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Indecomposable continua and the limit sets of Kleinian groups

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Abstract. A plane continuum is said to be decomposable if it is written as the union of two of its proper subcontinua. We investigate the structure of the limit set of a Kleinian group from a viewpoint of its decomposability.

§1. Introduction

An indecomposable continuum often appears as a singular set of a dynamical system. Its definition itself is very simple, however, the analysis of this concept sometimes makes a crucial step for understanding the dynamics.

Definition. A continuum (a compact connected set) Λ is *decomposable* if there exist proper subcontinua Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 \cup \Lambda_2$, and otherwise *indecomposable*.

We are concerned with the decomposability of singular sets for complex dynamics on the Riemann sphere. In this direction, Rogers [15], [16] studied the boundaries of local Siegel disks and obtained a dichotomy that either they are indecomposable or they have certain properties of Jorda n curves. Mayer and Rogers [10] also dealt with the Julia sets of polynomials. In this paper, we consider decomposability of limit sets of Kleinian groups, which may be regarded as a counterpart of their works. A Kleinian group is a discrete subgroup of Möbius transformations and its limit set is defined in the same way as the Julia set is defined for iteration of a rational map on the Riemann sphere.

Any simply connected hyperbolic domain D admits the canonical compactification through the Riemann mapping and an element on its boundary is called a prime end. Each prime end determines a corresponding subcontinuum of ∂D called the impression. The decomposability of the boundary of a simply connected domain D is closely related to the structure of the impressions of prime ends of D, which is a classical result by Rutt [18]. Moreover, if D is invariant under a Kleinian group, then distribution of the impressions has group invariance. In Section 3 of this paper, we investigate the structure of the impressions of prime ends of invariant domains.

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One of open problems in the theory of Kleinian groups is local connectivity of the limit set of a finitely generated Kleinian group. It is known that difficulty in solving this problem lies only in the case where Γ is a degenerate group, which is a Kleinian group with the connected and simply connected region of discontinuity. In general, if a non-degenerate continuum Λ on S^2 is locally connected, then Λ is decomposable. In Section 4, we survey several results on local connectivity and decomposability of limit sets of finitely generated Kleinian groups. Then in Section 5, we study the limit set of a degenerate group, not necessarily finitely generated, from a viewpoint of its decomposability.

In the last section, according to an argument due to Tukia [20] on continuous extendability of the Riemann map to a certain subset of the limit set, we try to refine our results obtained in the previous sections.

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§2. Preliminaries

A Kleinian group is a discrete subgroup of Möbius transformations of the Riemann sphere S^2. A Möbius transformation γ is said to be loxodromic if it is conjugate to $z \mapsto cz$ $(c \neq 0, |c| \neq 1)$. Its attracting fixed point is denoted by a_γ and its repelling fixed point by $a_{\gamma^{-1}}$. The set of the fixed points of γ is denoted by $\mathrm{Fix}(\gamma)$. If a Kleinian group Γ contains non-commuting loxodromic elements, we say that Γ is *non-elementary.* For a non-elementary Kleinian group Γ, the *limit set* $\Lambda(\Gamma)$ is the set of points where the orbit $\Gamma(z)$ of a point $z \in S^2$ accumulates. It is alternatively defined to be the smallest, non-empty, Γ-invariant, compact set on S^2, or the closure of the set of all fixed points of loxodromic elements. The complement $\Omega(\Gamma)$ of the limit set is called the *region of discontinuity.* It is the largest open set where Γ acts properly discontinuously. If a connected component D of $\Omega(\Gamma)$ is itself invariant under the action of Γ, then $\Lambda(\Gamma) = \partial D$. Indeed, $\Lambda(\Gamma) \supset \partial D$ follows from the facts that D is an open set and that $\Lambda(\Gamma)$ is the complement of $\Omega(\ Gamma)$, and $\Lambda(\Gamma) \subset \partial D$ follows from the facts that Γ acts properly discontinuously on D and that $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(z)$ for $z \in D$. See [9] for fundamental facts on Kleinian groups.

A continuum is a compact connected subset of S^2. For a Kleinian group Γ, a continuum Λ is said to be Γ-invariant if $\gamma(\Lambda) = \Lambda$ for every $\gamma \in \Gamma$. If Γ is non-elementary, then every Γ-invariant continuum contains the limit set $\Lambda(\Gamma)$.

We investigate decomposability of continua. The following proposition tells us a simple characterization for it. See Hocking and Young [5, p.139].

Proposition 1. *A continuum $\Lambda \subset S^2$ is decomposable if and only if Λ contains a proper subcontinuum λ having non-empty interior with respect to the relative topology on Λ.*

Next we introduce a concept of composant in a continuum. See Hocking and Young [5, pp.139-140].

Definition. Let Λ be a non-degenerate continuum, that is, a continuum with more than one point. For a point $x \in \Lambda$, the *composant* μ is the union of all proper subcontinua of Λ containing x.

PROPOSITION 2. *Let μ be a composant in a non-degenerate continuum Λ on S^2. Then the closure $\bar{\mu}$ is coincident with Λ. Moreover, if Λ is indecomposable, then Λ consists of uncountably many composants which are mutually disjoint.*

We consider the boundary ∂D of a simply connected domain D. When ∂D is a non-degenerate continuum, we call D hyperbolic. In this case, there exists a biholomorphic map (the Riemann map) $\varphi : \Delta \to D$ from the unit disk Δ to the domain D. The Carathéodory theorem says that the Riemann map φ extends to a homeomorphism $\bar{\varphi}$ of the Euclidean closure $\overline{\Delta}$ onto the compactification of D with the Carathéodory boundary $\bar{\partial} D$. Each point e in $\bar{\partial} D$ is called a *prime end*. For a prime end e, there corresponds a continuum $I(e) \subset \partial D$, which is called the *impression* of e. It can be defined as the cluster set of $\varphi(\zeta)$ as $\zeta \in \Delta$ tends to the boundary point $\xi = \bar{\varphi}^{-1}(e)$:

$$I(e) = \{x \in \partial D \mid \varphi(\zeta_n) \to x \ (\exists\, \zeta_n \to \xi)\, \}.$$

See Collingwood and Lohwater [4, Chap.9] for the above facts concerning prime ends.

As necessary and sufficient conditions for indecomposability of the Euclidean boundary ∂D, the following result due to Rutt [18, Th.1, Th.2] is of its final form. See also Rogers [17, Th.2.2, Th.3.1].

PROPOSITION 3. *Let D be a simply connected hyperbolic domain. If ∂D is indecomposable, then there exists a prime end $e \in \bar{\partial} D$ such that $I(e) = \partial D$. Conversely, if there exists a prime end $e \in \bar{\partial} D$ such that $I(e) = \partial D$, then ∂D is an indecomposable continuum or the union of two indecomposable continua.*

In this proposition, if ∂D coincides with the limit set of some Kleinian group, then the converse part of its statement becomes simpler: ∂D is indecomposable if and only if $I(e) = \partial D$ for some $e \in \bar{\partial} D$. This can be seen from the following lemma.

LEMMA 4. *The limit set $\Lambda(\Gamma)$ of a non-elementary Kleinian group Γ is not the union of two of its indecomposable proper subcontinua.*

PROOF. Suppose that $\Lambda(\Gamma)$ is written as the union of two indecomposable proper subcontinua Λ_1 and Λ_2 of $\Lambda(\Gamma)$. Then there exists an element $\gamma \in \Gamma$ such that $\{\Lambda_1, \Lambda_2\} \neq \{\gamma(\Lambda_1), \gamma(\Lambda_2)\}$. Indeed, if there exist no such elements, then γ^2 preserves both Λ_1 and Λ_2 for every $\gamma \in \Gamma$. In this case, consider a subgroup H of Γ generated by all the elements of the form γ^2. It is a non-trivial normal subgroup of Γ, and thus the limit set $\Lambda(H)$ is coincident with $\Lambda(\Gamma)$ (See [9, Lemma 2.22]). On the other hand, Λ_1 contains $\Lambda(H)$ because Λ_1 is an H-invariant compact set. This contradicts the assumption that Λ_1 is a proper subset of $\Lambda(\Gamma)$.

Consider the pairs of subcontinua $\{\Lambda_1, \Lambda_2\}$ and $\{\gamma(\Lambda_1), \gamma(\Lambda_2)\}$ for the element γ, which provide distinct decompositions of Λ. Then at least one of the four indecomposable continua, say Λ_1, intersects both $\gamma(\Lambda_1)$ and $\gamma(\Lambda_2)$ in such a way that $\Lambda_1 \cap \gamma(\Lambda_1)$ and $\Lambda_1 \cap \gamma(\Lambda_2)$ are properly contained in Λ_1. Since $\Lambda_1 - \gamma(\Lambda_2)$ is a non-empty open set with respect to the relative topology on Λ_1, the closed set $\Lambda_1 \cap \gamma(\Lambda_1)$ has an interior point x. Then the connected component of $\Lambda_1 \cap \gamma(\Lambda_1)$ containing x is a proper subcontinuum of Λ_1 with non-empty interior. Hence Λ_1 is decomposable by Proposition 1. This contradiction proves the assertion. $\square$

REMARK. If we do not require invariance under a Kleinian group, there are a lot of examples of simply connected hyperbolic domains D such that the impression $I(e)$ coincides with ∂D for some prime end $e \in \bar{\partial} D$. More strongly, there exists even such a domain D that $I(e) = \partial D$ for every prime end e. In Kuester [6], this is given in the case where ∂D is a pseudo-circle, and in Lewis [7] and Smith [19], in the case where ∂D is a pseudo-arc.

§3. Boundaries of invariant domains

Rogers [15], [16] investigated impressions of prime ends for local Siegel disks. In this section, we will follow his arguments and apply them to the case of simply connected invariant domains D for Kleinian groups.

Recall that we denote the extension of the Riemann map by $\bar{\varphi} : \partial\Delta \to \bar{\partial} D$ and the topology on the Carathéodory boundary $\bar{\partial} D$ is coincident with the topology induced by $\bar{\varphi}$. Intervals in $\bar{\partial} D$ are defined in this sense. The pull-back $g = \varphi^* \gamma$ of any biholomorphic automorphism γ of D is a Möbius transformation preserving Δ. The action of γ on D extends to $\bar{\partial} D$ and defines $\bar{\gamma}$ in such a way that

$$\bar{\gamma} \circ \bar{\varphi}(\xi) = \bar{\varphi} \circ g(\xi)$$

for every $\xi \in \partial\Delta$.

When a Kleinian group Γ acts on D, the pull-back $G = \varphi^* \Gamma$ is also a Kleinian group acting on Δ, which is called the *Fuchsian model* of the pair (Γ, D). We say that the Fuchsian model G is of *the first kind* if the limit set of G is coincident with $\partial\Delta$. We only consider the case where D is a connected component of the region of discontinuity $\Omega(\Gamma)$. Even in this case, the Fuchsian model G is not necessarily of the first kind (cf. [8]). If Γ is finitely generated, then G is of the first kind by the Ahlfors finiteness theorem.

We prove several results on the structure of impressions of prime ends of invariant domains for Kleinian groups.

DEFINITION. For a subset E of the Carathéodory boundary $\bar{\partial} D$ of a simply connected hyperbolic domain D, we define the impression of E as

$$I(E) = \bigcup_{e \in E} I(e).$$

First we formulate certain continuity of the correspondence of prime ends to their impressions. The following proposition is easily seen once we define the impression of a prime end as the cluster set of the Riemann map.

PROPOSITION 5. *The impression $I(e)$ for a prime end $e \in \bar{\partial} D$ is coincident with the intersection $\bigcap_{n=1}^{\infty} I(U_n)$ of the impressions $\{I(U_n)\}$, where $\{U_n\}_{n=1}^{\infty}$ is an arbitrary sequence of open intervals in $\bar{\partial} D$ satisfying*

$$U_1 \supset U_2 \supset \cdots \supset \bigcap_{n=1}^{\infty} U_n = \{e\}.$$

Our first two results provide sufficient conditions for $I(E)$ not to be the whole ∂D, which correspond to the results by Rogers [16, §2] for boundaries of local Siegel disks and by Mayer and Rogers [10, §3] for Julia sets of polynomials.

THEOREM 6. *Let D be a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$ for a non-elementary Kleinian group Γ. Assume that $\partial D = \Lambda(\Gamma)$ is decomposable. Then the impression $I(E)$ of every countable set $E \subset \bar{\partial} D$ does not have non-empty interior with respect to the relative topology on ∂D. In particular, $I(E) \subsetneqq \partial D$.*

PROOF. Suppose that $I(E)$ has non-empty interior in ∂D. Then the Baire category theorem implies that there exists a prime end $e \in E$ such that the interior $I(e)^\circ$ of the impression $I(e) \subset \partial D$ is not empty. Since loxodromic fixed points are dense in $\Lambda(\Gamma)$, there exists a loxodromic element $\gamma \in \Gamma$ such that $a_{\gamma^{-1}} \in I(e)^\circ$. By the group invariance, we see that $I(\bar{\gamma}^n(e))$ has the interior $\gamma^n(I(e)^\circ)$. Then the limit infimum of the sequence $\{I(\bar{\gamma}^n(e))\}_{n=1}^\infty$ contains ∂D except the attracting fixed point a_γ:

$$\bigcup_{n_0 \in \mathbf{N}} \bigcap_{n \geq n_0} I(\bar{\gamma}^n(e)) \supset \partial D - \{a_\gamma\}.$$

On the other hand, $\bar{\gamma}^n(e)$ converge to a prime end $e' \in \bar{\partial} D$. Let U be any open interval containing e'. Then there exists an integer n_0 such that $\bar{\gamma}^n(e) \in U$ for every $n \geq n_0$, which implies that $I(U) \supset \bigcup_{n \geq n_0} I(\bar{\gamma}^n(e))$. Since $I(e') = \bigcap I(U)$ by Proposition 5, it contains the limit supremum of the sequence:

$$I(e') \supset \bigcap_{n_0 \in \mathbf{N}} \bigcup_{n \geq n_0} I(\bar{\gamma}^n(e)).$$

Hence $I(e')$ contains $\partial D - \{a_\gamma\}$. Since $I(e')$ is closed, it must coincide with ∂D. However, Proposition 3 combined with Lemma 4 asserts that this is impossible when $\partial D = \Lambda(\Gamma)$ is decomposable. $\square$

THEOREM 7. *Let D be a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$ for a non-elementary Kleinian group Γ such that the Fuchsian model of (Γ, D) is of the first kind. Assume that $\partial D = \Lambda(\Gamma)$ is decomposable. Let E be a subset of $\bar{\partial} D$ such that the complement $E^c = \bar{\partial} D - E$ has non-empty interior. Then $I(E) \subsetneqq \partial D$.*

PROOF. Let $\varphi : \Delta \to D$ be the Riemann map and $G = \varphi^*(\Gamma)$ the Fuchsian model. Suppose that $I(E) = \partial D$. Set $X = \bar{\varphi}^{-1}(E) \subset \partial \Delta$ and $X^c = \partial \Delta - X = \bar{\varphi}^{-1}(E^c)$. The condition that E^c has non-empty interior is equivalent to that X^c has non-empty interior. Since the loxodromic fixed points are dense in the limit set, we can take a loxodromic element $g \in G$ such that $a_{g^{-1}} \in X^c$. Then $g^n(X)$ converge to the attracting fixed point a_g as $n \to \infty$. By the group invariance, we have

$$I(\bar{\varphi}(g^n(X)) = I(\bar{\gamma}^n(\bar{\varphi}(X))) = \gamma^n(I(E)) = \partial D$$

where $\varphi^*\gamma = g$. Take an arbitrary open interval U containing the prime end $\bar{\varphi}(a_g)$. Then $\bar{\varphi}(g^n(X)) \subset U$ for a sufficiently large n and hence $I(\bar{\varphi}(g^n(X)) \subset I(U)$. Since $I(\bar{\varphi}(a_g)) = \bigcap I(U)$ by Proposition 5, it must coincide with ∂D. Again by Proposition 3 and Lemma 4, this contradicts the assumption that ∂D is decomposable. $\square$

Next, we prove the existence of arbitrarily close prime ends whose impressions are disjoint. For a local Siegel disk, the Pommerenke-Rodin number is defined by considering the intersection of impressions (cf. [15], [16]). However, the group invariance does not allow to introduce such concept to the case of Kleinian groups. Remark that we will state a stronger result in the last section.

Theorem 8. *Let D be a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$ for a non-elementary Kleinian group Γ such that the Fuchsian model of (Γ, D) is of the first kind. Assume that $\partial D = \Lambda(\Gamma)$ is decomposable. In every open interval U of $\bar{\partial}D$, there exist distinct prime ends e_1 and e_2 such that the impressions $I(e_1)$ and $I(e_2)$ $(\subset \partial D)$ are disjoint.*

Proof. Let $\varphi : \Delta \to D$ be the Riemann map and $G = \varphi^*(\Gamma)$ the Fuchsian model. For a prime end $e \in \bar{\partial}D$, the impression $I(e)$ is a proper closed subset of ∂D because ∂D is decomposable. Hence we can choose a loxodromic element $\gamma \in \Gamma$ such that $a_\gamma \notin I(e)$. Let $g = \varphi^*\gamma$ be the corresponding element of G. Assume that the prime end $\bar{\varphi}(a_g)$ lies in the given interval U. Then $\bar{\gamma}^n(e)$ belongs to U for any sufficiently large n, and when m is larger enough than n, the impressions $I(\bar{\gamma}^n(e)) = \gamma^n(I(e))$ and $I(\bar{\gamma}^m(e)) = \gamma^m(I(e))$ are disjoint. In case $\bar{\varphi}(a_g)$ does not lie in U, take an element $\delta \in \Gamma$ such that $\bar{\delta}\bar{\varphi}(a_g) \in U$, and replace γ and e with $\delta\gamma\delta^{-1}$ and $\bar{\delta}(e)$. Then apply the same argument as above. □

On the contrary, if the boundary ∂D is indecomposable, then the impression of every prime end is equal to ∂D as we can see from Theorem 10 below, and in particular the conclusions of the above Theorems 7 and 8 are not satisfied. Hence these theorems give characterizations of the decomposability of the boundary ∂D.

Lemma 9. *Let D be a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$ for a non-elementary Kleinian group Γ such that the Fuchsian model of (Γ, D) is of the first kind. If $I(e_0) = \partial D$ for some prime end $e_0 \in \bar{\partial}D$, then $I(e) = \partial D$ for every prime end $e \in \bar{\partial}D$.*

Proof. By Proposition 5, if $I(U) = \partial D$ for every open interval U in $\bar{\partial}D$, then $I(e) = \partial D$ for every prime end e. By $I(e_0) = \partial D$, any open interval $U_0 \subset \bar{\partial}D$ containing e_0 satisfies $I(U_0) = \partial D$. For every interval U, there exists an element $\gamma \in \Gamma$ such that $\bar{\gamma}(U_0) \subset U$ because the Fuchsian model is of the first kind. By the group invariance, we have

$$I(U) \supset I(\bar{\gamma}(U_0)) = \gamma(I(U_0)) = \partial D.$$

Hence $I(U) = \partial D$ for every interval U. □

Lemma 9 applied to Proposition 3 concludes the following.

Theorem 10. *Let D be a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$ for a non-elementary Kleinian group Γ such that the Fuchsian model of (Γ, D) is of the first kind. If $\partial D = \Lambda(\Gamma)$ is indecomposable, then the impression $I(e)$ is coincident with ∂D for every prime end $e \in \bar{\partial}D$ (and the converse is also true).*

§4. Local connectivity and decomposability

In this section, we give a short expository concerning local connectivity and decomposability of the limit sets, mainly for finitely generated Kleinian groups. A continuum Λ is *locally connected* if, for every neighborhood V of each point $x \in \Lambda$, there exists a neighborhood $W \subset V$ of x such that $\Lambda \cap W$ is connected. First we note that decomposability of a continuum is a weaker condition than local connectivity in general (cf. [21, p.23]).

Proposition 11. *Let Λ be a non-degenerate continuum on S^2. Then it is locally connected if and only if, for every $\epsilon > 0$, there exist a finite number of subcontinua Λ_i $(i = 1, \ldots, n)$ such that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$ and the spherical diameter of each Λ_i is less than ϵ. In particular, if Λ is locally connected, then it is decomposable.*

Let G be a finitely generated Kleinian group with the connected limit set $\Lambda(G) \neq S^2$. Then by Anderson and Maskit [3], $\Lambda(G)$ is locally connected if and only if the limit set of the component subgroup $\Gamma = \mathrm{Stab}_G(D)$ for every connected component D of $\Omega(G)$ is locally connected. By the Ahlfors finiteness theorem, the component subgroup Γ is again finitely generated. Hence we can reduce the question on the local connectivity of $\Lambda(G)$ to the case when Γ is a finitely generated Kleinian group with a simply connected invariant component of the region of discontinuity $\Omega(\Gamma)$. It is also proved in [3] that the problem is further reduced to the case when such Γ is a degenerate group, in other words, $\Omega(\Gamma)$ is connected and simply connected.

Local connectivity of the limit set $\Lambda(\Gamma)$ for a finitely generated degenerate group Γ has been an open problem for a long time (see Abikoff [1]), and its solution is closely related to the ending lamination conjecture, which is one of the main problems in the theory of Kleinian groups. Minsky [12] proved that, if the associated hyperbolic 3-manifold to Γ has a lower bound on the injectivity radius, then it satisfies the ending lamination conjecture as well as the limit set $\Lambda(\Gamma)$ is locally connected. Later, it was shown by Minsky [13] and McMullen [11] that these results are always true for Γ such that $\Omega(\Gamma)/\Gamma$ is a once-punctured torus. Recently, Brock, Canary and Minsky have announced the resolution of the ending lamination conjecture in general. This may open a way to prove the local connectivity of connected limit sets of finitely generated Kleinian groups. On the other hand, there is very little known in general about the structure of the limit sets of infinitely generated Kleinian groups, and the results in this paper will be likely to applied mainly to infinitely generated groups.

Finally we remark the relationship between decomposability of Kleinian groups via the Klein-Maskit combination theorem and decomposability of limit sets in our sense. Let G be a finitely generated Kleinian group with the connected limit set $\Lambda(G) \neq S^2$. By Abikoff and Maskit [2], G can be decomposed into finitely many quasifuchsian groups, degenerate groups and web groups until accidental parabolic transformations disappear. If there exists an accidental parabolic transformation g in G, the limit set $\Lambda(G)$ is divided into two continua by the fixed point of g. Hence the decomposability of the limit set in their sense implies the decomposability in our sense. In other words, if a finitely generated degenerate group Γ contains an accidental parabolic transformation, then $\Lambda(\Gamma)$ is decomposable.

§5. Limit sets of degenerate groups

A *degenerate group* is, by definition within this section, a non-elementary Kleinian group Γ, not necessarily finitely generated, whose region of discontinuity $\Omega(\Gamma) \neq \emptyset$ is connected and simply connected. Further we always assume that the Fuchsian model of $(\Gamma, \Omega(\Gamma))$ is of the first kind. This is automatically satisfied for finitely generated degenerate groups.

First we prove the following theorem on the intersection of the impressions of prime ends, which is related to the arguments by Rogers [15, §7] for boundaries of local Siegel disks. In case $\Lambda(\Gamma)$ is locally connected, distinct impressions meet at a cut point of $\Lambda(\Gamma)$ and the following theorem implies density of such points.

THEOREM 12. *Let Γ be a degenerate Kleinian group. In every open interval U of the Carathéodory boundary of $\Omega(\Gamma)$, there exist distinct prime ends e_1 and e_2 such that the impressions $I(e_1)$ and $I(e_2)$ ($\subset \Lambda(\Gamma)$) have non-empty intersection.*

PROOF. Let $\varphi : \Delta \to \Omega(\Gamma)$ be the Riemann map and $G = \varphi^*(\Gamma)$ the Fuchsian model. Suppose that the impressions $I(e)$ are mutually disjoint for all $e \in U$. Let $X = \bar{\varphi}^{-1}(U) \subset \partial\Delta$ be the interval corresponding to U. There exists a loxodromic element $g \in G$ such that $\mathrm{Fix}(g) \subset X$. Then, by the group invariance, $I(e)$ are mutually disjoint for every $e \in \bar{\gamma}^n(U) = \bar{\varphi}(g^n(X))$ for each integer n. Since any two points on $\partial\Delta$ are contained in $g^n(X)$ for some n, the impressions are mutually disjoint for all prime ends $e \in \bar{\partial}\Omega(\Gamma)$.

Then the inverse function $\varphi^{-1} : \Omega(\Gamma) \to \Delta$ of the Riemann map has a continuous extension to the closure of $\Omega(\Gamma)$, which is denoted by $\hat{\varphi}^{-1} : S^2 \to \overline{\Delta}$. Take a point $z_0 \in \Omega(\Gamma)$ and a non-trivial closed curve ℓ in $\Omega(\Gamma) - \{z_0\}$. This is contractible in $S^2 - \{z_0\}$ and thus the continuous image $\hat{\varphi}^{-1}(\ell) = \varphi^{-1}(\ell)$ is also contractible in $\overline{\Delta} - \{\varphi^{-1}(z_0)\}$. However, $\varphi^{-1}(\ell)$ is a closed curve around $\varphi^{-1}(z_0)$, which is a contradiction. □

In general, a continuum $\Lambda \subset S^2$ with empty interior is called *tree-like* if the complement $\Omega = S^2 - \Lambda$ is connected. The limit set $\Lambda(\Gamma)$ of a degenerate group is a tree-like continuum.

In the remainder of this section, we will try to formulate a fact that the limit set of a degenerate group branches everywhere if it is decomposable. This will be done by considering an irreducible subcontinuum: for any points x and y on a continuum Λ, an *irreducible subcontinuum* λ about x and y is a continuum containing x and y such that no proper subcontinuum of λ has this property. Note that, every continuum has an irreducible subcontinuum about given points (cf. [5, p.44]).

PROPOSITION 13. *Let Λ be a tree-like continuum. Then, for any points x and y on Λ, an irreducible subcontinuum λ about x and y is unique. The irreducible subcontinuum λ is properly contained in Λ if and only if there exists a composant μ of Λ containing x and y.*

PROOF. It is known that a tree-like continuum Λ is unicoherent, that is, the intersection of any two subcontinua of Λ is connected (cf. [14, Th.1]). Let λ_1 and λ_2 be irreducible subcontinua about x and y. Then the intersection $\lambda_1 \cap \lambda_2$ is a continuum containing x and y. By irreducibility, we have $\lambda_1 = \lambda_1 \cap \lambda_2 = \lambda_2$.

If $\lambda \subsetneqq \Lambda$, then there is a composant μ containing λ. Conversely, if there is a composant μ with x and y, then there is a proper subcontinuum of Λ with x and y, which contains λ. Hence $\lambda \subsetneqq \Lambda$. □

DEFINITION. Let Ω be a simply connected hyperbolic domain in S^2. A point $x \in \partial\Omega$ is *accessible* if there exists an arc in Ω ending at x. A *crosscut* is an arc in Ω that has accessible points at both ends.

Let $\varphi : \Delta \to \Omega$ be the Riemann map. It can be proved that, for an accessible point x on $\partial\Omega$ by an arc ℓ in Ω, the arc $\varphi^{-1}(\ell)$ in Δ has the end point ξ in $\partial\Delta$, which is uniquely determined only by x. In this sense, we can identify an accessible point $x \in \partial\Omega$ with the prime end $e = \bar{\varphi}(\xi) \in \bar{\partial}\Omega$. The set of all accessible points has full Lebesgue measure on $\bar{\partial}\Omega \cong \partial\Delta$ as well as it is dense in $\partial\Omega$. See [4, Chap.9].

Proposition 14. *Let Ω be a simply connected hyperbolic domain in S^2 that is invariant under a loxodromic Möbius transformation γ. Then the fixed points a_γ and $a_{\gamma^{-1}}$ are accessible.*

Proof. Let ℓ be a closed arc in Ω whose end points are z and $\gamma(z)$ for some point z. A sequence of the arcs $\{\gamma^n(\ell)\}_{n\in\mathbf{Z}}$ converges to a_γ as $n \to +\infty$ and to $a_{\gamma^{-1}}$ as $n \to -\infty$. Then the union $\bigcup_{n\in\mathbf{Z}} \gamma^n(\ell)$ is an arc ending at a_γ and $a_{\gamma^{-1}}$. □

Definition. Let Ω be a simply connected hyperbolic domain in S^2. Let W be a subdomain of Ω divided by a crosscut. We say that W has *entire boundary* if $\overline{W}$ contains the whole $\partial\Omega$ and that W has *inner boundary* if $(\overline{W})^\circ$, the interior of the closure of W, is strictly larger than W. We call an element of $(\overline{W})^\circ - W$ an inner boundary point.

We state two elementary facts concerning the entire and inner boundary for the case that $\Lambda = S^2 - \Omega$ is a tree-like continuum.

Proposition 15. *Let Ω be a simply connected hyperbolic domain in S^2 such that $\Lambda = S^2 - \Omega$ is a tree-like continuum. Let W be a subdomain of Ω divided by a crosscut c and let $W^* = \Omega - \overline{W}$. Then W^* does not have entire boundary if and only if W has inner boundary.*

Proof. Assume that W^* does not have entire boundary. Then there exists a point $z \in \Lambda$ such that $z \notin \partial W^*$. This implies that z is contained in $S^2 - \overline{W^*}$, which is coincident with $(\overline{W})^\circ$ because $\overline{\Omega} = S^2$. Hence z is an inner boundary point of W.

For the opposite direction, we do not need the condition $\overline{\Omega} = S^2$. Assume that W has an inner boundary point $z \in \Lambda$. Then there exists a neighborhood U of z such that $U \subset W \cup \Lambda$. This implies that $z \notin \partial W^*$, and hence W^* does not have entire boundary. □

Proposition 16. *Let Ω be a simply connected hyperbolic domain in S^2 such that $\Lambda = S^2 - \Omega$ is a tree-like continuum. Let c be a crosscut of Ω, W a subdomain of Ω divided by c, and λ the irreducible subcontinuum of Λ about the end points of c. Then $\lambda = \partial\overline{W} - c$. Moreover, $\lambda = \Lambda$ if and only if W has entire boundary but not inner boundary.*

Proof. Let W' be the component of the complement of $\lambda \cup c$ that contains W. Since the difference between W and W' has no interior points, $\overline{W'}$ coincides with $\overline{W}$. Since λ is irreducible, $\partial\overline{W'} = \partial W'$. Hence $\partial\overline{W} = \lambda \cup c$, which implies the first assertion. If $\Lambda = \partial\overline{W} - c\,(= \lambda)$, then clearly W has entire boundary but not inner boundary, and vice versa. □

Summing up these facts, we obtain the following result concerning the structure of a tree-like continuum, in particular, the limit set of a degenerate group.

Theorem 17. *Let Ω be a simply connected hyperbolic domain in S^2 such that $\Lambda = S^2 - \Omega$ is a tree-like continuum. Then the following conditions are equivalent:*

1. *For every prime end $e \in \bar{\partial}\Omega$, the impression $I(e)$ coincides with Λ.*
2. *Every subdomain W of Ω divided by a crosscut has entire boundary.*
3. *Every subdomain W of Ω divided by a crosscut has no inner boundary.*
4. *For any distinct two accessible points on Λ, the irreducible subcontinuum λ about them coincides with Λ.*

(5) *For any distinct two accessible points on* Λ*, there exists no composant* μ *containing both of them.*

Proof. For any prime end $e \in \bar{\partial}\Omega$, choose a sequence of crosscuts $\{c_n\}$ of Ω with the end points e_n and e'_n in $\bar{\partial}\Omega$ such that the intervals (e_n, e'_n) satisfy

$$(e_1, e'_1) \supset (e_2, e'_2) \supset \cdots \supset \bigcap_{n=1}^{\infty}(e_n, e'_n) = \{e\}.$$

Consider the subdomain of Ω divided by each c_n. Then by Proposition 5, we can see that (1) is equivalent to (2). The equivalence of (2) and (3) follows from Proposition 15. Then the combination of (2) and (3) is equivalent to (4) by Proposition 16. Finally, (4) and (5) are equivalent by Proposition 13. □

Corollary 18. *For a degenerate Kleinian group* Γ*, the limit set* $\Lambda(\Gamma)$ *is indecomposable if and only if one (all) of the conditions in Theorem 17 are satisfied for* $\Omega = \Omega(\Gamma)$ *and* $\Lambda = \Lambda(\Gamma)$*. In particular, if* $\Lambda(\Gamma)$ *is indecomposable, then there exist no distinct loxodromic fixed points on a composant* μ *of* $\Lambda(\Gamma)$*.*

Proof. By Theorem 10, $\Lambda(\Gamma)$ is indecomposable if and only if $I(e) = \Lambda(\Gamma)$ for every prime end $e \in \bar{\partial}\Omega(\Gamma)$. Then the first assertion follows from Theorem 17. The second assertion can be seen from Proposition 14. □

§6. Continuous extension of the Riemann map

We compare decomposability and local connectivity of the boundary of a simply connected Γ-invariant component D of the region of discontinuity in terms of continuous extendability of the Riemann map $\varphi : \Delta \to D$. It is known that the boundary ∂D is locally connected if and only if the impression $I(e)$ is a singleton for every prime end $e \in \bar{\partial}D$. Since the impression can be represented by the cluster set of the Riemann map φ, this is equivalent to saying that φ extends continuously to the boundary $\partial\Delta$.

The following special kind of limit points of a Kleinian group were utilized in Tukia [20] to consider a problem on continuous extendability of the Riemann map.

Definition. Let Γ be a Kleinian group acting on S^2. (We regard a Fuchsian group as a Kleinian group.) A limit point $x \in \Lambda(\Gamma)$ is called a *Myrberg point* of Γ if, for any distinct points a and b in $\Lambda(\Gamma)$, there exists a sequence of elements $\{\gamma_n\}_{n=1}^{\infty} \subset \Gamma$ such that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ as $n \to \infty$ for all $y \in S^2 - \{x\}$ uniformly on compact sets. We denote the set of all Myrberg points by $M(\Gamma)$.

The following result by Tukia [20, Th. 3D] is closely related to our investigation on impressions of prime ends. Remark that decomposability of $\Lambda(G)$ implies the assumption in the following theorem that the impression is a proper subset of $\Lambda(G)$.

Theorem 19. *Let* D *be a simply connected invariant component of the region of discontinuity for a non-elementary Kleinian group* Γ*. Let* $\varphi : \Delta \to D$ *be the Riemann map and* $G = \varphi^*(\Gamma)$ *the Fuchsian model of* (Γ, D)*. Assume that there exists a limit point* $\zeta \in \Lambda(G)$ *such that the impression* $I(\bar{\varphi}(\zeta))$ *is a proper subset of* $\partial D = \Lambda(\Gamma)$*. Then* φ *extends continuously to* $M(G)$ *and defines a homeomorphism of* $M(G)$ *onto* $M(\Gamma)$*. Moreover, for any Myrberg point* $\xi \in M(G)$ *and any different limit point* $\zeta \in \Lambda(G)$*, the impressions* $I(\bar{\varphi}(\xi))$ *and* $I(\bar{\varphi}(\zeta))$ *are disjoint.*

It is known that $M(\Gamma)$ is dense in $\Lambda(\Gamma)$ for every non-elementary Kleinian group Γ. Therefore, this theorem in particular implies Theorem 8 in Section 3. Moreover, it is able to combine and refine Theorems 6 and 7 in the following stronger statement. In general, a set in a topological space said to be of *the first category* if it is a countable union of nowhere dense closed subsets.

THEOREM 20. *Let D be a simply connected invariant component of the region of discontinuity for a non-elementary Kleinian group Γ such that the Fuchsian model of (Γ, D) is of the first kind. Assume that $\partial D = \Lambda(\Gamma)$ is decomposable. Let E be a subset of the first category on the Carathéodory boundary $\bar{\partial} D$. Then the impression $I(E)$ does not have non-empty interior with respect to the relative topology on ∂D. In particular, $I(E) \subsetneqq \partial D$.*

PROOF. Let $E = \bigcup_{i=1}^{\infty} E_i$, where each E_i is a nowhere dense closed set in $\bar{\partial} D$. Suppose that $I(E)$ has non-empty interior in ∂D. Then the Baire category theorem implies that there exists, say $E_1 \subset E$ such that the interior $I(E_1)^\circ$ of the impression $I(E_1) \subset \partial D$ is not empty.

Let $\varphi : \Delta \to D$ be the Riemann map and $G = \varphi^*(\Gamma)$ the Fuchsian model. Since Myrberg points are dense in the limit set, since $X_1 = \bar{\varphi}^{-1}(E_1)$ is nowhere dense in $\Lambda(G) = \partial \Delta$ and since the extension $\hat{\varphi} : M(G) \to M(\Gamma)$ of φ is a homeomorphism by Theorem 19, we can find $x \in M(\Gamma)$ and $\xi \in M(G)$ such that $x \in I(E_1)^\circ$, $\xi \notin X_1$ and $\hat{\varphi}(\xi) = x$.

Take distinct points α and β in $M(G)$. Since ξ is a Myrberg point of G, there exists a sequence of elements $\{g_n\}$ in G such that $g_n(\xi) \to \alpha$ and $g_n(\eta) \to \beta$ for all $\eta \in S^2 - \{\xi\}$. Set $\gamma_n = \varphi g_n \varphi^{-1}$ $(g_n = \varphi^* \gamma_n)$, and consider the sequence $\{\gamma_n\}$ in Γ. Set $\hat{\varphi}(\alpha) = a$ and $\hat{\varphi}(\beta) = b$, which are Myrberg points of Γ. Then, by the continuity of $\hat{\varphi}$ and the convergence property of Γ (see [20, Lemma 2A & Th. 3B]), we can see that $\gamma_n(x) \to a$ and $\gamma_n(y) \to b$ as $n \to \infty$ for all $y \in S^2 - \{x\}$ uniformly on compact sets.

Consider the prime end $e' = \bar{\varphi}(\beta)$, whose impression $I(e')$ is a singleton $\{b\}$. Since $g_n(X_1)$ converge to β, Proposition 5 yields

$$I(e') \supset \bigcap_{n_0 \in \mathbf{N}} \bigcup_{n \geq n_0} I(\bar{\gamma}_n(E_1)) = \limsup_{n \to \infty} I(\bar{\gamma}_n(E_1)).$$

On the other hand, since $I(\bar{\gamma}_n(E_1))$ has the interior $\gamma_n(I(E_1)^\circ)$,

$$\liminf_{n \to \infty} I(\bar{\gamma}_n(E_1)) = \bigcup_{n_0 \in \mathbf{N}} \bigcap_{n \geq n_0} I(\bar{\gamma}_n(E_1)) \supset \partial D - \{b\}.$$

However this is a contradiction. □

When Γ is a finitely generated degenerate group, the Fuchsian model G is of cofinite area, and in this case, the set of all Myrberg points of G has full Lebesgue measure on $\partial \Delta$ (cf. [20, p.99]). Hence, as a corollary to Theorem 19, we obtain the following.

COROLLARY 21. *Let Γ be a finitely generated degenerate group. Assume that the limit set $\Lambda(\Gamma)$ is decomposable. Then the Riemann map $\varphi : \Delta \to \Omega(\Gamma)$ extends continuously to almost all points on $\partial \Delta$.*

Therefore we might say that the difference between the local connectivity and the decomposability of the limit set is of "null measure" in this sense.

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