \mathrm{Mod}(R) \to \mathrm{Mod}_{AT}(R)$ is well-defined by $\iota_{AT} \circ \iota_T^{-1}$, which sends $[g]_* \in \mathrm{Mod}(R)$ to $[g]_{**} = A([g]_*) \in \mathrm{Mod}_{AT}(R)$. In other words, we have $[g]_{**} \circ \pi(p) = \pi \circ [g]_*(p)$ for every $p \in T(R)$ and for every $[g]_* \in \mathrm{Mod}(R)$.

It is different from the case of the representation ι_T that the homomorphism ι_{AT} is not injective, namely, $\mathrm{Ker}\,\iota_{AT} \neq \{[id]\}$ unless R is either the unit disc or the once-punctured disc ([2]). We call an element of $\mathrm{Ker}\,\iota_{AT}$ *asymptotically trivial* and call $\mathrm{Ker}\,\iota_{AT}$ the *asymptotically trivial mapping class group.*

1.3. **Limit sets and regions of discontinuity.** For a subgroup $G \subset \mathrm{MCG}(R)$, it is said that $q \in T(R)$ is a *limit point* of $p \in T(R)$ for G if there exists a sequence $\{[g_n]_*\}_{n=1}^{\infty}$ of distinct elements of $\iota_T(G)$ such that $d_T([g_n]_*(p), q) \to 0$ as $n \to \infty$. The set of all limit points of p for G is denoted by $\Lambda_T(G,p)$, and the *limit set* for G is defined by $\Lambda_T(G) = \bigcup_{p\in T(R)} \Lambda_T(G,p)$. It is said that $p \in T(R)$ is a *recurrent point* for G if $p \in \Lambda_T(G,p)$, and the set of all recurrent points for G is called the *recurrent set* for G and is denoted by $\mathrm{Rec}_T(G)$. It is evident from the definition that $\mathrm{Rec}_T(G) \subset \Lambda_T(G)$ and these sets are G-invariant. Moreover, we have proved in [12, Proposition 2.2] that $\Lambda_T(G) = \mathrm{Rec}_T(G)$ and that they are closed.

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that the number of elements $[g]_* \in \iota_T(G)$ satisfying $[g]_*(U) \cap U \neq \emptyset$ is finite. This is equivalent to saying that the orbit $\iota_T(G)(p)$ is discrete and the stabilizer subgroup $\mathrm{Stab}_{\iota_T(G)}(p)$ is finite. We define $\Omega_T(G)$ as the set of all points $p \in T(R)$ where G acts discontinuously, and call $\Omega_T(G)$ the *region of discontinuity* for G. It is easy to see that $\Omega_T(G) = T(R) - \Lambda_T(G)$. See also [7] and [12].

Similarly, for a subgroup $G \subset \mathrm{MCG}(R)$, it is said that $\hat{q} \in AT(R)$ is a *limit point* of $\hat{p} \in AT(R)$ for G if there exists a sequence $\{[g_n]_{**}\}_{n=1}^{\infty}$ of distinct elements of $\iota_{AT}(G)$ such that $d_{AT}([g_n]_{**}(\hat{p}), \hat{q}) \to 0$ as $n \to \infty$. The set of all limit points of $\hat{p}$ for G is denoted by $\Lambda_{AT}(G,\hat{p})$, and the *limit set* for G is defined by $\Lambda_{AT}(G) = \bigcup_{\hat{p}\in AT(R)} \Lambda_{AT}(G,\hat{p})$.

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $\hat{p} \in AT(R)$ *discontinuously* if there exists a neighborhood $\hat{U}$ of $\hat{p}$ such that the number of elements $[g]_{**} \in \iota_{AT}(G)$ satisfying $[g]_{**}(\hat{U}) \cap \hat{U} \neq \emptyset$ is finite. We define the region of discontinuity $\Omega_{AT}(G)$ for $G \subset \mathrm{MCG}(R)$ as the open set of all points $\hat{p} \in AT(R)$ where G acts discontinuously. Then $\Omega_{AT}(G) = AT(R) - \Lambda_{AT}(G)$.

2. Statement of result

We investigate a relationship between the limit set on $T(R)$ and its projection to $AT(R)$. In [8], we have observed that a point in the region of discontinuity on $T(R)$ can be mapped into the limit set on $AT(R)$ by the projection $\pi : T(R) \to AT(R)$. On the other hand, we propose a problem whether $\pi(\Lambda_T(G)) \subset \Lambda_{AT}(G)$ for all subgroups $G \subset \mathrm{MCG}(R)$ and for all Riemann surfaces R. In [11], we have proved a partial solution of this problem.

Proposition 2.1. *Let R be a Riemann surface that admits a conformal automorphism g of infinite order. For the cyclic group $G = \langle [g] \rangle$ generated by $[g] \in \mathrm{MCG}(R)$, we have $\pi(\Lambda_T(G)) \subset \Lambda_{AT}(G)$.*

Note that the mapping class $[g] \in \mathrm{MCG}(R)$ induced by a conformal automorphism g of infinite order is not asymptotically trivial (see [22]), and the proof of Proposition 2.1 follows from this fact.

In this paper, we have another partial solution of the above problem under a certain geometric condition of Riemann surfaces.

Definition 2.2. We say that a Riemann surface R satisfies the *bounded geometry condition* if R satisfies the following two conditions:

(i) *lower bound condition*: there exists a constant $m > 0$ such that, for every point $x \in \dot{R}$, every homotopically non-trivial curve that starts from x and terminates at x has length greater than or equal to m. Here $\dot{R}$ is the non-cuspidal part of R obtained by removing all horocyclic cusp neighborhoods whose areas are 1:

(ii) *upper bound condition*: there exists a constant $M > 0$ such that, for every point $x \in R$, there exists a homotopically non-trivial simple closed curve that starts from x and terminates at x and whose length is less than or equal to M.

Every normal cover of a compact Riemann surface that is not the universal cover satisfies the bounded geometry condition. Moreover, if a Riemann surface admits such pants decomposition that the diameter of each pair of pants is uniformly bounded, then it satisfies the bounded geometry condition.

Then our result is the following, which was already announced in [11].

Theorem 2.3. *Let R be a Riemann surface satisfying the bounded geometry condition. Then $\pi(\Lambda_T(G)) \subset \Lambda_{AT}(G)$ for any subgroup G of* $\mathrm{MCG}(R)$.

We prove Theorem 2.3 in the next section.

Remark 2.4. We expect that the statement in Theorem 2.3 is true for all Riemann surfaces that does not necessarily satisfy the bounded geometry condition. In fact, it has proved in [8, Theorem 4.1] that $\Lambda_T(\mathrm{MCG}(R)) = T(R)$ and $\Lambda_{AT}(\mathrm{MCG}(R)) = AT(R)$ for all Riemann surfaces R that do not satisfy the lower bound condition.

We closely observe the inclusion in Theorem 2.3 in the following three examples. Note that if R satisfies the bounded geometry condition, then $\Omega(\mathrm{MCG}(R)) \neq \emptyset$. See [7, Theorem 3].

Example 2.5. Let R be a normal cover of a compact Riemann surface whose covering transformation group is a cyclic group $\langle g \rangle$ generated by a conformal automorphism of R of infinite order, and $G = \langle [g] \rangle$ the cyclic group generated by $[g] \in \mathrm{MCG}(R)$. Then $\Lambda_T(G) \neq \emptyset$. By the proof of [8, Proposition 4.3], we see that $\pi(\Lambda_T(G))$ is a proper subset of $\Lambda_{AT}(G)$. Moreover, [11, Theorem 4.6] states that $\Omega_{AT}(G) \neq \emptyset$.

Example 2.6. Let R be a Riemann surface in Example 2.5 and set $R' = R - \{p\}$ for a point $p \in R$. It is different from Example 2.5 that $\Lambda_T(\mathrm{MCG}(R')) = \emptyset$, which was proved by using the fact that $\mathrm{MCG}(R')$ is *stationary* (see [15, Theorem 2]). On the other hand, on the asymptotic Teichmüller space, we also have $\Lambda_{AT}(\mathrm{MCG}(R')) \neq \emptyset$ and $\Omega_{AT}(\mathrm{MCG}(R')) \neq \emptyset$. Indeed, the asymptotic Teichmüller spaces $AT(R)$ and $AT(R')$ are biholomorphic, and the subgroups $\mathrm{Mod}(R)$ of $\mathrm{Aut}(AT(R))$ and $\mathrm{Mod}_{AT}(R')$ of $\mathrm{Aut}(AT(R'))$ can be identified. For details, see the proof of [8, Theorem 4.2].

We also have another kind of example.

Example 2.7. Let R be a Riemann surface constructed in [14, Section 3], which is not a normal cover of a compact Riemann surface and does not satisfy the lower bound condition. By modifying the construction slightly as in Remark 3.4 of that paper, we see that R admits an asymptotically conformal automorphism g of infinite order such that it is not asymptotically trivial. Then $\Lambda_{AT}(G) \neq \emptyset$ for $G = \langle [g] \rangle$. We have proved that G acts on $T(R)$ discontinuously, namely $\Lambda_T(G) = \emptyset$. Note that this implies that $[g] \in \mathrm{MCG}(R)$ does not have a conformal representative on any Riemann surface quasiconformally equivalent to R. See [21, Theorem 3].

In the last of this section, we announce the following result, which follows from a similar argument in the proof of Proposition 3.2 below and will be proved in the forthcoming paper by authors.

Theorem 2.8. *Let R be a Riemann surface satisfying the bounded geometry condition, and G a subgroup of $\mathrm{MCG}(R)$ such that all the elements of $\iota_{AT}(G)$ other than the identity are of infinite order. Then $\Omega_{AT}(G) \neq \emptyset$.*

3. PROOF OF THEOREM

For a proof of Theorem 2.3, a topological characterization of $\mathrm{Ker}\,\iota_{AT}$ is crucial. To state the characterization, we define the following subgroup of the quasiconformal mapping class group.

Definition 3.1. The *stable quasiconformal mapping class group* $G_\infty(R)$ is the group of all essentially trivial mapping classes. Here $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* if there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$.

Then we have the following.

Proposition 3.2 ([14])**.** *Let R be a Riemann surface satisfying the bounded geometry condition. Then $G_\infty(R) = \mathrm{Ker}\,\iota_{AT}$.*

We also use the following observation on the action of $G_\infty(R)$ on $T(R)$.

Proposition 3.3 ([10])**.** *Let R be a topologically infinite Riemann surface satisfying the bounded geometry condition. Then the stable quasiconformal mapping class group $G_\infty(R)$ acts on $T(R)$ discontinuously.*

We are ready to prove our theorem.

Proof of Theorem 2.3. We take a limit point $p \in \Lambda_T(G)$ arbitrarily. Then there exists a sequence $[g_n]_*$ of distinct elements of $\iota_T(G)$ such that $d_T([g_n]_*(p), p) \to 0$ as $n \to \infty$. This implies that $d_{AT}([g_n]_{**}(\hat{p}), \hat{p}) \to 0$ for the projection $\hat{p} = \pi(p)$. We will show that $\{[g_n]_{**}\}_{n\in\mathbb{N}} \subset \iota_{AT}(G)$ contains infinitely many elements, from which we conclude that $\hat{p} \in \Lambda_{AT}(G)$. Suppose to the contrary that $\{[g_n]_{**}\}_{n\in\mathbb{N}}$ is a finite set $\{[h_1]_{**}, \ldots, [h_k]_{**}\}$ for some $k \geq 1$. Then there exists an integer i $(1 \leq i \leq k)$, say 1, such that $[g_n]_{**} = [h_1]_{**}$ for infinitely many n. Set $\gamma_n := g_n \circ h_1^{-1}$. Then $[\gamma_n] \in \mathrm{Ker}\,\iota_{AT}$ and $d_T([\gamma_n]_*([h_1]_*(p)), p) = d_T([g_n]_*(p), p) \to 0$. This means that the point $p \in T(R)$ is a limit point for the subgroup $\mathrm{Ker}\,\iota_{AT}$. On the other hand, by Propositions 3.2 and 3.3, the asymptotically trivial mapping class group $\mathrm{Ker}\,\iota_{AT}$ acts on $T(R)$ discontinuously. This contradiction shows that $\{[g_n]_{**}\}_{n\in\mathbb{N}}$ contains infinitely many elements. □

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