THE ISOMORPHISM THEOREM OF KLEINIAN GROUPS

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Abstract. A sufficient condition on a geometrically finite Kleinian group G is shown, under which any type-preserving isomorphism from G onto another geometrically finite one is induced by an automorphism of the Riemann sphere.

The Fenchel-Nielsen isomorphism theorem asserts that a type-preserving isomorphism $\varphi : \Gamma \to \Gamma'$ between cofinite volume Fuchsian groups Γ and Γ' is induced by an automorphism f of the unit disk Δ, that is, there is f such that $\varphi(\gamma) = f \circ \gamma \circ f^{-1}$ for every $\gamma \in \Gamma$. Roughly speaking, this means that an algebraic isomorphism between such Fuchsian groups is geometric. In this note, we extend this result to Kleinian groups and investigate a sufficient condition for an algebraic isomorphism to be geometric. Along this line, there is a result due to Marden and Maskit [7]. Their theorem works under certain assumptions on both the Kleinian group G and the isomorphism φ. Our theorem assumes nothing about φ but that it is type-preserving, and provides a sharp sufficient condition for G under which any type-preserving isomorphism φ is geometric.

A fundamental result is the following Marden isomorphism theorem [6].

Proposition. *Let G be a geometrically finite torsion-free Kleinian group, and let $\varphi : G \to G'$ be a type-preserving isomorphism onto another Kleinian group. Suppose there is a homeomorphism $f : \Omega(G) \to \Omega(G')$ of the region of discontinuity where $f \circ g = \varphi(g) \circ f$ for all $g \in G$. Then f extends to $\hat{\mathbb{C}}$ as an automorphism conjugating g into $\varphi(g)$.*

We may regard this proposition as a translation of the following topological result due to Waldhausen (cf. [2] Chap.13) into the Kleinian group theory: for compact orientable irreducible 3-manifolds M and M' with incompressible boundary components, an isomorphism $\varphi : \pi_1(M) \to \pi_1(M')$ is geometric (i.e. there exists a homeomorphism $f : M \to M'$ which induces φ) whenever it preserves the peripheral structure (i.e. for each component S of ∂M, there is a component S' of $\partial M'$ such that φ maps $\pi_1(S)$ to a conjugate of $\pi_1(S')$ in $\pi_1(M')$). Hence the problem is reduced to a problem when the peripheral structure is preserved. Johannson [4] proved that if M is acylindrical, then the peripheral structure is preserved. Our result may be regarded as a translation of Johannson's into the Kleinian group theory. However, without assuming his theorem, we exhibit in this note a simple proof relying on the intersection property of the limit sets of Kleinian groups, which was studied by Susskind [11].

Now, letting $\Omega(G)$ be the region of discontinuity of a Kleinian group G, $\Lambda(G)$ the limit set, and $\mathrm{Stab}_G(\Delta)$ the component subgroup for a component Δ of $\Omega(G)$, we state our result:

Theorem. *Let G and G' be geometrically finite Kleinian groups possibly with torsion. We assume that G satisfies the following three conditions:*

(0) *each component Δ of $\Omega(G)$ is simply connected;*
(1) *G has no APT;*
(2) *for any distinct components Δ_1 and Δ_2 of $\Omega(G)$, $\mathrm{Stab}_G(\Delta_1) \cap \mathrm{Stab}_G(\Delta_2)$ contains no loxodromic elements.*

Then, for any type-preserving isomorphism $\varphi : G \to G'$, there is an automorphism f of $\hat{\mathbb{C}}$ such that $\varphi(g) = f \circ g \circ f^{-1}$.

Remark. The combination of the assumptions (0) and (1) is equivalent to the following condition:

(1′) G is a web group, i.e. every component of $\Omega(G)$ is a Jordan domain.

We can rewrite Theorem as a statement for hyperbolic manifolds. Let $\mathbb{H}^3$ be the hyperbolic 3-space, and N_G a complete hyperbolic 3-manifold $\mathbb{H}^3/G$ divided by a finitely generated torsion-free Kleinian group G. When the convex core of N_G has finite hyperbolic volume, we say that G and N_G are geometrically finite. We may regard $\Omega(G)/G$ as boundary at infinity of the hyperbolic manifold N_G. Consider the topological manifold $M_G = (\mathbb{H}^3 \cup \Omega(G))/G$ with boundary. Then the assumption (0) is equivalent to the condition that every boundary component S of M_G is incompressible, that is, the homomorphism $\pi_1(S) \to \pi_1(M_G)$ induced by the inclusion $S \hookrightarrow M_G$ is injective. In virtue of the loop theorem, we may say that M_G has no essential disks when this condition is satisfied. The assumption (1) is equivalent to the following condition: if a loop in ∂M_G is freely homotopic to a loop round a cusp in M_G, then the homotopy can be performed in ∂M_G. The assumption (2) is equivalent to the condition that M_G is acylindrical: if two loops in ∂M_G are freely homotopic in M_G, then the homotopy can be performed in ∂M_G or they are freely homotopic to a loop round a cusp. In virtue of the annulus theorem, we may say that M_G has no essential punctured-disks when the former condition is satisfied and no essential annuli when the latter is.

Theorem′. *Let N_G and $N_{G'}$ be geometrically finite hyperbolic 3-manifolds. We assume that N_G has neither essential disks, essential punctured-disks nor essential annuli. Then for any isomorphism $\varphi : \pi_1(N_G) \to \pi_1(N_{G'})$ which preserves the cusps, there is a (quasi-isometric) homeomorphism $f : N_G \to N_{G'}$ which induces φ.*

Remark. If we drop any of three assumptions in the above Theorem, we can find a counterexample to the statements. In this sense, our theorem is sharp.

Proof of Theorem. Suppose that G is torsion-free. If G or G' is of the first kind, namely, N_G or $N_{G'}$ is of finite volume, then the Mostow rigidity theorem implies ours (cf. [6]). Hence we may further assume that G and G' are of the second kind. Let Δ be any component of $\Omega(G)$ and H the component subgroup $\mathrm{Stab}_G(\Delta)$. By the assumptions (0) and (1), we know H is quasifuchsian. We shall prove that $H' = \varphi(H)$ is also a component subgroup of G'. This means that the peripheral structure is preserved by φ. Then our claim follows from the Marden isomorphism theorem (See [5] p.218 for a detailed argument to apply Marden's theorem).

First, we see that H' is also quasifuchsian. Indeed, the image H' under the type-preserving isomorphism is either quasifuchsian or totally degenerate ([9] Theorem 6), but it cannot be totally degenerate because G' is geometrically finite and of the second kind (cf. [10] p.134).

Next, we will show that $g'(\Lambda(H')) \cap \Lambda(H')$ is empty or consists of one parabolic fixed point for any $g' \in G' - H'$. When $\Lambda(H')$ satisfies this (and $h'(\Lambda(H')) = \Lambda(H')$ for any $h' \in H'$), we say that $\Lambda(H')$ is precisely H'-invariant except for a parabolic fixed point. We investigate the intersection of the limit sets of two subgroups in a Kleinian group. By the following lemma, which is a corollary to Susskind's result, we know that $\Lambda(H')$ satisfies the above property.

Lemma. *Under the assumptions of Theorem, let H_1 and H_2 be distinct component subgroups of G. Then $\Lambda(\varphi(H_1)) \cap \Lambda(\varphi(H_2))$ is empty or consists of one parabolic fixed point.*

Proof. Since $\varphi(H_1)$ and $\varphi(H_2)$ are geometrically finite subgroups of a Kleinian group G', we know from Theorem 3 in [11] that

$$\Lambda(\varphi(H_1)) \cap \Lambda(\varphi(H_2)) = \Lambda(\varphi(H_1) \cap \varphi(H_2)) \cup P',$$

where P' is a set of points fixed by a parabolic abelian group of rank 2 generated by an element of $\varphi(H_1)$ and another element of $\varphi(H_2)$. However P' is empty in our case. In fact, a parabolic element of H_1 and another of H_2 cannot generate an abelian group of rank 2 because H_1 and H_2 are component subgroups. Accordingly, one of $\varphi(H_1)$ and another of $\varphi(H_2)$ cannot, which implies that $P' = \emptyset$. As a consequence, we have

$$\Lambda(\varphi(H_1)) \cap \Lambda(\varphi(H_2)) = \Lambda(\varphi(H_1) \cap \varphi(H_2)) = \Lambda(\varphi(H_1 \cap H_2)).$$

Here, $H_1 \cap H_2$ is an elementary group without a loxodromic element by the assumption (2), and so is $\varphi(H_1 \cap H_2)$. Therefore $\Lambda(\varphi(H_1 \cap H_2))$ consists of one parabolic fixed point at most, which proves the statement of the lemma. □

Proof continued. We will see that H' is embedded, namely, there is a properly embedded incompressible surface S' in $N_{G'}$ whose fundamental group is H' under the identification $\pi_1(N_{G'}) \cong G'$. Since $\Lambda(H')$ is precisely H'-invariant except for a parabolic fixed point, we can construct an H'-invariant and G'-equivariant contractible surface in $\mathbb{H}^3$ with the boundary $\Lambda(H')$ (cf. [10] VII.B.16). Then its projection to $N_{G'}$ yields the desired S'.

As the final step of the torsion-free case, we will show that $H' = \varphi(H)$ is a component subgroup of G'. If not, the properly embedded incompressible surface S' induces a non-trivial amalgamated or HNN free product decomposition of G'. It is

$$G' = \Gamma'_1 \underset{H'}{*} \Gamma'_2 \qquad \text{or} \qquad G' = \Gamma' \underset{H'}{*}$$

according as $M_{G'} - S'$ is disconnected or connected. Then, operating φ^{-1}, we have a non-trivial decomposition

$$G = \Gamma_1 \underset{H}{*} \Gamma_2 \qquad \text{or} \qquad G = \Gamma \underset{H}{*} \ .$$

Let $(M_G)_0$ be the pared manifold $M_G - \{\text{cusp neighborhoods}\}$. It is a compact topological manifold with boundary whose interior is homeomorphic to N_G. Using the above free product decomposition of $G \cong \pi_1((M_G)_0)$, we have a properly embedded incompressible surface S in $(M_G)_0$ such that $\pi_1(S)$ corresponds to a subgroup of H and S induces a non-trivial decomposition of G (cf. [3] p.35). Further, since φ is type-preserving, all parabolic elements of G are contained in conjugates of the factors of this decomposition. Hence, by moving S by a homotopy if necessary, we may assume that ∂S is in the non-cuspidal boundary $\partial_n(M_G)_0 = \partial(M_G)_0 \cap \partial M_G$. If ∂S is not empty, every component of ∂S must be in the surface Δ/H because an annulus $(A, \partial A)$ in $((M_G)_0, \partial_n(M_G)_0)$ is not essential due to the assumptions (1) and (2). If S were a disk, then $\partial_n(M_G)_0$ would be compressible. This contradicts the assumption (0), and thus S is not a disk. Since every non-trivial loop in S is freely homotopic to a loop in Δ/H, we can see that S divides $(M_G)_0$ into two parts, one of which is homeomorphic to $S \times [0, 1]$. This contradicts the fact that S induces a non-trivial amalgamated free product decomposition of G. Thus the proof of the torsion-free case completes.

In case G contains elliptic elements, we take a torsion-free subgroup Γ of G with finite index by the Selberg lemma. Since $\Lambda(\Gamma) = \Lambda(G)$, Γ also satisfies the assumptions (0), (1) and (2). We restrict the isomorphism φ to Γ. Then $\varphi|_\Gamma : \Gamma \to \Gamma'$ is geometric by the result in the torsion-free case; there is an automorphism f of $\hat{\mathbb{C}}$ which induces $\varphi|_\Gamma$. In particular, f determines the correspondence between the components Δ of $\Omega(\Gamma) = \Omega(G)$ and Δ' of $\Omega(\Gamma') = \Omega(G')$. In each component Δ, we modify $f|_\Delta$ so that it may compatible with $H = \mathrm{Stab}_G(\Delta)$. This is possible because the type-preserving isomorphism $\varphi|_H$ is geometric by the original Fenchel-Nielsen isomorphism theorem for Fuchsian groups. Thus we can construct a homeomorphism $\tilde{f} : \Omega(G) \to \Omega(G')$ which induces $\varphi : G \to G'$. Consider $f^{-1} \circ \tilde{f}$. It is defined on $\Omega(\Gamma)$ and induces the identity isomorphism $\Gamma \to \Gamma$. Then by the Maskit identity theorem [8], $f^{-1} \circ \tilde{f}$ is extendable to an automorphism of $\hat{\mathbb{C}}$, and so is $\tilde{f}$. This completes the proof of the general case. □

Remark. In [5], Keen, Maskit and Series have shown that for a geometrically finite web group G such that every component of $\Omega(G)$ is a round disk, the peripheral structure is preserved under any type-preserving isomorphism onto another geometrically finite Kleinian group G'. It is evident that such a Kleinian group G satisfies our assumptions (0), (1) and (2). The authors use two facts to prove their result: a lemma due to Otal and a theorem by Floyd [1]. The former lemma characterizes whether a quasifuchsian subgroup of G' is peripheral or not in terms of the topology of the limit set. The latter theorem asserts that $\Lambda(G)$ and $\Lambda(G')$ are homeomorphic if G and G' are isomorphic under a type-preserving map. We can see that their arguments extend to another proof of our theorem.

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