Inclusion relations between the Bers embeddings of Teichmüller spaces

Katsuhiko Matsuzaki

Department of Mathematics, Ochanomizu University

Abstract. We prove that if the Bers embeddings of the Teichmüller spaces of infinitely generated Fuchsian groups are coincident, then these Fuchsian groups are the same.

§1. Introduction

Recent researches on infinite dimensional Teichmüller spaces contribute to a problem whether any biholomorphic isomorphism between Teichmüller spaces is induced by a geometric mapping between Riemann surfaces. One can consult a monograph by Gardiner and Lakic [6] for a detailed exposition on this topic. In this present paper, we deal with the Bers embeddings of Teichmüller spaces and prove that if the Bers embeddings of the Teichmüller spaces of non-exceptional Fuchsian groups are coincident, then these Fuchsian groups are the same. This is a special case of the above mentioned problem. Actually, we prove our result for the Banach spaces of the bounded holomorphic automorphic 2-forms for Fuchsian groups, and hence the statement itself can be understood without considering Teichmüller spaces.

Acknowledgment. The author would like to thank the referee for careful reading of the previous version of this paper, pointing out several mistakes, suggesting improvement of the arguments and providing useful information. In particular, the proof of Theorem 2 relying on Lemma 3 was introduced by the referee, which shortened the original proof significantly. He also informed the author of a recent preprint by Markovic, which solves the problem on biholomorphic isomorphisms between Teichmüller spaces completely.

1991 *Mathematics Subject Classification.* Primary 30F60, 30F35; Secondary 30F30, 32G15.

Key words and phrases. Riemann surface, Fuchsian group, automorphic form, Teichmüller space, quadratic differential, Bers embedding.

§2. Preliminaries

A Fuchsian group is a discrete subgroup of $\mathrm{Aut}(\Delta)$, the group of all biholomorphic automorphisms of the unit disk Δ. When the hyperbolic metric $ds = \rho(z)|dz|$ is provided on Δ where $\rho(z) = 2/(1-|z|^2)$, a biholomorphic automorphism of Δ is an orientation preserving isometric automorphism of the hyperbolic plane (Δ, ds), and vice versa.

The push-forward of a function ϕ on Δ by $f \in \mathrm{Aut}(\Delta)$ is defined as

$$(f_*\phi)(z) = \phi(f^{-1}(z))(f^{-1})'(z)^2.$$

Let $\mathrm{Hol}(\Delta)$ denote the set of all holomorphic functions on Δ. If $\phi \in \mathrm{Hol}(\Delta)$ satisfies $\gamma_*\phi = \phi$ for every element γ of a Fuchsian group Γ, we say that ϕ is a holomorphic automorphic 2-form for Γ.

Consider the following complex Banach space of holomorphic automorphic 2-forms φ with the hyperbolic L^∞-norm:

$$B(\Gamma) = \{\varphi \in \mathrm{Hol}(\Delta) |\ \gamma_*\varphi = \varphi\ (\forall\gamma \in \Gamma),\ \|\varphi\|_\infty < \infty\},$$

where

$$\|\varphi\|_\infty = \sup_{z\in\Delta} \rho^{-2}(z)|\varphi(z)|.$$

An element of $B(\Gamma)$ is called a **bounded** holomorphic automorphic 2-form. Also consider the complex Banach space of holomorphic automorphic 2-forms φ with the L^1-norm:

$$A(\Gamma) = \{\varphi \in \mathrm{Hol}(\Delta) |\ \gamma_*\varphi = \varphi\ (\forall\gamma \in \Gamma),\ \|\varphi\|_1 < \infty\},$$

where

$$\|\varphi\|_1 = \int_{\Delta/\Gamma} |\varphi(z)| dxdy.$$

An element of $A(\Gamma)$ is called an **integrable** holomorphic automorphic 2-form.

Any element $f \in \mathrm{Aut}(\Delta)$ induces an inner automorphism of $\mathrm{Aut}(\Delta)$ by conjugation. Namely, if we set $f_*\gamma = f\gamma f^{-1}$ for $\gamma \in \mathrm{Aut}(\Delta)$, we have an isomorphism $f_*\colon \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta)$. Moreover, $f \in \mathrm{Aut}(\Delta)$ also induces linear isometric bijections $f_*\colon B(1) \to B(1)$ and $f_*\colon A(1) \to A(1)$ by $\varphi \mapsto f_*\varphi$. For a Fuchsian group Γ, the f_* maps $B(\Gamma)$ onto $B(f_*(\Gamma))$ isometrically and $A(\Gamma)$ onto $A(f_*(\Gamma))$ isometrically.

Let R be a hyperbolic Riemann orbifold which is represented by Δ/Γ for a Fuchsian group Γ. Let Δ_0 be a subdomain of Δ obtained by removing all the elliptic fixed points of Γ from Δ and set $R_0 = \Delta_0/\Gamma$. Namely, R_0 is a Riemann surface obtained by removing all the cone singular points from the orbifold R. Then any integrable holomorphic automorphic 2-form for Γ on Δ is identified with the corresponding integrable holomorphic quadratic differential on R_0. Indeed, any integrable holomorphic automorphic 2-form for Γ defines a holomorphic quadratic differential on R_0, which is of course integrable. Conversely, any integrable holomorphic quadratic differential on R_0 has at most a simple pole at a puncture p of R_0. Let φ be its lift to Δ, which is an integrable meromorphic automorphic 2-form for Γ. If p is a cone point of the orbifold R with the branch order $n \geq 2$, then the order of φ at $\tilde{p}$ is at least $n-2$, where $\tilde{p} \in \Delta$ is a lift of p. Hence φ is holomorphic on Δ. See [5, Section 3.1] and [7, Section III.8]. We denote the Banach space of all integrable holomorphic quadratic differentials on R_0 by $A(R_0)$.

§3. Banach spaces of quadratic differentials

We determine the coincidence of Fuchsian groups by the corresponding Banach spaces of holomorphic automorphic forms. First we define exceptional groups for which our determination does not work.

Definition. A Fuchsian group Γ is **exceptional** if there exists an element g of $\mathrm{Aut}(\Delta)$ such that g does not belong to Γ and $g_*(\varphi) = \varphi$ for every $\varphi \in A(\Gamma)$. The corresponding Riemann orbifold $R = \Delta/\Gamma$ is also called exceptional.

From this definition, it is clear that non-exceptional Fuchsian groups Γ_1 and Γ_2 satisfy $A(\Gamma_1) = A(\Gamma_2)$ if and only if $\Gamma_1 = \Gamma_2$. We will prove the same result for the Banach spaces of the bounded holomorphic automorphic forms. Before getting into this, we describe non-exceptional Fuchsian groups more concretely. First, we prove the following.

Lemma 1. *If $R_0 = \Delta_0/\Gamma$ can be embedded in the complex plane as a bounded domain, then a Fuchsian group Γ is not exceptional. In particular, an elementary Fuchsian group is not exceptional.*

Proof. Suppose that $g \in \mathrm{Aut}(\Delta)$ satisfies $g_*(\varphi) = \varphi$ for every $\varphi \in A(\Gamma)$. Take a convergent sequence $\{\tilde{p}_n\}_{n=1}^{\infty}$ of distinct points in $\Delta_0 \cap g^{-1}(\Delta_0)$. Let $p_n \in R_0$ be the projection of $\tilde{p}_n$, where R_0 is regarded as a bounded domain in the complex ζ-plane. We consider $\hat{\varphi}_n(\zeta) = \zeta - p_n$ for each integer n and regard $\hat{\varphi}_n(\zeta)d\zeta^2$ as an integrable holomorphic quadratic differential on R_0. Lifting $\hat{\varphi}_n$ to Δ_0 and extending it to Δ, we have an integrable holomorphic automorphic form $\varphi_n \in A(\Gamma)$. Since g keeps the zero set of φ_n invariant and $g(\tilde{p}_n)$ is in Δ_0, there exists an element $\gamma_n \in \Gamma$ such that $g(\tilde{p}_n) = \gamma_n(\tilde{p}_n)$. This implies that a subsequence $\{\gamma_{n'}\}$ of $\{\gamma_n\}$ converges to some $\gamma \in \mathrm{Aut}(\Delta)$, and thus $\gamma_{n'} = \gamma$ for all sufficiently large n' by discreteness of Γ. Then g must be coincident with $\gamma \in \Gamma$. □

The following lemma is a slight modification of the results due to Earle, Gardiner and Lakic [3], which characterizes non-exceptional hyperbolic Riemann orbifolds geometrically. A geodesic on the orbifold R means the projection of a geodesic in Δ.

Lemma 2. *Let Γ be a Fuchsian group and $R = \Delta/\Gamma$ the corresponding hyperbolic Riemann orbifold having disjoint simple closed geodesics ℓ_1 and ℓ_2 such that the* *distance between ℓ_1 and ℓ_2 is attained by a unique geodesic segment α. Then Γ is not exceptional.*

Remark. A geodesic ℓ connecting two cone points of order 2 is regarded as a simple closed geodesic. The free homotopy class represented by this simple closed geodesic consists of all simple closed curves c in R_0 such that ℓ and c bound an annulus in R_0.

Proof. We regard ℓ_1 and ℓ_2 as non-trivial simple closed curves on R_0 which are not contractible to a puncture and which are not freely homotopic to one another. There exists a simple Jenkins-Strebel quadratic differential $\hat{\varphi}_i \in A(R_0)$ corresponding to each ℓ_i $(i = 1, 2)$, that is, an integrable holomorphic quadratic differential all of whose horizontal regular trajectories are simple closed curves that are freely homotopic to ℓ_i (see [6, Chap. 11] and [11, Chap. VI]). Since $A(R_0)$ is identified with $A(\Gamma)$, we lift $\hat{\varphi}_i$ to Δ as an integrable holomorphic automorphic 2-form for Γ, which is denoted by $\varphi_i \in A(\Gamma)$.

Consider the union of ℓ_1, ℓ_2 and α and develop it on Δ from the midpoint of α. Then we have the union H of geodesic lines L_1 and L_2 connected with a geodesic segment $\tilde{\alpha}$ such that $\pi(L_i) = \ell_i$ and $\pi(\tilde{\alpha}) = \alpha$, where π: $\Delta \to R$ is the universal orbifold covering. Each horizontal regular trajectory of φ_i is parallel to $\gamma(L_i)$ for some $\gamma \in \Gamma$, in other words, it has the same end points on $\partial\Delta$ as $\gamma(L_i)$. Let λ_i be a horizontal regular trajectory of φ_i that is parallel to L_i.

Assume that $g \in \mathrm{Aut}(\Delta)$ satisfies $g_*\varphi = \varphi$ for every $\varphi \in A(\Gamma)$. Then $g(\lambda_i)$ is a horizontal regular trajectory of $g_*\varphi_i = \varphi_i$. Hence $g(L_i)$ is an image of L_i by some element of Γ, or equivalently $\pi(g(L_i)) = \ell_i$. Since the shortest geodesic segment connecting ℓ_1 and ℓ_2 is uniquely α, we can see that $\pi(g(\tilde{\alpha})) = \alpha$ and thus $g(H)$ is an image of H by some $\gamma \in \Gamma$. Since g and γ map the H with four endpoints onto the same image keeping the correspondence of L_i, we conclude that $g = \gamma$ and hence $g \in \Gamma$. $\square$

Using these lemmas, we list up the possibility of Fuchsian groups to be exceptional as follows.

Proposition 1. *Let Γ be an exceptional Fuchsian group. Then Γ is a finitely generated group of the first kind that has one of the types (g, n) in the following list, where g is the genus of the orbifold $R = \Delta/\Gamma$ and n is the number of singular points including punctures:*

$$(0,3),\ (0,4),\ (1,1),\ (1,2),\ (2,0)$$

Remark. For torsion-free Fuchsian groups, this is the complete list. In other words, every finitely generated torsion-free Fuchsian group of the first kind that has the type (g, n) in this list is exceptional. For example, for a Fuchsian group Γ of type $(2, 0)$, there exists a lift of the hyperelliptic involution $g \in \mathrm{Aut}(\Delta) - \Gamma$ such that $g_*\varphi = \varphi$ for every $\varphi \in A(\Gamma)$. Hence exceptional torsion-free Fuchsian groups by our definition are coincident with those defined in other literatures such as [5, Section 9.2] and [6, Section 8.3].

Proof of Proposition 1. By Lemma 1, we have only to consider non-elementary Fuchsian groups Γ. Suppose that the orbifold $R = \Delta/\Gamma$ does not have the type (g, n) in the above list. Then, excluding one possibility, we can find disjoint simple closed geodesics ℓ_1 and ℓ_2 in R such that each of them divides R into two pieces. A simple closed geodesic connecting two cone points of order 2 (mentioned in the remark just after Lemma 2) is assumed to be dividing in this sense. The excluded possibility is the case where R is a disk with two singular points. However Lemma 1 asserts that Γ is not exceptional either in this case.

Let W be the subdomain of R whose boundary consists of ℓ_1 and ℓ_2. In case W is an annulus with only one singular point, we take the unique shortest geodesic segment α in W that connects ℓ_1 and ℓ_2. Otherwise, we choose a simple closed geodesic ℓ_3 in W so that ℓ_i $(i = 1, 2, 3)$ bound a pair of pants P. Then take the shortest α in P among the geodesic segments connecting two of $\{\ell_i\}_{i=1}^3$. Without loss of generality, we may assume that α connects ℓ_1 and ℓ_2. Since there is no geodesic segment connecting ℓ_1 and ℓ_2 outside of P, any other one connecting them must be strictly longer than α and hence α is the unique shortest geodesic segment in R. Then Γ is not exceptional by Lemma 2. $\square$

Remark. Another proof of Proposition 1 was remarked by the referee, which uses the following result of Markovic [10, Section 3]: Unless $R = \Delta/\Gamma$ is of the type (g, n) in the above list, there is a discrete set E in R_0 such that, for an arbitrarily given point $p \in R_0 - E$ and for any point $q \in R_0$ distinct from p, there exists $\hat{\varphi}_q \in A(R_0)$ that takes a zero exactly one of the points p and q. Suppose that $g \in \mathrm{Aut}(\Delta)$ satisfies $g_*(\varphi) = \varphi$ for every $\varphi \in A(\Gamma)$. Take a point $\tilde{p} \in \Delta_0 \cap g^{-1}(\Delta_0)$ that is a lift of $p \in R_0 - E$, and consider the lifts $\varphi_q \in A(\Gamma)$ of $\hat{\varphi}_q$ for each $q \in R_0$. Then Markovic's theorem implies that $g(\tilde{p})$ must lie in the Γ-orbit of $\tilde{p}$. Next we consider a convergent sequence consisting of such points $\tilde{p}$. Since Γ is discrete, we can see that g belongs to Γ as in the proof of Lemma 1.

Now we state our main theorem, though the proof itself turns to be very simple once we adapt the definition of exceptional Fuchsian groups as above.

Theorem 1. *If non-exceptional Fuchsian groups Γ_1 and Γ_2 satisfy $B(\Gamma_1) = B(\Gamma_2)$, then $\Gamma_1 = \Gamma_2$.*

Theorem 1 is an immediate consequence of Theorem 2 below. A proof of Theorem 2 follows directly from Lemma 3, which appeared in Drasin and Earle [1, Section 5]. The proof using Lemma 3 was pointed out by the referee, which made the original proof much shorter.

Lemma 3. *The intersection $A(\Gamma) \cap B(\Gamma)$ is dense in $A(\Gamma)$.*

Theorem 2. *Let Γ be a non-exceptional Fuchsian group. If $g \in \mathrm{Aut}(\Delta)$ satisfies $g_*\varphi = \varphi$ for every $\varphi \in B(\Gamma)$, then g belongs to Γ.*

Proof. Since $g_*\varphi = \varphi$ for every $\varphi \in B(\Gamma)$ and since $A(\Gamma) \cap B(\Gamma)$ is dense in $A(\Gamma)$ by Lemma 3, we have $g_*\varphi = \varphi$ for all $\varphi \in A(\Gamma)$. For a non-exceptional Fuchsian group Γ, this implies $g \in \Gamma$. $\square$

§4. Biholomorphic maps between Teichmüller spaces

In this section, we consider connections of our results on the Banach spaces of holomorphic automorphic forms with the problem of determining the biholomorphic maps between Teichmüller spaces.

The universal Teichmüller space T of the complementary disk $\Delta^* = \{z|\ |z| > 1\} \cup \{\infty\}$ is the set of all equivalence classes of quasiconformal automorphisms of Δ^*, where two such quasiconformal automorphisms are assumed to be equivalent if they have the same boundary value. The Bers model of the universal Teichmüller space T is

$$T_B = \{\varphi \in \mathrm{Hol}(\Delta)|\ \varphi(z) = S_f(z)\},$$

where f is a holomorphic function on the unit disk Δ that is extendable to a quasiconformal automorphism of the Riemann sphere and S_f is the Schwarzian derivative of f. It can be proved that T_B is a bounded contractible domain in the complex Banach space $B(1)$ for the trivial Fuchsian group 1. Concerning those facts, see [9, Chap. III].

For any Fuchsian group Γ acting properly discontinuously on Δ and Δ^*, we set $T_B(\Gamma) = T_B \cap B(\Gamma)$. Then it can be also proved that $T_B(\Gamma)$ coincides with the set of the Schwarzian derivatives of holomorphic functions on Δ that are extendable to Γ-compatible quasiconformal automorphisms of the Riemann sphere (see [9, Section V.4]). This is called the Bers embedding of the Teichmüller space for the Fuchsian group Γ: there exists a canonical bijection β: $T(\overline{R}) \to T_B(\Gamma)$ from the Teichmüller space of $\overline{R} = \Delta^*/\Gamma$, the conjugate orbifold to $R = \Delta/\Gamma$. On the other hand, the reflection with respect to the unit circle induces the anti-holomorphic bijection K: $T(R) \to T(\overline{R})$. Hence an anti-holomorphic bijection $J = K^{-1} \circ \beta^{-1}$: $T_B(\Gamma) \to T(R)$ enables us to identify these two spaces.

For $f \in \mathrm{Aut}(\Delta)$, the linear isometric bijection f_*: $B(1) \to B(1)$ preserves T_B. The restriction of f_* to T_B is a biholomorphic automorphism of T_B fixing the origin, which is called a **geometric** automorphism of T_B. Conversely, it is known that every biholomorphic automorphism of T_B fixing the origin is geometric. See Theorem 6 and Section 2 in [2].

For a Fuchsian group Γ, the f_* maps $B(\Gamma)$ onto $B(f_*(\Gamma))$, and hence it maps $T_B(\Gamma)$ onto $T_B(f_*(\Gamma))$ biholomorphically. We say that Fuchsian groups Γ_1 and Γ_2 are **isomorphic** by $f \in \mathrm{Aut}(\Delta)$ if $f_*\Gamma_1 = \Gamma_2$, and that the Bers embeddings $T_B(\Gamma_1)$ and $T_B(\Gamma_2)$ are **isomorphic** by $f \in \mathrm{Aut}(\Delta)$ if $f_*(T_B(\Gamma_1)) = T_B(\Gamma_2)$.

As a corollary to Theorem 1, we can see the following.

Corollary 1. *Let Γ_1 and Γ_2 be non-exceptional Fuchsian groups. The Bers embeddings $T_B(\Gamma_1)$ and $T_B(\Gamma_2)$ are isomorphic by $f \in \mathrm{Aut}(\Delta)$ if and only if* *the Fuchsian groups Γ_1 and Γ_2 are isomorphic by f. In particular, if a non-exceptional Fuchsian group Γ is contained in a Fuchsian group G as a proper subgroup, then $T_B(G)$ is a proper subspace of $T_B(\Gamma)$.*

Proof. Since $f_*(T_B(\Gamma_1)) = T_B(f_*(\Gamma_1))$, the assertion is equivalent to saying that $T_B(f_*(\Gamma_1)) = T_B(\Gamma_2)$ if and only if $f_*(\Gamma_1) = \Gamma_2$. The "if" part is evident. Conversely, $T_B(f_*(\Gamma_1)) = T_B(\Gamma_2)$ implies $B(f_*(\Gamma_1)) = B(\Gamma_2)$ by our definition of the Bers embedding and then Theorem 1 implies $f_*(\Gamma_1) = \Gamma_2$. $\square$

Next, imposing the hypothesis that the Fuchsian groups Γ_1 and Γ_2 are torsion-free, we consider a stronger assertion than this corollary: only having a biholomorphic map between the Bers embeddings $T_B(\Gamma_1)$ and $T_B(\Gamma_2)$ fixing the origin, not assuming it to be induced by $f \in \mathrm{Aut}(\Delta)$, we ask whether Γ_1 and Γ_2 are isomorphic by some $f \in \mathrm{Aut}(\Delta)$.

Problem 1. *Let Γ_1 and Γ_2 be non-exceptional torsion-free Fuchsian groups. There exists a biholomorphic map $F\colon T_B(\Gamma_1) \to T_B(\Gamma_2)$ fixing the origin 0 if and only if the Fuchsian groups Γ_1 and Γ_2 are isomorphic by $f \in \mathrm{Aut}(\Delta)$, where F and f are related as $f_*|_{T_B(\Gamma_1)} = F$.*

Remark. If we do not assume that Γ_1 and Γ_2 are torsion-free, then Problem 1 is not true. Indeed, let Γ be a Fuchsian group with torsion, that is, the orbifold $R = \Delta/\Gamma$ has a cone singurality. Let R_0 be a Riemann surface obtained by removing all the cone singuralities from R and Γ_0 a torsion-free Fuchsian group such that $R_0 = \Delta/\Gamma_0$. Then, by the Bers-Greenberg theorem (see Gardiner [5, Section 9.1]), there exists a biholomorphic map between $T_B(\Gamma)$ and $T_B(\Gamma_0)$ fixing the origin, while the Fuchsian groups Γ and Γ_0 are not isomorphic.

Hereafter, until the end of this paper, we assume that a Fuchsian group Γ is torsion-free and a Riemann orbifold $R = \Delta/\Gamma$ has no cone singularities, that is, R is a Riemann surface.

We can transfer the statement of Problem 1 to a statement on the Teichmüller spaces of the Riemann surfaces $R_1 = \Delta/\Gamma_1$ and $R_2 = \Delta/\Gamma_2$. If F is a biholomorphic map of $T_B(\Gamma_1)$ onto $T_B(\Gamma_2)$ fixing the origin 0, then $\tilde{F} = J \circ F \circ J^{-1}$ is a biholomorphic map of $T(R_1)$ onto $T(R_2)$ fixing the origin, and vice versa. If $f \in \mathrm{Aut}(\Delta)$ induces an isomorphism f_* of Γ_1 onto Γ_2, then it induces a conformal homeomorphism $\hat{f}$ of R_1 onto R_2, and vice versa.

Assume that there exists a biholomorphic map $\tilde{H}\colon T(R_1) \to T(R_2)$ between Teichmüller spaces. Let ω be a representative of a homotopy class of quasiconformal homeomorphisms R_2 onto another Riemann surface R_2' such that the Teichmüller class $[\omega] \in T(R_2)$ is equal to the image of the origin of $T(R_1)$ under $\tilde{H}$. Consider a biholomorphic map $\omega_*\colon T(R_2) \to T(R_2')$ that is induced geometrically by the quasiconformal homeomorphism ω as

$$[\tau] \mapsto \omega_*[\tau] := [\tau \circ \omega^{-1}].$$

Then $\tilde{F} = \omega_* \circ \tilde{H}\colon T(R_1) \to T(R_2')$ is a biholomorphic map that sends the origin to the origin. If Problem 1 has an affirmative answer, then we have a conformal homeomorphism $\hat{f}$: $R_1 \to R_2'$ that induces $\tilde{F}$, and hence a quasiconformal homeomorphism $\hat{h} := \omega^{-1} \circ \hat{f}$: $R_1 \to R_2$ that induces $\tilde{H}$. The converse argument is also true and thus Problem 1 is equivalent to the following.

Problem 2. *Let R_1 and R_2 be non-exceptional hyperbolic Riemann surfaces. There exists a biholomorphic map $\tilde{H}$: $T(R_1) \to T(R_2)$ between the Teichmüller spaces if and only if there exists a quasiconformal homeomorphism $\hat{h}$: $R_1 \to R_2$ that induces $\tilde{H}$, namely $\hat{h}_* = \tilde{H}$.*

Recently, Markovic [10] solved Problem 2 affirmatively. He proved that every non-exceptional Riemann surface (torsion-free Fuchsian group) has the following isometry property. Though we state it in terms of the integrable holomorphic automorphic forms for Fuchsian groups, of course this is equivalent to the statement on the quadratic differentials on Riemann surfaces.

Theorem (isometry property). *Let Γ_1 and Γ_2 be non-exceptional torsion-free Fuchsian groups. Then, for every surjective linear isometry Φ: $A(\Gamma_1) \to A(\Gamma_2)$, there exists a biholomorphic automorphism $f \in \mathrm{Aut}(\Delta)$ and a unimodular constant c such that $f_*\Gamma_1 = \Gamma_2$ and $\Phi = cf_*$.*

The isometry property was first proved by Royden for compact Riemann surfaces (see [5, Section 9.6]). Earle and Gardiner [2] extended it to topologically finite Riemann surfaces and Lakic [8] for Riemann surfaces of finite genus. In [2], it is proved that Problem 2 is true for all Riemann surfaces that satisfy the isometry property. See also [6, Chap. 8].

Since Problems 1 and 2 are equivalent, Problem 1 also follows from Markovic's result. Hence so does our Theorem 2 in the case where the Fuchsian groups are restricted to be torsion-free.

§5. Teichmüller modular groups

We see that our Theorem 2 is related to the faithfulness of the action of Teichmüller modular groups, which was first proved by Earle, Gardiner and Lakic [3]. A different proof was given by Epstein [4].

Set the torsion-free Fuchsian groups Γ_1 and Γ_2 in the previous section to be identical, in other words, the Riemann surfaces R_1 and R_2 identical. Then Problem 2 becomes a problem asking whether any biholomorphic automorphism of the Teichmüller space is induced by a quasiconformal automorphism of the Riemann surface. To formulate this more precisely, let us define the Teichmüller modular group $\mathrm{Mod}(R)$ of a Riemann surface R as the group of all homotopy classes $[\omega]$ of quasiconformal automorphisms ω: $R \to R$, where homotopy is assumed to be relative to the boundary at infinity if the Fuchsian group is of the second kind. Then $[\omega] \in \mathrm{Mod}(R)$ acts on the Teichmüller space $T(R)$ as a biholomorphic automorphism ω_*, namely, a homomorphism θ: $\mathrm{Mod}(R) \to \mathrm{Aut}(T(R))$ is defined by $\omega_*[\tau] = [\tau \circ \omega^{-1}]$ for any Teichmüller class $[\tau] \in T(R)$.

Problem 3. *Let R be a non-exceptional Riemann surface. Then θ: $\mathrm{Mod}(R) \to \mathrm{Aut}(T(R))$ is a surjective isomorphism.*

Markovic's positive solution of Problem 2 also implies that Problem 3 is valid.

Injectivity of θ: $\mathrm{Mod}(R) \to \mathrm{Aut}(T(R))$ was previously proved in [3] and [4]. We remark that our Theorem 2 implies the injectivity.

Theorem (faithfulness). *Let $R = \Delta/\Gamma$ be a non-exceptional Riemann surface. Then the homomorphism θ: $\mathrm{Mod}(R) \to \mathrm{Aut}(T(R))$ is injective.*

Proof. If $[\omega]$ belongs to the kernel of θ, then $\omega_* = \theta([\omega])$ fixes every point of $T(R)$. In particular, ω_* fixes the origin of $T(R)$. This is equivalent to saying that there exists a conformal automorphism $\hat{g}$ of R in the homotopy class $[\omega]$. A lift of $\hat{g}$ to Δ, denoted by g, is an element of $\mathrm{Aut}(\Delta)$ that satisfies $g_*\Gamma = \Gamma$. Since $\hat{g}_* = \omega_*$ fixes every point of $T(R)$, the conjugation by the anti-holomorphic bijection J: $T_B(\Gamma) \to T(R)$ yields $g_*\varphi = \varphi$ for every $\varphi \in T_B(\Gamma)$. This is also valid for every $\varphi \in B(\Gamma)$. Then Theorem 2 concludes that g belongs to Γ. This implies that $[\omega] = [\hat{g}] = [id]$. $\square$

References

1. D. Drasin and C. Earle, *On the boundedness of automorphic forms*, Proc. Amer. Math. Soc. **19** (1968), 1039–1042.
2. C. Earle and F. Gardiner, *Geometric isomorphisms between infinite dimensional Teichmüller spaces*, Trans. Amer. Math. Soc. **348** (1996), 1163–1190.
3. C. Earle, F. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence* (preprint).
4. A. Epstein, *Effectiveness of Teichmüller modular groups*, In the tradition of Ahlfors and Bers, Contemporary Math. 256, AMS, 2000, pp. 69–74.
5. F. Gardiner, *Teichmüller theory and quadratic differentials*, Wiley-Interscience, 1987.
6. F. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, SURV 76, American Mathematical Society, 1999.
7. I. Kra, *Automorphic forms and Kleinian groups*, W. A. Benjamin, 1972.
8. N. Lakic, *An isometry theorem for quadratic differentials on Riemann surfaces of finite genus*, Trans. Amer. Math. Soc. **349** (1997), 2951–2967.
9. O. Lehto, *Univalent functions and Teichmüller spaces*, GTM 109, Springer, 1986.
10. V. Markovic, *Biholomorphic maps between Teichmüller spaces*, Duke Math. J. (to appear).
11. K. Strebel, *Quadratic differentials*, EMG 5, Springer, 1984.

Otsuka 2-1-1, Bunkyo-ku, Tokyo 112-8610, JAPAN
E-mail address: matsuzak@math.ocha.ac.jp