The hyperbolic metric on the complement of the integer lattice points in the plane

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Abstract. A domain in the plane obtained by removing all integer lattice points admits the hyperbolic metric, which is the rank 2 abelian cover of the once-punctured square tours. We compare the hyperbolic metric of this domain with a scaled Euclidean metric in the complement of the cusp neighborhoods. They are quasi-isometric. We investigate the best possible quasi-isometry constant relying on numerical experiment by computer.

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1. Euclidean metric vs. hyperbolic metric

In this note, we consider metrics on a planar domain

$$\Omega = \mathbb{C} - \mathbb{Z} \times \mathbb{Z}.$$

Take the square torus $T = \mathbb{C}/\langle z \mapsto z+1, z \mapsto z+i \rangle$ and remove the point $[0]$ from T to make a once-punctured torus T^*. It admits a complete hyperbolic metric by the uniformization theorem. The universal cover $\mathbb{C} \to T$ with the deck transformation group $\mathbb{Z} \times \mathbb{Z}$ induces an Abelian cover $\pi : \Omega \to T^*$. The hyperbolic metric on Ω is defined by the pull-back of that on T^*. The Euclidean metric on Ω is the restriction of the Euclidean metric on $\mathbb{C}$ with scaling (defined later).

We compare these two metrics on the complement of the cusp neighborhoods. The hyperbolic metric gets much larger near to the punctures and there is no comparison there. For a given open neighborhood A of T^*, set

$$T_0^* := T^* - A; \quad \Omega_0 := \Omega - \pi^{-1}(A).$$

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The hyperbolic metric and the Euclidean metric on T_0^* are comparable (bi-Lipschitz) since T_0^* is compact, and so are on the covering space Ω_0. Hence the inner distances induced by the integration of these metrics along the paths in Ω_0 are also comparable. However, the distances we are interested in are just the restriction of the hyperbolic and the Euclidean distances on Ω to Ω_0.

2. A problem on the optimal quasi-isometry constant

We denote the hyperbolic distance and the Euclidean distance on Ω by d_H and d_E respectively, and use the same notation for their restriction to Ω_0.

Definition 2.1. For metric spaces (X, d_X) and (Y, d_Y) in general, a map $f : X \to Y$ is called a K-quasi-isometry ($K \geq 1$) if there is a constant $C \geq 0$ such that

$$\frac{1}{K}(d_X(x_1, x_2) - C) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for any $x_1, x_2 \in X$.

From the fact that any geodesic curve on (Ω, d_H) connecting any two points in Ω_0 cannot go deeply into the cusp, we see the following.

Proposition 2.2. *The identity map* id : $(\Omega_0, d_H) \to (\Omega_0, d_E)$ *is a K-quasi-isometry with the constant $C \geq 0$ depending on the cusp neighborhood A.*

We try to find the best possible constant K in this proposition. Note that this is independent of the choice of the cusp neighborhood A. We put the following normalization. The Euclidean metric is scaled so that the length of the unit interval is equal to the hyperbolic length of the simple closed geodesic on the punctured torus T^* corresponding to the covering transformation $z \mapsto z + 1$ on Ω. (This also coincides with that for $z \mapsto z + i$ by the symmetry of the square torus.)

Due to the additive constant C, we can ignore small errors in distance without changing the quasi-isometry constant K. Hence we do not have to consider any two points in Ω_0 for the comparison of the distances. Only the following measurement is enough to determine K: for any coprime $p, q \in \mathbb{N}$, the distances between a fixed $z_0 \in \Omega_0$ and $z_0 + pi + q \in \Omega_0$. The Euclidean distance is simply given by $\sqrt{p^2 + q^2}$ without scaling and the hyperbolic distance is comparable with the hyperbolic length of the (p/q)-simple closed geodesic on the once-punctured torus T^*.

3. The computation of lengths of simple closed geodesics

It is known that the hyperbolic length of the (p/q)-simple closed geodesic on T^*, which is denoted by Length(p/q), can be computed recursively by the trace identity from the lengths of $(1/0)$- and $(0/1)$-simple closed geodesics, which are $2 \operatorname{arccosh}\sqrt{2}$ for the square torus. The information about how many times we should apply the recursive relations alternatively is represented by the coefficients of the regular continued fraction of p/q, which is

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{\cdots}{a_{n-1} + \cfrac{1}{a_n}}}}} = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n].$$

The idea of this algorithm can be found in Mumford, Series and Wright [2, Chapter 9]. For example, $30/13 = [2, 3, 4]$. Then, in the order of

$$\frac{0}{1} \xrightarrow{\frac{1}{0}} \frac{1}{1} \xrightarrow{\frac{1}{0}} \frac{2}{1}, \quad \frac{1}{0} \xrightarrow{\frac{2}{1}} \frac{3}{1} \xrightarrow{\frac{2}{1}} \frac{5}{2} \xrightarrow{\frac{2}{1}} \frac{7}{3}, \quad \frac{2}{1} \xrightarrow{\frac{7}{3}} \frac{9}{4} \xrightarrow{\frac{7}{3}} \frac{16}{7} \xrightarrow{\frac{7}{3}} \frac{23}{10} \xrightarrow{\frac{7}{3}} \frac{30}{13},$$

we derive the lengths of their simple closed geodesics.

To obtain the desired estimate, we consider when the supremum of

$$\frac{\mathrm{Length}(p/q)}{2\,\mathrm{arccosh}\sqrt{2} \cdot \sqrt{p^2 + q^2}}$$

is achieved, where $p, q \in \mathbb{N}$ run over coprime integers. At first, we expected that it should be when p/q tend to the golden ratio $\phi = (1 + \sqrt{5})/2 = [1, 1, 1, \ldots]$ and its inverse $\phi^{-1} = [0, 1, 1, \ldots]$.

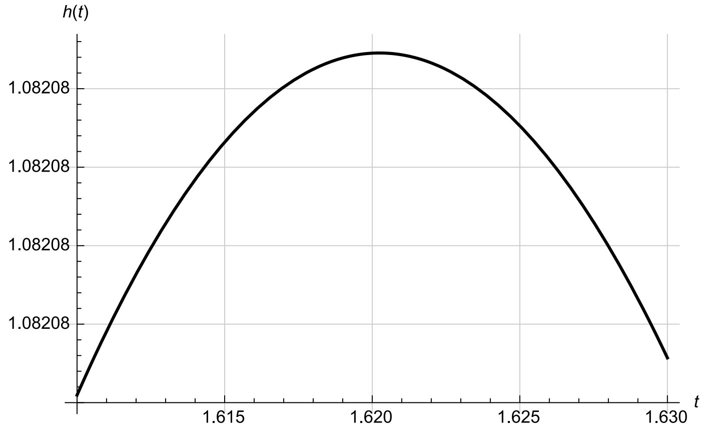

FIGURE 1. Maximum of function $h(t)$

However, a numerical experiment tells us that this expectation is false. As Figure 1 by Mathematica shows, the supremum $1.082085\cdots$ is achieved when p/q converge to $1.62024\cdots$ and its inverse, which is slightly different from the golden ratio $\phi = 1.61803\cdots$.

We also observe the following state (Figure 2) by Mathematica besides the fact we have mentioned above. In this note, statements of experimental results without rigorous proof are called "Claim".

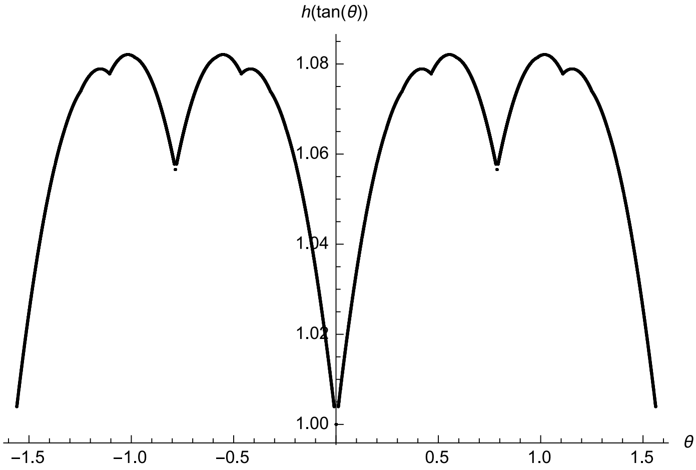

FIGURE 2. Graph of $h(\tan(\theta))$ on $[-\pi/2, \pi/2]$ for square torus

Claim 3.1. *For a rational number p/q with coprime $p, q \in \mathbb{N}$, define a function*

$$h(p/q) := \frac{\mathrm{Length}(p/q)}{2\,\mathrm{arccosh}\sqrt{2} \cdot \sqrt{p^2+q^2}}.$$

Then h satisfies the following:

1. *h is bounded and extends continuously to any $t \in \mathbb{R} \cup \{-\infty, \infty\}$;*
2. *the range of h is $1 \leq h(t) < 1.08209$.*

By these numerical experiments, we can obtain the optimal quasi-isometry constant.

Claim 3.2. *There is a constant $C \geq 0$ depending on the cusp neighborhood A such that*

$$\widetilde{d}_E(z_1, z_2) - C \leq d_H(z_1, z_2) < 1.08209 \cdot \widetilde{d}_E(z_1, z_2) + C$$

for any $z_1, z_2 \in \Omega_0 = \Omega - \pi^{-1}(A)$, where $\widetilde{d}_E = (2\,\mathrm{arccosh}\sqrt{2}) d_E$.

4. Absolute norm and rough-isometry

We introduce a new real norm to $\mathbb{C} = \mathbb{R}^2$ by using the above function h, which is equivalent to the Euclidean norm $\|\cdot\|_2$. We use a general result concerning absolute norm.

For a positive continuous function $\varphi : [0, \pi/2] \to (0, \infty)$ with $\varphi(0) = \varphi(\pi/2) = 1$, we define

$$\|(x, y)\|_\varphi := \|(x, y)\|_2 \cdot \varphi(\arctan(y/x))$$

for every non-trivial $(x, y) \in \mathbb{R}^2$ with $x, y \geq 0$, and then extend it to $\mathbb{R}^2$ by $\|(x, y)\|_\varphi = \|(|x|, |y|)\|_\varphi$ and $\|(0, 0)\|_\varphi = 0$. The following fact is known by Bonsall and Duncan [1, Section 21, Lemma 3].

Proposition 4.1. *Under the above notation,* $\|(x,y)\|_\varphi$ *gives a real norm on* $\mathbb{R}^2$ *if and only if* $\psi(t) = \|(1-t,t)\|_\varphi$ $(0 \le t \le 1)$ *is a convex continuous function such that* $\psi(0) = \psi(1) = 1$ *and* $\max\{1-t,t\} \le \psi(t) \le 1$.

Now we set $\varphi(\theta) = h(\tan\theta)$ by using our function h. A numerical experiment by Mathematica gives the following graph (Figure 3) of $\psi(t) = \|(1-t,t)\|_\varphi$, which satisfies the condition in the above proposition. Then $\|\cdot\|_{h\circ\tan}$ is a norm on $\mathbb{R}^2$.

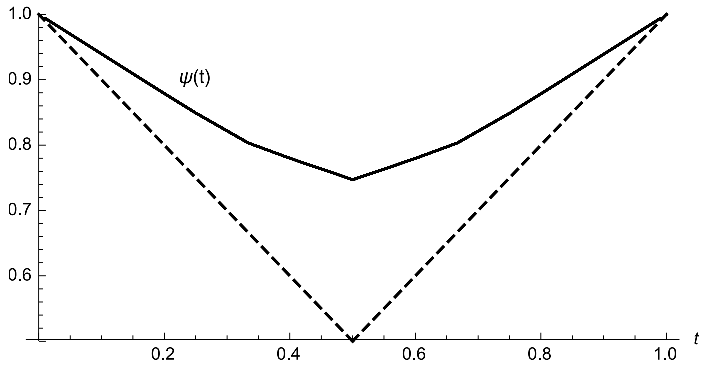

FIGURE 3. Graph of $\psi(t) = \|(1-t,t)\|_\varphi$ for $\varphi(\theta) = h(\tan\theta)$

If the above observation is true, we will have another claim from the ones in the previous section.

Claim 4.2. *The hyperbolic distance* d_H *on* Ω_0 *is rough-isometric to the distance defined by the norm* $(2\,\mathrm{arccosh}\sqrt{2})\|\cdot\|_{h\circ\tan}$*, where rough-isometry means* K*-quasi-isometry for* $K=1$.

5. Generalization: another example

We can consider the similar problem starting from a torus in general

$$T = \mathbb{C}/\langle z \mapsto z+1, z \mapsto z+\tau\rangle$$

for $\tau \in \mathbb{H}$. A difficulty in this case is to describe explicitly the correspondence between τ and the hyperbolic structure on T^*. Here, we only deal with another special case: $\tau = (-1+\sqrt{3}i)/2$.

In this case, our function h becomes

$$h(t) = \frac{\mathrm{Length}(p/q)}{2\,\mathrm{arccosh}(3/2)\cdot\sqrt{p^2+q^2-pq}} \qquad (t = \sqrt{3}p/(2q-p))$$

and its range is $1 \le h(t) < 1.06453$ and the supremum is taken at $t \approx 0.42949$, 0.74692, 8.44047 and their symmetric points (Figure 4).

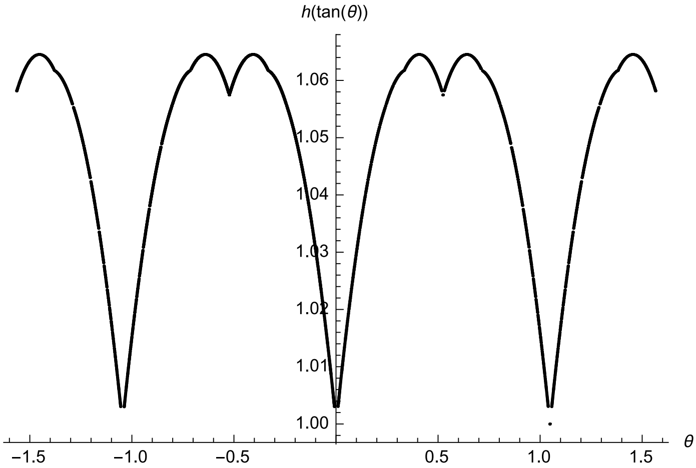

FIGURE 4. Graph of $h(\tan(\theta))$ on $[-\pi/2, \pi/2]$ for equilateral torus

References

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