

# STABLE QUASICONFORMAL MAPPING CLASS GROUPS AND ASYMPTOTIC TEICHMÜLLER SPACES

EGE FUJIKAWA AND KATSUHIKO MATSUZAKI

ABSTRACT. The stable quasiconformal mapping class group is a group of quasiconformal mapping classes of a Riemann surface that are homotopic to the identity outside some topologically finite subsurface. Its analytic counterpart is a group of mapping classes that act on the asymptotic Teichmüller space trivially. We prove that the stable quasiconformal mapping class group is coincident with the asymptotically trivial mapping class group for every Riemann surface satisfying a certain geometric condition. Consequently, the intermediate Teichmüller space, which is the quotient space of the Teichmüller space by the asymptotically trivial mapping class group, has a complex manifold structure, and its automorphism group is geometrically isomorphic to the asymptotic Teichmüller modular group. The proof utilizes a condition for an asymptotic Teichmüller modular transformation to be of finite order, and this is given by the consideration of hyperbolic geometry of topologically infinite surfaces and its deformation under quasiconformal homeomorphisms. Also these arguments enable us to show that every asymptotic Teichmüller modular transformation of finite order has a fixed point on the asymptotic Teichmüller space, which can be regarded as an asymptotic version of the Nielsen theorem.

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## 1. INTRODUCTION

The stable mapping class group is regarded as a mapping class group of infinite genus and its algebraic and topological structures have been investigated by many authors (see e.g. [31], [33]). This is defined as the limit of an inductive system  $\{G_{g,1}, \theta_{hg}\}$ , where  $G_{g,1}$  is the mapping class group of a compact surface  $\Sigma_{g,1}$  of genus  $g \geq 1$  with one boundary component and  $\theta_{hg} : G_{g,1} \rightarrow G_{h,1}$  is an injective homomorphism induced by an inclusion map  $\iota_{hg} : \Sigma_{g,1} \rightarrow \Sigma_{h,1}$  for any  $g \leq h$ . We denote this inductive limit  $\varinjlim G_{g,1}$  by  $G_\infty$ .

The stable mapping class group  $G_\infty$  can be realized as a group of the mapping classes of a surface  $\Sigma_\infty$  of infinite genus. In this realization, each mapping class has a representative that is the identity outside some compact subsurface of  $\Sigma_\infty$ . On the other hand, in complex analytic theories of Teichmüller spaces, it is natural to deal with Riemann surfaces of topologically infinite type including those of infinite genus, and accordingly, the concept of the quasiconformal mapping class group  $\text{MCG}(R)$  comes up for an arbitrary Riemann surface  $R$ , which is a group of the mapping classes that has a representative of a quasiconformal automorphism of  $R$ . Then we can define the *stable quasiconformal mapping class group*  $G_\infty(R)$  for  $R$  as a subgroup of  $\text{MCG}(R)$  whose element has a representative that is the identity outside some topologically finite subsurface. We call such a mapping class *essentially trivial*. Since  $\text{MCG}(R)$  acts on the Teichmüller space  $T(R)$  biholomorphically and isometrically, the stable quasiconformal mapping class group  $G_\infty(R)$  acquires the space where it should act naturally.

The moduli space  $M_g$  of the Riemann surfaces of genus  $g$  is the quotient space of the Teichmüller space  $T_g$  by the mapping class group  $G_g$ . For an arbitrary Riemann surfaces  $R$  not necessarily topologically finite, the moduli space  $M(R)$  may be also defined as the quotient space  $T(R)/\text{MCG}(R)$  but this does not have a nice structure as a topological space in general ([26]). Instead, we introduce the *enlarged moduli space*  $\widetilde{M}(R)$ , which is the quotient space  $T(R)/G_\infty(R)$  by the stable quasiconformal mapping class group with the projection  $\widetilde{M}(R) \rightarrow M(R)$  onto the moduli space. We have seen in [14] that the action of  $G_\infty(R)$  on  $T(R)$  is properly discontinuous and free for any topologically infinite Riemann surface  $R$  satisfying a certain boundedness condition on hyperbolic geometry and hence the complex structure is induced to  $\widetilde{M}(R)$  from  $T(R)$ .

The analytic counterpart of the stable quasiconformal mapping class group is defined by using a representation of the quasiconformal mapping class group  $\text{MCG}(R)$  in the automorphism group of the asymptotic Teichmüller space  $AT(R)$ . The asymptotic Teichmüller space  $AT(R)$  is the space of all asymptotic equivalence classes of quasiconformal homeomorphisms of  $R$ , and this

equivalence, which is weaker than Teichmüller equivalence, is defined by using asymptotically conformal homeomorphisms instead of conformal ones ([7], [8]). There is the quotient map  $\pi : T(R) \rightarrow AT(R)$  that is holomorphic with respect to the corresponding complex structures on both spaces. Since each quasiconformal mapping class of  $R$  acts on  $T(R)$  in such a way that the fibers of  $\pi$  are preserved, it induces a biholomorphic automorphism of  $AT(R)$ . Thus we have a representation  $\iota_{AT} : \text{MCG}(R) \rightarrow \text{Aut}(AT(R))$ . This is not injective in almost all cases and there is no reason for its being surjective. We define the asymptotic Teichmüller modular group  $\text{Mod}_{AT}(R)$  as the image of  $\iota_{AT}$ . We also consider the kernel  $\text{Ker } \iota_{AT}$  of the representation  $\iota_{AT}$ , which is defined as the *asymptotically trivial mapping class group*. We call an element of  $\text{Ker } \iota_A$  asymptotically trivial.

We introduce the *intermediate Teichmüller space* between  $T(R)$  and  $M(R)$ , which is the quotient space  $IT(R) = T(R)/\text{Ker } \iota_{AT}$ . Then  $IT(R)$  is the smallest quotient space of  $T(R)$  by a subgroup of  $\text{MCG}(R)$  such that the projection  $\pi : T(R) \rightarrow AT(R)$  is factored by the quotient map. In particular, the intermediate Teichmüller space  $IT(R)$  also lies between  $T(R)$  and  $AT(R)$ . If  $R$  is analytically finite, then  $IT(R)$  is coincident with the moduli space  $M(R)$ . If  $R$  is the unit disk  $\mathbb{D}$ , then  $IT(\mathbb{D})$  is coincident with the universal Teichmüller space  $T(\mathbb{D})$ .

In the complex analytic theory of Teichmüller space, it had been a central problem to determine the group  $\text{Aut}(T(R))$  of all biholomorphic automorphisms of  $T(R)$ . Recently, this problem has been completely solved ([20]), and now we know that, except for few cases of lower dimensions,  $\text{Aut}(T(R))$  is coincident with the Teichmüller modular group  $\text{Mod}(R)$ , which is the group of all automorphisms that are induced by quasiconformal automorphisms of  $R$ . This means that the representation of the quasiconformal mapping class group  $\iota_T : \text{MCG}(R) \rightarrow \text{Aut}(T(R))$  is bijective. However, as is mentioned above, this is not true for the representation in  $\text{Aut}(AT(R))$ . The corresponding problem for this case will be characterizing  $\text{Mod}_{AT}(R)$  and  $\text{Ker } \iota_{AT}$  instead of the whole  $\text{Aut}(AT(R))$ .

In this paper, we give an answer to this problem under an assumption that  $R$  satisfies a certain boundedness condition on hyperbolic geometry. This condition is geometrically natural. For example, every non-universal normal cover of a compact Riemann surface satisfies this boundedness condition. We will prove that every asymptotically trivial mapping class belongs to the stable quasiconformal mapping class group, that is,  $\text{Ker } \iota_{AT} \subset G_\infty(R)$ . In fact, since the converse inclusion is obvious, we have  $G_\infty(R) = \text{Ker } \iota_{AT}$  (Theorem 2.5). This means that asymptotic triviality, an analytic property of modular transformations, can be characterized by a topological property of the corresponding mapping classes. Consequently, the intermediate Teichmüller space

$IT(R)$  is coincident with the enlarged moduli space  $\widetilde{M}(R)$  and it is endowed with the complex structure. Hence, we obtain that the group  $\text{Aut}(IT(R))$  of all biholomorphic automorphisms of  $IT(R)$  is geometrically isomorphic to  $\text{Mod}_{AT}(R)$ .

One of the ingredients of the proof of the theorem  $G_\infty(R) = \text{Ker } \iota_{AT}$  is to give a sufficient condition for an elliptic element of  $\text{Mod}_{AT}(R)$  to be of finite order. We say that an element of  $\text{Mod}_{AT}(R)$  is *elliptic* if it has a fixed point on  $AT(R)$ . This definition is made after the ellipticity of an element of  $\text{Mod}(R)$ , meaning that it has a fixed point on  $T(R)$ . Every elliptic element of  $\text{Mod}_{AT}(R)$  is realized as an asymptotically conformal automorphism of the Riemann surface corresponding to its fixed point. We prove that, under the boundedness assumption on  $R$ , if an elliptic element of  $\text{Mod}_{AT}(R)$  is induced by an asymptotically conformal automorphism that fixes the free homotopy classes of infinitely many simple closed geodesics satisfying certain properties, then it is of finite order (Theorem 2.9).

On the other hand, every element of  $\text{Mod}(R)$  of finite order is elliptic even for an analytically infinite Riemann surface  $R$ . When  $R$  is analytically finite, this follows from the classical result of Nielsen. As a consequence of our theorem  $G_\infty(R) = \text{Ker } \iota_{AT}$ , we prove the corresponding result for the asymptotic Teichmüller space, which asserts that every element of  $\text{Mod}_{AT}(R)$  of finite order is elliptic under the boundedness assumption on  $R$  (Theorem 2.8). Then we will come to know a necessary and sufficient condition for an element of  $\text{Mod}_{AT}(R)$  to be of finite order. This is a consequence from our main results, which will be shown as the final theorem of this paper.

The organization of this paper is as follows. In Section 2, we review definitions concerning Teichmüller spaces and then precisely state our results (Theorems 2.5, 2.8 and 2.9) mentioned in the previous three paragraphs. In Section 3, we demonstrate a major application of our main theorem. We introduce the enlarged moduli space with the complex structure, and then determine the biholomorphic automorphism group of the intermediate Teichmüller space.

We devote the succeeding two sections to presenting our tools for the arguments on hyperbolic geometry. In Section 4, we define a frame of geodesics on a hyperbolic surface, observe the change of their hyperbolic lengths under asymptotically conformal deformation, and choose appropriate frames moved by the action of a non-trivial mapping class. In Section 5, we construct hyperbolic right-angled hexagons from those frames and give an estimate of the variation of the moduli of such hexagons under their non-trivial movement. After these preparations, in Section 6, we prove Theorem 6.1 ( $\doteq$  Theorem 2.9) concerning a sufficient condition under which some power of a quasiconformal mapping class is essentially trivial.

We will give a proof of Theorem 2.5 as an application of Theorem 2.9. The next two sections make crucial steps towards it. In Section 7, we prove that, if a mapping class moves a sequence of mutually disjoint simple closed geodesics of uniformly bounded lengths in a certain manner, then it is not asymptotically trivial. In Section 8, we give a midway result which states that, if some power of an asymptotically trivial mapping class is essentially trivial, then so is itself. Summing up all these results, we complete the proof of Theorem 9.1 ( $\equiv$  Theorem 2.5) in Section 9 with the aid of Theorem 6.1. Also, as an application, we prove the Nielsen theorem on the asymptotic Teichmüller space (Theorem 2.8) as well as the characterization of asymptotic Teichmüller modular transformations of finite order in Theorem 9.3.

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## 2. PRELIMINARIES AND STATEMENT OF RESULTS

Throughout this paper, we assume that a Riemann surface  $R$  admits a hyperbolic structure. The *Teichmüller space*  $T(R)$  of  $R$  is the set of all equivalence classes  $[f]$  of quasiconformal homeomorphisms  $f$  of  $R$ . Here we say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  of  $R$  are *equivalent* if there exists a conformal homeomorphism  $h : f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1$  is homotopic to the identity on  $R$ . Here the homotopy is considered to be relative to the ideal boundary at infinity. A distance between two points  $[f_1]$  and  $[f_2]$  in  $T(R)$  is defined by  $d_T([f_1], [f_2]) = (1/2) \log K(f)$ , where  $f$  is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation  $K(f)$  is minimal in the homotopy class of  $f_2 \circ f_1^{-1}$ . Then  $d_T$  is a complete distance on  $T(R)$  which is called the Teichmüller distance. The Teichmüller space  $T(R)$  can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on  $R'$ , where  $R'$  is the complex conjugate of  $R$ . In this way,  $T(R)$  is endowed with the complex structure. It is known that the Teichmüller distance is coincident with the Kobayashi distance on  $T(R)$ . For details, see [16], [19] and [27].

A quasiconformal mapping class is the homotopy equivalence class  $[g]$  of quasiconformal automorphisms  $g$  of a Riemann surface  $R$ , and the *quasiconformal mapping class group*  $\text{MCG}(R)$  of  $R$  is the group of all quasiconformal mapping classes of  $R$ . Here the homotopy is again considered to be relative to the ideal boundary at infinity. Every element  $[g] \in \text{MCG}(R)$  induces a biholomorphic automorphism  $[g]_*$  of  $T(R)$  by  $[f] \mapsto [f \circ g^{-1}]$ , which is also isometric with respect to the Teichmüller distance. Let  $\text{Aut}(T(R))$  denote the group of

all biholomorphic automorphisms of  $T(R)$ . Then we have a homomorphism

$$\iota_T : \text{MCG}(R) \rightarrow \text{Aut}(T(R))$$

given by  $[g] \mapsto [g]_*$ , and we define the *Teichmüller modular group* for  $R$  by

$$\text{Mod}(R) = \iota_T(\text{MCG}(R)).$$

We call an element of  $\text{Mod}(R)$  a Teichmüller modular transformation. It is proved in [6] that the homomorphism  $\iota_T$  is injective (faithful) for all Riemann surfaces  $R$  of non-exceptional type. See also [10] and [22] for other proofs. Here we say that a Riemann surface  $R$  is of *exceptional type* if  $R$  has finite hyperbolic area and satisfies  $2g + n \leq 4$ , where  $g$  is the genus of  $R$  and  $n$  is the number of punctures of  $R$ . It was a problem to determine the homomorphism  $\iota_T$  is also surjective, especially for an analytically infinite Riemann surface. By a combination of the results of [5] and [20], this problem has been solved affirmatively, namely,  $\text{Mod}(R) = \text{Aut}(T(R))$ . See [13] for a survey of the proof.

The asymptotic Teichmüller space has been introduced in [18] for the hyperbolic plane and in [7] and [8] for an arbitrary Riemann surface. We say that a quasiconformal homeomorphism  $f$  of  $R$  is *asymptotically conformal* if, for every  $\epsilon > 0$ , there exists a compact subset  $V$  of  $R$  such that the maximal dilatation  $K(f|_{R-V})$  of the restriction of  $f$  to  $R - V$  is less than  $1 + \epsilon$ . We say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  of  $R$  are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism  $h : f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1$  is homotopic to the identity on  $R$  relative to the ideal boundary at infinity. The *asymptotic Teichmüller space*  $AT(R)$  of  $R$  is the set of all asymptotic equivalence classes  $[[f]]$  of quasiconformal homeomorphisms  $f$  of  $R$ . The asymptotic Teichmüller space  $AT(R)$  is of interest only when  $R$  is analytically infinite. Otherwise  $AT(R)$  is trivial, that is, it consists of just one point. Conversely, if  $R$  is analytically infinite, then  $AT(R)$  is not trivial. Since a conformal homeomorphism is asymptotically conformal, there is a projection  $\pi : T(R) \rightarrow AT(R)$  that maps each Teichmüller equivalence class  $[f] \in T(R)$  to the asymptotic Teichmüller equivalence class  $[[f]] \in AT(R)$ . The asymptotic Teichmüller space  $AT(R)$  has a complex structure such that  $\pi$  is holomorphic. See also [9] and [17].

For a quasiconformal homeomorphism  $f$  of  $R$ , the boundary dilatation of  $f$  is defined by  $H^*(f) = \inf K(f|_{R-V})$ , where the infimum is taken over all compact subsets  $V$  of  $R$ . Furthermore, for a Teichmüller equivalence class  $[f] \in T(R)$ , the boundary dilatation of  $[f]$  is defined by  $H([f]) = \inf H^*(f')$ , where the infimum is taken over all elements  $f' \in [f]$ . A distance between two points  $[[f_1]]$  and  $[[f_2]]$  in  $AT(R)$  is defined by  $d_{AT}([f_1], [f_2]) = (1/2) \log H([f_2 \circ f_1^{-1}])$ , where  $[f_2 \circ f_1^{-1}]$  is the Teichmüller equivalence class of  $f_2 \circ f_1^{-1}$  in  $T(f_1(R))$ . Then  $d_{AT}$  is a complete distance on  $AT(R)$ , which is called the asymptotic

Teichmüller distance. For every point  $[[f]] \in AT(R)$ , there exists an asymptotically extremal element  $f_0 \in [[f]]$  satisfying  $H([f]) = H^*(f_0)$ . It is different from the case of Teichmüller space that we do not know yet whether the asymptotic Teichmüller distance is coincident with the Kobayashi distance on  $AT(R)$  or not.

Every element  $[g] \in \text{MCG}(R)$  induces a biholomorphic automorphism  $[g]_{**}$  of  $AT(R)$  by  $[[f]] \mapsto [[f \circ g^{-1}]]$ , which is also isometric with respect to  $d_{AT}$ . See [8]. Let  $\text{Aut}(AT(R))$  be the group of all biholomorphic automorphisms of  $AT(R)$ . Then we have a homomorphism

$$\iota_{AT} : \text{MCG}(R) \rightarrow \text{Aut}(AT(R))$$

given by  $[g] \mapsto [g]_{**}$ , and we define the *asymptotic Teichmüller modular group* for  $R$  (the geometric automorphism group of  $AT(R)$ ) by

$$\text{Mod}_{AT}(R) = \iota_{AT}(\text{MCG}(R)).$$

We call an element of  $\text{Mod}_{AT}(R)$  an asymptotic Teichmüller modular transformation. It is different from the case of the representation  $\iota_T : \text{MCG}(R) \rightarrow \text{Aut}(T(R))$  that the homomorphism  $\iota_{AT}$  is not injective, namely,  $\text{Ker } \iota_{AT} \neq \{\text{id}\}$  unless  $R$  is either the unit disc or the once-punctured disc ([6]). We call an element of  $\text{Ker } \iota_{AT}$  asymptotically trivial and call  $\text{Ker } \iota_{AT}$  the *asymptotically trivial mapping class group*.

In this paper, we completely characterize  $\text{Ker } \iota_{AT}$  topologically. To state our theorem, we define the following subgroup of the quasiconformal mapping class group  $\text{MCG}(R)$ .

**Definition 2.1.** The *stable quasiconformal mapping class group*  $G_\infty(R)$  is a subgroup of  $\text{MCG}(R)$  consisting of all essentially trivial mapping classes  $[g]$  of a Riemann surface  $R$ . Here a quasiconformal mapping class  $[g] \in \text{MCG}(R)$  is said to be *essentially trivial* (or trivial near the infinity) if there exists a topologically finite subsurface  $V_g$  of  $R$  such that, for each connected component  $W$  of  $R - V_g$ , the restriction  $g|_W : W \rightarrow R$  is homotopic to the inclusion map  $\text{id}|_W : W \hookrightarrow R$  relative to the ideal boundary at infinity.

Note that  $G_\infty(R)$  is a countable group whereas  $\text{MCG}(R)$  is not a countable group in most cases when  $R$  is topologically infinite. Also  $G_\infty(R)$  is a normal subgroup of  $\text{MCG}(R)$ .

For a topologically infinite Riemann surface  $R$ , a regular exhaustion  $\{R_n\}_{n=1}^\infty$  of  $R$  is an increasing sequence of compact subsurfaces  $R_n$  satisfying that  $R = \bigcup_{n=1}^\infty R_n$  and each connected component of the complement of  $R_n$  is not relatively compact. See [1, Chapter II, 12D] for the existence of regular exhaustion.

Let  $\{W_n^{(i)}\}_{i=1}^{N(n)}$  be the set of all connected components of the complement of  $R_n$ . A determining sequence for a regular exhaustion  $\{R_n\}_{n=1}^\infty$  is a sequence

$\{W_n^{(i_n)}\}_{n=1}^\infty$  such that  $W_n^{(i_n)} \supset W_{n+1}^{(i_{n+1})}$  for all  $n$ . Another regular exhaustion  $\{R'_n\}_{n=1}^\infty$  can give another determining sequence  $\{W_n^{(j_n)}\}_{n=1}^\infty$ . We say that two determining sequences  $\{W_n^{(i_n)}\}_{n=1}^\infty$  and  $\{W_n^{(j_n)}\}_{n=1}^\infty$  are equivalent if, for every  $n$ , there exists an  $m$  such that  $W_n^{(i_n)} \supset W_m^{(j_m)}$ , and vice versa.

A *topological end* of a Riemann surface  $R$  is an equivalence class of determining sequences, and the end compactification  $R^*$  of  $R$  is the union of  $R$  and the set of all topological ends endowed with canonical topology. For details, see [29, Chapter IV, 5D]. The cardinality of the topological ends is a topological invariant of a Riemann surface. In particular, there are infinitely many Riemann surfaces that are not mutually homeomorphic. Every (homeomorphic) automorphism of  $R$  extends to an automorphism of  $R^*$  by the correspondence of determining sequences. The extension of an automorphism to the boundary  $R^* - R$  is determined by the mapping class of the automorphism. Hence the mapping class group of  $R$  acts on  $R^* - R$ .

The *pure mapping class group*  $P(R)$  is a subgroup of  $\text{MCG}(R)$  consisting of all quasiconformal mapping classes  $[g]$  such that  $g$  fixes all non-cuspidal topological ends of  $R$ , where we say that a topological end is *non-cuspidal* if it does not correspond to a puncture. It is clear that  $G_\infty(R)$  is contained in  $P(R)$ . The following result says that  $\text{Ker } \iota_{AT}$  sits between these topologically characterized subgroups.

**Proposition 2.2** ([14]). *The inclusion relation  $G_\infty(R) \subset \text{Ker } \iota_{AT} \subset P(R)$  holds for an arbitrary Riemann surface  $R$ .*

Each inclusion in Proposition 2.2 is not necessarily equality. See [14, Remark 4.1] for the difference. However, under a certain condition on hyperbolic geometry of Riemann surfaces, we will give a complete characterization of  $\text{Ker } \iota_{AT}$ .

For a Riemann surface  $R$ , let  $\dot{R}$  be the non-cuspidal part of  $R$  obtained by removing all horocyclic cusp neighborhoods whose hyperbolic areas are 1.

**Definition 2.3.** We say that a Riemann surface  $R$  satisfies the *bounded geometry condition* if  $R$  satisfies the following two conditions:

- (i) *lower bound condition*: there exists a constant  $m > 0$  such that, for every point  $x \in \dot{R}$ , every homotopically non-trivial closed curve that starts from  $x$  and terminates at  $x$  has hyperbolic length greater than or equal to  $m$ ;
- (ii) *upper bound condition*: there exists a constant  $M > 0$  such that, for every point  $x \in R$ , there exists a homotopically non-trivial simple closed curve that starts from  $x$  and terminates at  $x$  and whose hyperbolic length is less than or equal to  $M$ .



If  $R$  satisfies the lower bound condition for a constant  $m$  and the upper bound condition for a constant  $M$ , we say that  $R$  satisfies  $(m, M)$ -bounded geometry condition.

**Remark 2.4.** The upper bound condition defined above is stronger than the one we have used in our previous papers, say, [11], [12], [14] and [15]. Note that, if  $R$  satisfies the present upper bound condition, then  $R$  has no ideal boundary at infinity. We believe that the statements of this paper are valid even for the previous definition, but for the sake of arguments, we use the stronger one.

Every normal cover of a compact Riemann surface that is not the universal cover satisfies the bounded geometry condition. Moreover, if a Riemann surface  $R$  admits such pants decomposition that the diameter of each pair of pants is uniformly bounded, then  $R$  satisfies the bounded geometry condition.

The bounded geometry condition is preserved under quasiconformal homeomorphisms. Thus, this can be regarded as a condition for the Teichmüller space. More precisely, if  $R$  satisfies  $(m, M)$ -bounded geometry condition, then  $R' = f(R)$  for a  $K$ -quasiconformal homeomorphism  $f$  of  $R$  satisfies  $(m', M')$ -bounded geometry condition for other constants  $m'$  and  $M'$ . Indeed, in the homotopy class of  $f$ , we take an extremal quasiconformal homeomorphism  $f_*$  and the quasiconformal diffeomorphism  $f_\#$  induced by the barycentric extension given in [4]. The maximal dilatation  $K(f_*)$  of  $f_*$  is obviously not greater than  $K$  and the biLipschitz constant  $L(f_\#)$  of  $f_\#$  with respect to the hyperbolic metric can be estimated from above in terms of  $K$ . Then  $R'$  satisfies the lower bound condition for  $m' = m/K(f_*)$  and the upper bound condition for  $M' = L(f_\#)M$ . See [15, Lemma 8].

Now we are ready to state our main theorem.

**Theorem 2.5.** *Let  $R$  be a Riemann surface satisfying the bounded geometry condition. Then*

$$G_\infty(R) = \text{Ker } \iota_{AT}.$$

As is mentioned above, Theorem 2.5 is not true without the bounded geometry condition.

**Remark 2.6.** If  $R$  satisfies the bounded geometry condition, then  $\text{Ker } \iota_{AT}$  is a proper subgroup of  $\text{MCG}(R)$ . Namely, the action of  $\text{MCG}(R)$  on  $AT(R)$  is not trivial. See [12, Corollary 3.5]. However, there exists a Riemann surface  $R$  that does not satisfy the bounded geometry condition and that  $G_\infty(R) = \text{Ker } \iota_{AT} = \text{MCG}(R)$ . See [24].

Next, we consider periodicity and a fixed point property of (asymptotic) Teichmüller modular transformations.

**Definition 2.7.** We say that a Teichmüller modular transformation in  $\text{Mod}(R)$  is *elliptic* if it has a fixed point on  $T(R)$ , and an asymptotic Teichmüller modular transformation in  $\text{Mod}_{AT}(R)$  is *elliptic* if it has a fixed point on  $AT(R)$ .

Every elliptic element of  $\text{Mod}(R)$  is realized as a conformal automorphism of the Riemann surface corresponding to its fixed point, and every elliptic element of  $\text{Mod}_{AT}(R)$  is realized as an asymptotically conformal automorphism of the Riemann surface corresponding to its fixed point. We say that  $[g] \in \text{MCG}(R)$  is a *conformal mapping class* if  $[g]_*$  is elliptic and an *asymptotically conformal mapping class* if  $[g]_{**}$  is elliptic. There is a Teichmüller modular transformation  $[g]_*$  such that  $[g]_*$  is not elliptic but  $[g]_{**}$  is elliptic and non-trivial ([28]). Also there is an elliptic element  $[g]_*$  and a point  $p \in T(R)$  such that  $p$  is not a fixed point of  $[g]_*$  but  $\pi(p) \in AT(R)$  is a fixed point of  $[g]_{**}$  ([25]).

It is known that, for an analytically finite Riemann surface  $R$ , an element of  $\text{Mod}(R)$  is elliptic if and only if it is of finite order (periodic). This is a consequence of the theorem due to Nielsen. Even for an analytically infinite Riemann surface  $R$ , every element of finite order of  $\text{Mod}(R)$  is elliptic. In fact, it is proved in [21] that, if the orbit of a subgroup of  $\text{Mod}(R)$  is bounded, then it has a common fixed point on  $T(R)$ . We prove the corresponding result for the asymptotic Teichmüller space  $AT(R)$ .

**Theorem 2.8.** *Let  $R$  be a Riemann surface satisfying the bounded geometry condition. If  $[g]_{**} \in \text{Mod}_{AT}(R)$  is of finite order, then it is elliptic.*

On the other hand, elliptic elements are not necessarily of finite order. However, if  $[g]_* \in \text{Mod}(R)$  is elliptic and if  $g(c)$  is freely homotopic to  $c$  for some simple closed geodesic  $c$  on  $R$ , then  $[g]_*$  is of finite order. This follows from the fact that the group of conformal automorphisms of a Riemann surface  $R$  acts on  $R$  properly discontinuously. We also prove the corresponding result for an elliptic element of  $\text{Mod}_{AT}(R)$ .

**Theorem 2.9.** *Let  $R$  be a Riemann surface satisfying the bounded geometry condition, and  $[g]_{**} \in \text{Mod}_{AT}(R)$  an elliptic element. Suppose that, for some constant  $\ell > 0$  and in any topologically infinite neighborhood of each topological end of  $R$ , there exists a simple closed geodesic  $c$  with  $\ell(c) \leq \ell$  such that  $g(c)$  is freely homotopic to  $c$ . Then  $[g]_{**}$  is of finite order.*

Finally, as a consequence of all Theorems 2.5, 2.8 and 2.9 above, we have our final result Theorem 9.3, which gives a necessary and sufficient condition for an asymptotic Teichmüller modular transformation to be of finite order.

We will prove Theorem 2.9 in Section 6 and Theorems 2.5 and 2.8 in Section 9. Actually we take a way to Theorem 2.5 via Theorem 2.9 in order to present various methods of investigating these objects. This route makes us aware of Theorem 2.8 and brings us to the final destination Theorem 9.3.

In the next section, we first give an application of Theorem 2.5 to the arguments on the enlarged moduli space and the asymptotic Teichmüller modular group, which have been focused on in the introduction.

### 3. THE INTERMEDIATE TEICHMÜLLER SPACE

For a Riemann surface  $R$  not necessarily topologically finite, the moduli space  $M(R)$  may be defined as the quotient space of the Teichmüller space  $T(R)$  by the quasiconformal mapping class group  $\text{MCG}(R)$  and we have the projection  $\rho : T(R) \rightarrow M(R)$ . Here we regard  $\text{MCG}(R)$  as acting on  $T(R)$  through the representation in  $\text{Mod}(R)$ . However, the moduli space does not have a nice structure as a topological space in general. In this section, we define another moduli space and another Teichmüller space which are quotient spaces by certain subgroups of  $\text{MCG}(R)$ . By using our main theorem, we see that these spaces coincide and have a complex structure under the assumption that  $R$  satisfies the bounded geometry condition.

**Definition 3.1.** The *enlarged moduli space* of a Riemann surface  $R$  is defined by

$$\widetilde{M}(R) = T(R)/G_\infty(R),$$

which is the quotient space of the Teichmüller space  $T(R)$  by the stable quasiconformal mapping class group  $G_\infty(R)$ .

The enlarged moduli space  $\widetilde{M}(R)$  factorizes the projection  $\rho : T(R) \rightarrow M(R)$  into two projections  $\tau : T(R) \rightarrow \widetilde{M}(R)$  and  $\rho_1 : \widetilde{M}(R) \rightarrow M(R)$  so that  $\rho = \rho_1 \circ \tau$ .

We have another equivalent definition of the enlarged moduli space by using a certain equivalence relation in the space of quasiconformal homeomorphisms of  $R$ .

**Definition 3.2.** We say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  of a Riemann surface  $R$  are *weakly equivalent* if there exist a conformal homeomorphism  $h : f_1(R) \rightarrow f_2(R)$  and a topologically finite subsurface  $V_h$  in  $R$  such that, for each connected component  $W$  of  $R - V_h$ , the quasiconformal homeomorphism  $f_2^{-1} \circ h \circ f_1|_W$  restricted to  $W$  is homotopic to the inclusion map  $\text{id}|_W : W \hookrightarrow R$  relative to the ideal boundary at infinity.

It is obvious that, if two quasiconformal homeomorphisms are equivalent, then they are weakly equivalent.

**Proposition 3.3.** The enlarged moduli space  $\widetilde{M}(R)$  of a Riemann surface  $R$  is coincident with the set of all weak equivalence classes ( $f$ ) of quasiconformal homeomorphisms  $f$  of  $R$ .

*Proof.* For the projection  $\tau : T(R) \rightarrow \widetilde{M}(R)$  and for any two points  $[f_1]$  and  $[f_2]$  in  $T(R)$ ,  $\tau([f_1]) = \tau([f_2])$  if and only if there exists an element  $[g] \in G_\infty(R)$  such that  $[g]_*([f_1]) = [f_2]$ . This is equivalent to saying that there exists a conformal homeomorphism  $h : f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1 \circ g^{-1}$  is homotopic to the identity on  $R$  relative to the ideal boundary at infinity. In other words,  $f_2^{-1} \circ h \circ f_1$  is homotopic to  $g$  in this sense. Since  $[g]$  is essentially trivial, there exists a topologically finite subsurface  $V$  of  $R$  such that, for each connected component  $W$  of  $R - V$ , the restriction  $g|_W : W \rightarrow R$  is homotopic to the inclusion map  $\text{id}|_W : W \hookrightarrow R$ . Thus  $f_2^{-1} \circ h \circ f_1|_W$  is homotopic to  $\text{id}|_W$  relative to the ideal boundary at infinity. This means that  $f_1$  and  $f_2$  are weakly equivalent. The converse is also proved along the same lines.  $\square$

The Teichmüller space  $T(R)$  can be regarded as the space of all marked Riemann surfaces that are obtained by quasiconformal deformation of  $R$ . In this context, the moduli space  $M(R)$  is the space of all Riemann surfaces quasiconformally equivalent to  $R$  and the quotient map  $\rho : T(R) \rightarrow M(R)$  is given by forgetting the marking. By Proposition 3.3, the projection  $\tau : T(R) \rightarrow \widetilde{M}(R)$  corresponds to forgetting the marking only on topologically finite subsurfaces of Riemann surfaces.

The enlarged moduli space  $\widetilde{M}(R)$  is endowed with a pseudo-distance induced from the Teichmüller distance  $d_T$ , which is defined by

$$d_{\widetilde{M}(R)}(\tau(p), \tau(q)) = \inf\{d_T(p', q') \mid \tau(p') = \tau(p), \tau(q') = \tau(q)\}$$

for  $p, q \in T(R)$ . By Proposition 3.3, the pseudo-distance between two points  $\tau(p) = (f_1)$  and  $\tau(q) = (f_2)$  in  $\widetilde{M}(R)$  is also represented by

$$d_{\widetilde{M}(R)}((f_1), (f_2)) = \frac{1}{2} \inf \log K(f),$$

where the infimum is taken over all quasiconformal homeomorphisms  $f$  that are homotopic to  $f_2 \circ f_1^{-1}$  outside of some topologically finite subsurface of  $f_1(R)$  relative to the ideal boundary at infinity.

We consider a complex structure of the enlarged moduli space. However, for this purpose, we have to assume that Riemann surfaces satisfy the bounded geometry condition.

**Theorem 3.4.** *Let  $R$  be a topologically infinite Riemann surface satisfying the bounded geometry condition. Then the enlarged moduli space  $\widetilde{M}(R)$  has a complex structure and the pseudo-distance  $d_{\widetilde{M}(R)}$  is coincident with the Kobayashi distance on  $\widetilde{M}(R)$ .*

To see Theorem 3.4, we observe the action of  $G_\infty(R)$  on  $T(R)$ . We say that a subgroup  $G \subset \text{MCG}(R)$  acts at a point  $p \in T(R)$  *discontinuously* if there exists a neighborhood  $U$  of  $p$  such that the number of elements  $[g]_* \in \iota_T(G)$

satisfying  $[g]_*(U) \cap U \neq \emptyset$  is finite. This is equivalent to saying that the orbit  $\iota_T(G)(p)$  is a discrete set and the stabilizer subgroup  $\text{Stab}_{\iota_T(G)}(p)$  is finite. We say that a subgroup  $G \subset \text{MCG}(R)$  acts at  $p \in T(R)$  *freely* if  $\text{Stab}_{\iota_T(G)}(p)$  consists only of the identity.

The following result is crucial in our arguments. Theorem 3.4 immediately follows from this.

**Proposition 3.5** ([14]). *Let  $R$  be a topologically infinite Riemann surface satisfying the bounded geometry condition. Then the stable quasiconformal mapping class group  $G_\infty(R)$  acts on  $T(R)$  discontinuously and freely.*

The analytic counterpart of the stable quasiconformal mapping class group is given by using the representation of the quasiconformal mapping class group  $\text{MCG}(R)$  in the automorphisms group of the asymptotic Teichmüller space  $AT(R)$ . We also introduce another Teichmüller space corresponding to the kernel  $\text{Ker } \iota_{AT}$  of the representation  $\iota_{AT} : \text{MCG}(R) \rightarrow \text{Aut}(AT(R))$ .

**Definition 3.6.** The *intermediate Teichmüller space* of a Riemann surface  $R$  is defined by

$$IT(R) = T(R)/\text{Ker } \iota_{AT},$$

which is the quotient space of the Teichmüller space  $T(R)$  by the asymptotically trivial mapping class group  $\text{Ker } \iota_{AT}$ .

The projection  $\sigma : T(R) \rightarrow IT(R)$  factorizes the projection  $\rho : T(R) \rightarrow M(R)$  into  $\rho_2 : IT(R) \rightarrow M(R)$  so that  $\rho = \rho_2 \circ \sigma$ . We also have a projection from  $IT(R)$  onto  $AT(R)$ . Indeed, the subgroup  $\text{Ker } \iota_{AT}$  acts on  $AT(R)$  trivially, and hence  $\sigma$  factorizes the projection  $\pi : T(R) \rightarrow AT(R)$  into  $\sigma' : IT(R) \rightarrow AT(R)$  so that  $\pi = \sigma' \circ \sigma$ . Here is the diagram of those projections:

$$\begin{array}{ccccccc} T(R) & \xrightarrow{\tau} & \widetilde{M}(R) & \longrightarrow & IT(R) & \xrightarrow{\rho_2} & M(R) \\ & & & & \downarrow \sigma' & & \\ & & & & AT(R) & & \end{array}$$

The enlarged moduli space  $\widetilde{M}(R)$  lies between  $T(R)$  and  $IT(R)$ . This is seen from the inclusion  $G_\infty(R) \subset \text{Ker } \iota_{AT}$  in Proposition 2.2. The fact that there is a projection  $\tau'$  from  $\widetilde{M}(R)$  onto  $AT(R)$  is also seen from the fact that weakly equivalent quasiconformal homeomorphisms are asymptotically equivalent.

If  $R$  is analytically finite, then  $\widetilde{M}(R) = IT(R) = M(R)$ , and the asymptotic Teichmüller space  $AT(R)$  is just one point. On the other hand, if  $R$  is the unit disk  $\mathbb{D}$ , then  $T(\mathbb{D}) = \widetilde{M}(\mathbb{D}) = IT(\mathbb{D})$ . Indeed,  $\text{Ker } \iota_{AT}$  is trivial for the unit disk  $\mathbb{D}$ . Thus we have  $IT(\mathbb{D}) = T(\mathbb{D})/\text{Ker } \iota_{AT} = T(\mathbb{D})$ .

The intermediate Teichmüller space is located at the middle of the way down from  $T(R)$ , which is at the branch point of the projections  $\rho$  and  $\pi$ . The following theorem states this situation precisely.

**Theorem 3.7.** *The intermediate Teichmüller space  $IT(R) = T(R)/\text{Ker } \iota_{AT}$  is the smallest quotient space  $T(R)/G$  by a subgroup  $G$  of  $\text{MCG}(R)$  such that the projection  $\pi_G : T(R) \rightarrow T(R)/G$  splits the projection  $\pi : T(R) \rightarrow AT(R)$  into  $\pi'_G : T(R)/G \rightarrow AT(R)$  with  $\pi = \pi'_G \circ \pi_G$ . Here the smallest quotient space means that if there is another quotient space  $T(R)/G$  satisfying this property, then  $G \subset \text{Ker } \iota_{AT}$ .*

*Proof.* Suppose to the contrary that there exists a subgroup  $G$  of  $\text{MCG}(R)$  such that  $\pi_G$  splits  $\pi$  into  $\pi'_G$  with  $\pi = \pi'_G \circ \pi_G$  but  $G$  is not contained in  $\text{Ker } \iota_{AT}$ . We take an element  $[g] \in G$  that does not belong to  $\text{Ker } \iota_{AT}$ . Since  $[g]$  acts on  $AT(R)$  non-trivially, there exists a point  $\hat{p} \in AT(R)$  such that  $[g]_{**}(\hat{p}) \neq \hat{p}$ . We take a lift  $p \in T(R)$  of  $\hat{p}$  against the projection  $\pi : T(R) \rightarrow AT(R)$ . Since  $[g]_{**}(\hat{p}) = [g]_{**}(\pi(p)) = \pi([g]_*(p))$ , we can take  $[g]_*(p) \in T(R)$  as one of the lifts of  $[g]_{**}(\hat{p})$  to  $T(R)$ . Here the two points  $p$  and  $[g]_*(p)$  project to the same point on  $T(R)/G$ . Since there is a projection  $\pi'_G : T(R)/G \rightarrow AT(R)$ , these two points also project to the same point on  $AT(R)$ , namely,  $\hat{p} = [g]_{**}(\hat{p})$ . However this is a contradiction.  $\square$

Now we assume that a Riemann surface  $R$  satisfies the bounded geometry condition and apply Theorem 2.5. Then we have  $G_\infty(R) = \text{Ker } \iota_{AT}$ , and hence the enlarged moduli space  $\widetilde{M}(R)$  is coincident with the intermediate Teichmüller space  $IT(R)$ . In particular,  $IT(R)$  has the complex structure by Theorem 3.4. We consider the group  $\text{Aut}(IT(R))$  of all biholomorphic automorphisms of  $IT(R)$ .

The quasiconformal mapping class group  $\text{MCG}(R)$  acts on the intermediate Teichmüller space  $IT(R) = T(R)/\text{Ker } \iota_{AT}$ . This action is biholomorphic since  $G_\infty(R) = \text{Ker } \iota_{AT}$  acts discontinuously and freely on  $T(R)$  by Proposition 3.5. Hence we have a representation

$$\iota_{IT} : \text{MCG}(R) \rightarrow \text{Aut}(IT(R)).$$

This homomorphism  $\iota_{IT}$  is surjective. Indeed, every element  $\varphi \in \text{Aut}(IT(R))$  can be lifted to a biholomorphic automorphism of  $T(R)$  since  $T(R)$  is simply connected. From the fact that any biholomorphic automorphism of  $T(R)$  is induced by an element of  $\text{MCG}(R)$ , we see that  $\varphi$  is also induced by the element of  $\text{MCG}(R)$ .

From the definition  $IT(R) = T(R)/\text{Ker } \iota_{AT}$ , it follows that  $\text{Ker } \iota_{IT} \supset \text{Ker } \iota_{AT}$ . On the other hand,  $\text{Ker } \iota_{IT} \subset \text{Ker } \iota_{AT}$  since there is the projection  $\sigma' : IT(R) \rightarrow AT(R)$ . Hence we conclude that  $\text{Ker } \iota_{IT} = \text{Ker } \iota_{AT}$ . Therefore we are able to state the following.

**Theorem 3.8.** *For a topologically infinite Riemann surface satisfying the bounded geometry condition, there exists a geometric isomorphism of  $\text{Mod}_{AT}(R)$  onto  $\text{Aut}(IT(R))$ .*

This theorem says that the asymptotic Teichmüller modular group  $\text{Mod}_{AT}(R)$  can be represented as the automorphism group of some Teichmüller space, which corresponds to the fact that the Teichmüller modular group  $\text{Mod}(R)$  is equal to  $\text{Aut}(T(R))$  for the ordinary Teichmüller space.

#### 4. FRAMES OF GEODESICS ON RIEMANN SURFACES

In this section, we give preparatory results on frames of geodesics in a hyperbolic surface, which are fundamental skeletons dominating local geometry. We will utilize them in our proof of Theorem 2.9 later.

First, we give the following remark concerning the upper bound condition on a Riemann surface. See [15, Proposition 1].

**Proposition 4.1.** *Let  $R$  be a Riemann surface satisfying the upper bound condition for a constant  $M$ . Then there exists a constant  $M' (\geq M)$  depending only on  $M$  that satisfies the following: for every point  $x$  in the non-cuspidal part  $\dot{R}$ , there exists a homotopically non-trivial simple closed curve based at  $x$  not surrounding a puncture of  $R$  whose hyperbolic length is less than or equal to  $M'$ .*

Thus, replacing  $M$  with  $M'$ , we may regard that the constant  $M$  for the upper bound condition also satisfies the condition for the constant  $M'$  in Proposition 4.1. Namely, the upper bound condition for the constant  $M$  also implies that, for every point  $x \in \dot{R}$ , there is a non-trivial and non-cuspidal simple closed curve based at  $x$  whose hyperbolic length is less than or equal to  $M$ .

Let  $d$  denote the hyperbolic distance on a Riemann surface  $R$  and  $\ell(c)$  denote the hyperbolic length of a curve  $c$  on  $R$ . For a non-trivial and non-cuspidal simple closed curve  $c$  on  $R$ , let  $c_*$  be the unique simple closed geodesic that is freely homotopic to  $c$ . For a subsurface  $V$  of  $R$  whose relative boundary  $\partial V$  consists of simple closed curves, let  $V_*$  be a subsurface of  $R$  each of whose relative boundary components is the simple closed geodesic that is freely homotopic to the corresponding component of  $\partial V$ . We call such  $V_*$  a *geodesic subsurface*. Remark that if a relative boundary component of  $\partial V$  is trivial or cuspidal, then we assume that the corresponding component of  $\partial V_*$  is degenerated.

**Definition 4.2.** A *frame*  $X$  in a Riemann surface  $R$  is an ordered triple  $(c_1, c_2, \eta)$  satisfying the following: (i)  $c_1$  and  $c_2$  are oriented simple closed geodesics on  $R$  possibly intersecting or coincident; (ii)  $\eta$  is a non-degenerate oriented geodesic arc connecting  $c_1$  with  $c_2$  perpendicularly, possibly having

self-intersection or other intersection with  $c_1, c_2$  different from the end points; (iii)  $\eta$  is initiated from  $c_1$  on the right of its orientation and terminated to  $c_2$  on the right of its orientation. For a frame  $X = (c_1, c_2, \eta)$ , we define its inverse  $X^{-1}$  by  $(c_2, c_1, \eta^{-1})$ . Furthermore, we say that  $X = (c_1, c_2, \eta)$  is a  $D$ -frame for a constant  $D > 0$  if the hyperbolic lengths of  $c_1, c_2$  and  $\eta$  are not greater than  $D$ . In the special case where  $c_1$  and  $c_2$  are coincident counting the orientation, we call  $X = (c, c, \eta)$  particularly a *thetaframe*. Then  $\eta$  must be initiated from and terminated to  $c$  on the same (right) side. See Figure 1.

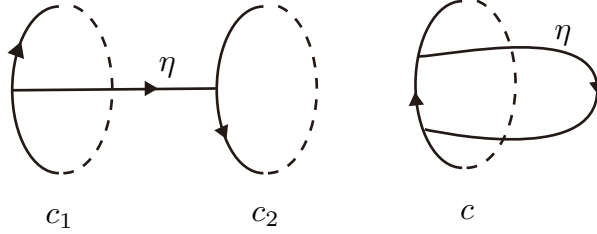


FIGURE 1. Frames

Equivalently, we can say that a frame  $X$  in  $R$  is the image of the following  $H$ -shape in the hyperbolic plane  $\mathbb{D}$  under the universal covering projection  $\mathbb{D} \rightarrow R$ . Let  $\tilde{c}_1$  and  $\tilde{c}_2$  be disjoint oriented axes on  $\mathbb{D}$  corresponding to simple closed geodesics such that each one lies on the right side of the other. Let  $\tilde{\eta}$  be the shortest geodesic arc connecting  $\tilde{c}_1$  with  $\tilde{c}_2$ , in other words, the common perpendicular geodesic arc, that has the orientation from  $\tilde{c}_1$  to  $\tilde{c}_2$ . Then the  $H$ -shape consists of these  $\tilde{c}_1, \tilde{c}_2$  and  $\tilde{\eta}$ .

A non-degenerate geodesic arc  $\eta$  connecting simple closed geodesics  $c_1$  with  $c_2$  perpendicularly is called a *bridge*. A bridge is uniquely determined by  $c_1, c_2$  and a homotopy class of an arc from  $c_1$  to  $c_2$ , where we regard the homotopy as preserving the ends points of the moving arcs in  $c_1$  and  $c_2$  throughout. In this sense, we can define the bridge by specifying a homotopy class.

For a quasiconformal homeomorphism  $f$  of  $R$  onto another Riemann surface  $R'$  and for a frame  $X = (c_1, c_2, \eta)$  in  $R$ , we denote by  $f(X)_*$  the frame in  $R'$  that is homotopic to  $f(X) = (f(c_1), f(c_2), f(\eta))$ . More precisely, the frame  $f(X)_* = (f(c_1)_*, f(c_2)_*, f(\eta)_*)$  consists of the simple closed geodesics  $f(c_1)_*, f(c_2)_*$  freely homotopic to  $f(c_1), f(c_2)$  and the bridge  $f(\eta)_*$  defined as follows. Let  $f(x_i)$  be the end point of  $f(\eta)$  in  $f(c_i)$  and let  $H_t^{(i)}$  ( $0 \leq t \leq 1$ ) be a homotopy deforming  $f(c_i)$  to  $f(c_i)_*$  for  $i = 1, 2$ . Then  $H_t^{(i)}(f(x_i))$  defines an arc  $s_i$  from  $f(x_i)$  to a point in  $f(c_i)_*$ . The bridge  $f(\eta)_*$  connects  $f(c_1)_*$  with  $f(c_2)_*$  in the homotopy class of  $s_1^{-1} \cdot f(\eta) \cdot s_2$  in the above sense.

For a frame  $X = (c, c', \eta)$ , we can make a thetaframe  $\theta(X) = (c, c, \tilde{\eta})$ , where the new bridge  $\tilde{\eta}$  is in the homotopy class of  $\eta \cdot c' \cdot \eta^{-1}$ . For a quasiconformal



homeomorphism  $f$ , we have  $\theta(f(X)_*) = f(\theta(X))_*$ . We also see the following facts easily.

**Proposition 4.3.** *The thetaframe  $\theta(X)$  for a frame  $X$  satisfies the following:*

- (1) *if  $X$  is a  $D$ -frame, then  $\theta(X)$  is a  $3D$ -frame;*
- (2) *if  $X_1$  and  $X_2$  are distinct frames, then  $\theta(X_1)$  and  $\theta(X_2)$  are distinct.*

The following proposition gives an estimate of the ratio of the hyperbolic length of a simple closed geodesic to that of the image under a quasiconformal homeomorphism, which is an improvement of the well-known result given in [30] and [32]. In particular, this can be applied to simple closed geodesics in frames.

**Proposition 4.4** ([12]). *Let  $c$  be a simple closed geodesic on a Riemann surface  $R$ . For a subset  $V$  of  $R$ , let  $d = d(c, V)$  be the hyperbolic distance between  $c$  and  $V$ . If  $f$  is a  $K$ -quasiconformal homeomorphism of  $R$  onto another Riemann surface such that the restriction of  $f$  to  $R - V$  is  $(1 + \epsilon)$ -quasiconformal for some  $\epsilon \geq 0$ , then an inequality*

$$\frac{1}{\alpha} \cdot \ell(c) \leq \ell(f(c)_*) \leq \alpha \cdot \ell(c)$$

*is satisfied for a constant*

$$\alpha = \alpha(K, \epsilon, d) = K + (1 + \epsilon - K) \frac{2 \arctan(\sinh d)}{\pi}$$

*with  $1 \leq \alpha \leq K$  and  $\lim_{d \rightarrow \infty} \alpha = 1 + \epsilon$ .*

On the other hand, the following result gives an estimate for bridges of frames, which is essentially proved in [2].

**Proposition 4.5.** *If  $f$  is a  $K$ -quasiconformal homeomorphism of a Riemann surface  $R$  onto another Riemann surface, then, for every frame  $(c_1, c_2, \eta)$  in  $R$ , an inequality*

$$\frac{1}{K} \ell(\eta) - C/K \leq \ell(f(\eta)_*) \leq K \ell(\eta) + C$$

*is satisfied, where  $C = C(K) \geq 0$  is a constant depending only on  $K$ .*

The estimate obtained in Proposition 4.5 is linear with respect to  $K$  having the additive constant  $C$ . This is useful when  $\ell(\eta)$  is large but otherwise not. In the next lemma, we consider the same situation as in Proposition 4.4, and obtain a linear estimate without a constant term though the multiplier does not depend only on  $K$ .

**Lemma 4.6.** *Let  $W_*$  be a geodesic subsurface in a Riemann surface  $R$ , and  $V$  a subset of  $R$  such that  $d = d(W_*, V) > 0$ . If  $f$  is a  $K$ -quasiconformal homeomorphism of  $R$  onto another Riemann surface such that the restriction*

of  $f$  to  $R - V$  is  $(1 + \epsilon)$ -quasiconformal for  $\epsilon \geq 0$ , then, for every  $D$ -frame  $(c_1, c_2, \eta)$  in  $W_*$ , an inequality

$$\{1 - (\alpha - 1)\beta\}\ell(\eta) \leq \ell(f(\eta)_*) \leq \{1 + (\alpha - 1)\beta\}\ell(\eta)$$

is satisfied, where  $\alpha = \alpha(K, \epsilon, d)$  is the constant as in Proposition 4.4 and  $\beta = \beta(K, D) > 0$  is a constant depending only on  $K$  and  $D$ .

*Proof.* Let  $c_3$  be a closed geodesic that is freely homotopic to a closed curve  $c_1 \cdot \eta \cdot c_2 \cdot \eta^{-1}$  based at the initial point of  $\eta$ . Then it is contained in  $W_*$ . By a formula for right-angled hexagons on the universal cover (see [3, Theorem 2.4.1]), we know that

$$\cosh \frac{\ell(c_3)}{2} = \cosh \ell(\eta) \sinh \frac{\ell(c_1)}{2} \sinh \frac{\ell(c_2)}{2} - \cosh \frac{\ell(c_1)}{2} \cosh \frac{\ell(c_2)}{2}.$$

By Proposition 4.4, we have the inequalities  $(1/\alpha) \cdot \ell(c_i) \leq \ell(f(c_i)_*) \leq \alpha \cdot \ell(c_i)$  for  $i = 1, 2, 3$ .

We define an increasing function  $\phi(t)$  of the variable  $t \geq 1$  by

$$\phi(t) = \operatorname{arccosh} \left( \frac{\cosh \frac{t\ell(c_3)}{2} + \cosh \frac{t\ell(c_1)}{2} \cosh \frac{t\ell(c_2)}{2}}{\sinh \frac{\ell(c_1)}{2t} \sinh \frac{\ell(c_2)}{2t}} \right).$$

Then  $\ell(\eta) = \phi(1)$ . Applying the same formula to the image under  $f$ , we have  $\ell(f(\eta)_*) \leq \phi(\alpha)$ . The mean value theorem for  $\phi(t)$  yields an inequality

$$\phi(\alpha) - \phi(1) \leq \max_{1 \leq \xi \leq \alpha} \phi'(\xi)(\alpha - 1).$$

By setting  $\beta_1 = \max_{1 \leq \xi \leq \alpha} \phi'(\xi)/\phi(1) > 0$ , we have  $\phi(\alpha) \leq \{1 + (\alpha - 1)\beta_1\}\phi(1)$ .

Similarly, we define a decreasing function  $\psi(t)$  of the variable  $t \geq 1$  by

$$\psi(t) = \operatorname{arccosh} \left( \frac{\cosh \frac{\ell(c_3)}{2t} + \cosh \frac{\ell(c_1)}{2t} \cosh \frac{\ell(c_2)}{2t}}{\sinh \frac{t\ell(c_1)}{2} \sinh \frac{t\ell(c_2)}{2}} \right).$$

Then  $\ell(\eta) = \psi(1)$  and  $\ell(f(\eta)_*) \geq \psi(\alpha)$ . Again the mean value theorem for  $\psi(t)$  yields an inequality

$$\psi(\alpha) - \psi(1) \geq \min_{1 \leq \xi \leq \alpha} \psi'(\xi)(\alpha - 1).$$

By setting  $\beta_2 = \min_{1 \leq \xi \leq \alpha} \psi'(\xi)/\psi(1) < 0$ , we have  $\psi(\alpha) \geq \{1 + (\alpha - 1)\beta_2\}\psi(1)$ .

We define  $\beta = \max\{\beta_1, -\beta_2\}$ , which depends continuously on  $\ell(c_1)$ ,  $\ell(c_2)$ ,  $\ell(\eta)$  ( $\leq D$ ) and  $\alpha$  ( $\leq K$ ). Hence  $\beta$  depends on  $K$  and  $D$  and we have the required inequalities given by this  $\beta$ .  $\square$

Now we will choose appropriate frames for our purpose. Under the bounded geometry condition, we can take a frame of bounded size as follows.

**Proposition 4.7.** *Let  $R$  be a Riemann surface satisfying  $(m, M)$ -bounded geometry condition and  $\ell > 0$  a positive constant. For any simple closed geodesic  $c$  on  $R$  whose hyperbolic length  $\ell(c)$  is bounded by  $\ell$ , there exists a  $D$ -thetaframe  $(c, c, \eta)$  or  $(c^{-1}, c^{-1}, \eta)$ , where  $D$  depends only on  $m, M$  and  $\ell$ .*

*Proof.* It has been proved in [11, Proposition 3.1] that we can choose a bridge  $\eta'$  connecting  $c$  with itself whose hyperbolic length is not greater than some constant  $D/3 \geq \ell$ , where  $D$  depends only on  $m, M$  and  $\ell$ . Then either  $X = (c, c, \eta')$  or  $X' = (c, c^{-1}, \eta')$  becomes a  $(D/3)$ -frame after changing the orientation of  $c$  if necessary. Thus the thetaframe  $X$  or  $\theta(X')$  is the desired one by Proposition 4.3.  $\square$

Also we have an estimate of the number of such frames. This can be obtained by [11, Proposition 3.2].

**Proposition 4.8.** *The number of  $D$ -thetaframes  $(c, c, \eta)$  in an arbitrary Riemann surface  $R$  based on a fixed simple closed geodesic  $c$  is not greater than*

$$-\frac{2D}{\log(\tanh(D/2))} > 0.$$

In the proof of our main theorem, we estimate the variation of certain values associated with frames under their movement (Theorem 5.5). The next lemma ensures that we can take two adjacent frames such that one is fixed and the other is not fixed by a given non-trivial mapping class.

**Lemma 4.9.** *Let  $R$  be a Riemann surface satisfying  $(m, M)$ -bounded geometry condition and  $U_*$  a geodesic subsurface in  $R$ . Let  $g$  be a quasiconformal automorphism of  $R$  such that the restriction  $g|_{U_*}$  is not homotopic to the inclusion map  $U_* \hookrightarrow R$ . Suppose that there exists a  $D$ -frame  $X_0 = (c_0, c'_0, \eta_0)$  in  $U_*$  for some constant  $D > 0$  with  $g(X_0)_* = X_0$ . Then there exist  $\tilde{D}$ -frames  $Y = (c_Y, c'_Y, \eta_Y)$  and  $Z = (c_Z, c'_Z, \eta_Z)$  in the  $\tilde{B}$ -neighborhood of  $U_*$  that satisfy the following properties: (i) the oriented simple closed geodesics  $c_Y$  and  $c_Z$  satisfy either  $c_Y = c_Z$  or  $c_Y = (c_Z)^{-1}$ ; (ii)  $g(Y)_* = Y$  and  $g(Z)_* \neq Z$ . Here  $\tilde{D}(> D)$  is a constant depending only on  $m, M$  and  $D$ , and  $\tilde{B} > 0$  is a constant depending only on  $m$  and  $M$ .*

*Proof.* We will prove this lemma by dividing the arguments into four claims. First, we have the following claim possibly by changing the orientation of  $c_0$ .

*Claim 1.* *There exists a thetaframe  $X = (c_0, c_0, \eta)$  such that  $g(c_0)_* = c_0$  and  $g(X)_* \neq X$ .*

Since  $g$  is not homotopic to the inclusion map on  $U_*$ , there exists a simple closed geodesic  $\hat{c}$  on  $U_*$  such that  $g(\hat{c})$  is not freely homotopic to  $\hat{c}$ , though we cannot control the hyperbolic length of  $\hat{c}$ . See [10, Lemma 3]. By changing the orientation of  $c_0$  and  $\hat{c}$  if necessary, we can choose a bridge  $\eta'$  in  $U_*$  connecting  $c_0$

with  $\hat{c}$  so that  $(c_0, \hat{c}, \eta')$  is a frame. Moreover, we take the thetaframe  $\theta(c_0, \hat{c}, \eta')$  in  $U_*$ , which is denoted by  $X = (c_0, c_0, \eta)$ . Then  $g(\hat{c})_* \neq \hat{c}$  yields  $g(X)_* \neq X$  by Proposition 4.3. Thus  $X$  is the desired one.

Note that we may assume that  $\eta$  is contained in the non-cuspidal part  $\dot{R}$ . Indeed, if  $\eta$  has the intersection with a cusp neighborhood, we have only to make a detour avoiding it in the same homotopy class. In this case,  $\eta$  is no longer geodesic, but this does not cause a trouble in the arguments below.

The next claim follows from the proof of [15, Lemma 6]. Recall Proposition 4.1 and the remark just below concerning the constant  $M$ .

*Claim 2. There exists a constant  $B > 0$  depending only on  $m$  and  $M$  such that, for every point  $z \in \dot{R}$ , there is a simple closed geodesic  $c$  with  $d(z, c) \leq B$  and  $\ell(c) \leq M$ .*

Set  $D_1 = 2B + 2M + D$ ,  $\tilde{D} = 2D_1 + B$  and  $\tilde{B} = B + M/2$ . We will find  $\tilde{D}$ -frames  $Y$  and  $Z$  in the  $\tilde{B}$ -neighborhood of  $U_*$  as in the statement of the lemma. If  $\ell(\eta) \leq 2D_1$ , then we have nothing to prove, for  $Y = X_0$  and  $Z = X$  are the desired ones. Thus we may assume that  $\ell(\eta) > 2D_1$ . We denote a subarc of  $\eta$  between two points  $z$  and  $z'$  by  $\eta(z, z')$ . Let  $z_0 \in c_0$  be the initial point and  $z_\infty \in c_0$  the terminal point of  $\eta$ . We will construct a ring of frames as the following claim describes.

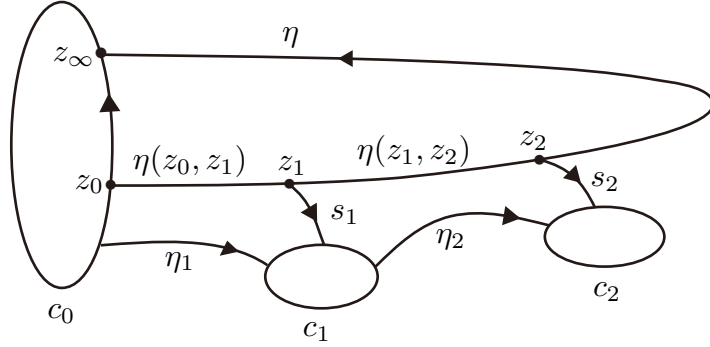


FIGURE 2. A ring of frames

*Claim 3. There exist a set  $\{z_i\}_{i=1}^n$  of points  $z_i$  on  $\eta$ , a set  $\{c_i\}_{i=1}^n$  of simple closed geodesics  $c_i$  on  $R$ , a set  $\{s_i\}_{i=1}^n$  of arcs  $s_i$  from  $z_i$  to  $c_i$ , and a set  $\{\eta_i\}_{i=1}^n$  of bridges  $\eta_i$  connecting  $c_{i-1}$  with  $c_i$  such that they satisfy the following properties: (i)  $z_n = z_\infty$ ,  $c_n = c_0$  or  $c_0^{-1}$ , and  $s_n = \{z_\infty\}$ ; (ii)  $\eta_i$  is in the homotopy class of  $s_{i-1}^{-1} \cdot \eta(z_{i-1}, z_i) \cdot s_i$  in the sense that the homotopy keeps the initial point in  $c_{i-1}$  and the terminal point in  $c_i$ , where we assume  $s_0 = \{z_0\}$ ; (iii) either  $(c_{i-1}, c_i, \eta_i)$  or  $(c_{i-1}^{-1}, c_i, \eta_i)$  constitutes a  $\tilde{D}$ -frame  $X_i$  in the  $\tilde{B}$ -neighborhood of  $U_*$  for each  $i = 1, \dots, n$ .*

First, we take a point  $z_1 \in \eta$  so that the length of the subarc  $\eta(z_0, z_1)$  is  $D_1$ . By Claim 2, there exists a simple closed geodesic  $c_1$  and an arc  $s_1$  from  $z_1$  to  $c_1$  such that  $\ell(s_1) \leq B$  and  $\ell(c_1) \leq M$ . Note that  $c_1$  is in the  $\tilde{B}$ -neighborhood of  $U_*$ . Let  $\eta_1$  be the bridge connecting  $c_0$  with  $c_1$  in the homotopy class of  $\eta(z_0, z_1) \cdot s_1$ . Then  $\ell(\eta_1) \geq D_1 - B - D - M > 0$  and  $\ell(\eta_1) \leq D_1 + B < \tilde{D}$ . By changing the orientation of  $c_1$  if necessary, we have a  $\tilde{D}$ -frame  $X_1 = (c_0, c_1, \eta_1)$  in the  $\tilde{B}$ -neighborhood of  $U_*$ .

Next, we assume that there is a point  $z_2 \in \eta$  such that the length of the subarc  $\eta(z_1, z_2)$  is  $D_1$ . There are a simple closed geodesic  $c_2$  and an arc  $s_2$  from  $z_2$  to  $c_2$  such that  $\ell(s_2) \leq B$  and  $\ell(c_2) \leq M$ . As before,  $c_2$  is in the  $\tilde{B}$ -neighborhood of  $U_*$ . Take the bridge  $\eta_2$  connecting  $c_1$  with  $c_2$  in the homotopy class of  $s_1^{-1} \cdot \eta(z_1, z_2) \cdot s_2$ . Then  $\ell(\eta_2) \geq D_1 - 2B - 2M > 0$  and  $\ell(\eta_2) \leq D_1 + 2B < \tilde{D}$ . By changing the orientation of  $c_2$  if necessarily, either a triple  $(c_1, c_2, \eta_2)$  or a triple  $(c_1^{-1}, c_2, \eta_2)$  is a  $\tilde{D}$ -frame  $X_2$ . See Figure 2.

Repeating this argument, we obtain the dividing points  $z_i \in \eta$ , the arcs  $s_i$  and the triples  $(c_{i-1}, c_i, \eta_i)$  inductively until the length of the subarc  $\eta(z_{n-1}, z_\infty)$  is not greater than  $2D_1$ . We set  $z_n = z_\infty$  and  $c_n = c_0$  or  $c_0^{-1}$ . Let  $\eta_n$  be the bridge connecting  $c_{n-1}$  with  $c_n$  in the homotopy class of  $s_{n-1}^{-1} \cdot \eta(z_{n-1}, z_n)$ . Then  $\ell(\eta_n) \geq D_1 - B - M - D > 0$  and  $\ell(\eta_n) \leq 2D_1 + B = \tilde{D}$ . Therefore, for each  $i$  ( $1 \leq i \leq n$ ), either  $(c_{i-1}, c_i, \eta_i)$  or  $(c_{i-1}^{-1}, c_i, \eta_i)$  is a  $\tilde{D}$ -frame in the  $\tilde{B}$ -neighborhood of  $U_*$ , which we define as  $X_i$ . This completes the proof of Claim 3.

We have constructed a ring of frames. The following claim says that we can choose a desired one among these frames.

*Claim 4. There exists some  $i$  ( $1 \leq i \leq n$ ) such that  $g(X_i)_* \neq X_i$ .*

Suppose to the contrary that there is no such  $i$ . That is,  $g(c_i)_* = c_i$  and  $g(\eta_i)_* = \eta_i$  for all  $i$ . Here we note that  $s_{i-1}^{-1} \cdot \eta(z_{i-1}, z_i) \cdot s_i$  is homotopic to  $\eta_i$  by a homotopy keeping the initial point in  $c_{i-1}$  and the terminal point in  $c_i$ , and  $g(s_{i-1}^{-1} \cdot \eta(z_{i-1}, z_i) \cdot s_i)$  is homotopic to  $g(\eta_i)$  by the corresponding homotopy in the image under  $g$ . Also the product of all the arcs  $s_{i-1}^{-1} \cdot \eta(z_{i-1}, z_i) \cdot s_i$  taken over  $i = 1, \dots, n$  is equal to  $\eta$ . From all these conditions, we see that the bridge  $g(\eta)_*$  of the frame  $g(X)_*$  must satisfy  $g(\eta)_* = \eta$ . However, since  $g(c_0)_* = c_0$  by the assumption, this contradicts Claim 1.

Take the smallest number  $i$  such that  $g(X_i)_* \neq X_i$  and set  $Y = X_{i-1}^{-1}$  and  $Z = X_i$ . These are  $\tilde{D}$ -frames and satisfy the desired condition.  $\square$

## 5. MODULI OF HYPERBOLIC RIGHT-ANGLED HEXAGONS

In this section, we define the moduli of hyperbolic right-angled hexagons and estimate the variation of the moduli from below when a quasiconformal automorphism moves a hexagon in a non-trivial way.

A right-angled hexagon in a Riemann surface  $R$  is, by definition, the image of a hyperbolic right-angled hexagon in  $\mathbb{D}$  under the universal covering projection  $\mathbb{D} \rightarrow R$ . A right-angled hexagon in  $\mathbb{D}$  is called *regular* if it does not have self-intersection (and is called *twisted* otherwise). Its image in  $R$  is also called regular. A regular right-angled hexagon  $H$  having the sides  $\eta_0, \delta_2, \eta_1, \delta_0, \eta_2$  and  $\delta_1$  in cyclic order is denoted by  $(\eta_i, \delta_i)_{i=0,1,2}$ . Consider the hyperbolic lengths  $(a_0, a_1, a_2) = (\ell(\eta_0), \ell(\eta_1), \ell(\eta_2))$  and  $(b_0, b_1, b_2) = (\ell(\delta_0), \ell(\delta_1), \ell(\delta_2))$ . The isometry class of regular right-angled hexagons in  $\mathbb{D}$  is determined by the triple either  $(a_0, a_1, a_2)$  or  $(b_0, b_1, b_2)$ . We call them the *moduli triples*. If all the lengths  $a_i = \ell(\eta_i)$  and  $b_i = \ell(\delta_i)$  are bounded by a constant  $D > 0$ , then we say that  $H = (\eta_i, \delta_i)_{i=0,1,2}$  is a  $D$ -hexagon.

**Lemma 5.1.** *If  $H = (\eta_i, \delta_i)_{i=0,1,2}$  is a regular right-angled  $D$ -hexagon for a constant  $D > 0$ , then there exists a constant  $d > 0$  depending only on  $D$  such that all the lengths  $a_i = \ell(\eta_i)$  and  $b_i = \ell(\delta_i)$  are bounded from below by  $d$ .*

*Proof.* By the formula on regular right-angled hexagons, we have

$$b_i = \operatorname{arccosh} \left( \frac{\cosh a_i + \cosh a_j \cosh a_k}{\sinh a_j \sinh a_k} \right)$$

for any distinct  $i, j, k \in \{0, 1, 2\}$ , and the same formula for  $a_i$  by exchanging the roles of  $\{a_i\}$  and  $\{b_i\}$ . Although the assertion of the lemma may come directly from these formulae, we prove this here in the following way.

We prepare a mirror image  $H'$  of  $H$  and glue  $H$  and  $H'$  along the corresponding three  $\eta$ -sides to obtain a pair of pants whose geodesic boundaries have lengths  $2b_i$  ( $i = 0, 1, 2$ ). By Lemma 5.2 below, we have  $\operatorname{arcsinh}(1/\sinh b_i) \leq D$ . Set  $d = \operatorname{arcsinh}(1/\sinh D)$ . Then we have  $b_i \geq d$ . By exchanging the roles of  $\{a_i\}$  and  $\{b_i\}$ , we also obtain  $a_i \geq d$ . This proves the assertion.  $\square$

The following claim so called the collar lemma serves as the foundation of our entire arguments on hyperbolic geometry. See [3] for instance.

**Lemma 5.2.** *Let  $c$  be a simple closed geodesic on a Riemann surface with hyperbolic length  $\ell(c)$ . Then there exists an annular neighborhood  $A(c)$  of  $c$  with width  $\omega$  (i.e. an  $\omega$ -neighborhood of  $c$ ) for  $\sinh \omega = 1/\sinh(\ell(c)/2)$ , which is called a collar. Moreover, if  $c_1$  and  $c_2$  are disjoint simple closed geodesics, then the collars  $A(c_1)$  and  $A(c_2)$  are disjoint.*

Let  $c$  be an oriented simple closed geodesic on a Riemann surface and let  $\eta_i$  ( $i = 0, 1, 2$ ) be a bridge connecting  $c$  with itself. Note that  $\eta_i$  can have the

intersection with itself, another  $\eta_j$  or  $c$ . We denote the initial point of  $\eta_i$  on  $c$  by  $x_i^-$  and the terminal point of  $\eta_i$  on  $c$  by  $x_i^+$ . Let  $\delta_i$  ( $i = 0, 1, 2$ ) be a subarc of  $c$  starting from  $x_{i+1}^+$  and going along  $c$  for its orientation and ending at  $x_{i+2}^-$ . Here we use the convention that  $x_{i+3}^\pm = x_i^\pm$ .

**Definition 5.3.** Under the notation above, a quadruple  $\hat{X} = (c, \eta_0, \eta_1, \eta_2)$  in a Riemann surface  $R$  is called a *hexapod* based on  $c$  if a closed curve  $\eta_0 \cdot \delta_2 \cdot \eta_1 \cdot \delta_0 \cdot \eta_2 \cdot \delta_1$  is homotopically trivial in  $R$  and if all the six ends of the bridges  $\eta_i$  ( $i = 0, 1, 2$ ) lie on the right side of  $c$ . See Figure 3. For a constant  $D > 0$ ,  $\hat{X} = (c, \eta_0, \eta_1, \eta_2)$  is called a  $D$ -hexapod if  $\ell(c) \leq D$  and  $\ell(\eta_i) \leq D$  for all  $i = 0, 1, 2$ .

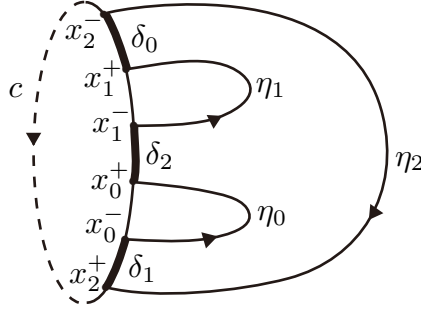


FIGURE 3. The hexapod

For a hexapod  $\hat{X} = (c, \eta_0, \eta_1, \eta_2)$  in  $R$ , the right-angled hexagon  $(\eta_i, \delta_i)_{i=0,1,2}$  is determined by the bridges  $\eta_i$  and the subarcs  $\delta_i$  of  $c$ , which is denoted by  $H(\hat{X})$ . The condition that all the six ends of  $\{\eta_i\}$  lie on the same side of  $c$  implies that the right-angled hexagon  $H(\hat{X})$  is regular.

A hexapod can be constructed from two thetaframes based on a common simple closed geodesic possibly having different orientation. We will explain this procedure here. First we note the following. Consider an oriented simple closed geodesic  $c$  and two bridges  $\eta_0$  and  $\eta_1$  connecting  $c$  with itself in general. Let  $x_0^+$  be the terminal point of  $\eta_0$ ,  $x_1^-$  the initial point of  $\eta_1$ , and  $\delta_2$  the subarc of  $c$  from  $x_0^+$  to  $x_1^-$  for its orientation. Then we have a bridge connecting  $c$  with itself that is homotopic to  $(\eta_0 \cdot \delta_2 \cdot \eta_1)^{-1}$  by a homotopy keeping the end points of the arcs on  $c$ . We denote this bridge by  $\eta(c, \eta_0, \eta_1)$ .

As the easier case, suppose that two distinct thetaframes  $X_0 = (c, c, \eta_0)$  and  $X_1 = (c, c, \eta_1)$  are given. Set  $\eta_2$  to be the bridge  $\eta(c, \eta_0, \eta_1)$  defined as above. Then the ends of  $\eta_2$  also lie on the right side of  $c$  and thus we have a hexapod represented by

$$\hat{X}(X_0, X_1) = (c, \eta_0, \eta_1, \eta_2).$$

If  $X_0$  and  $X_1$  are  $D$ -thetaframes, then the length of  $\eta_2$  is bounded by  $5D$  and thus  $\hat{X}(X_0, X_1)$  is a  $5D$ -hexapod.

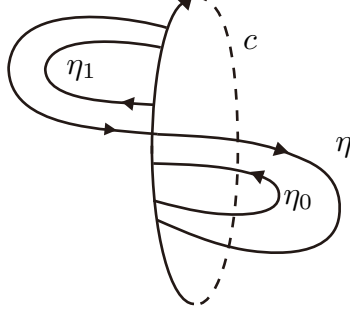


FIGURE 4. Twisted bridge

However, if the given thetaframes are  $X_0 = (c, c, \eta_0)$  and  $X_1 = (c^{-1}, c^{-1}, \eta_1)$ , then we have to replace  $X_1$  with another thetaframe  $X_1^\# = (c, c, \eta_1^\#)$  in order to construct the hexapod as above. In this case, the ends of  $\eta_0$  and  $\eta_1$  lie on different sides of  $c$  as in Figure 4. Set  $\eta = \eta(c, \eta_0, \eta_1)$ . Since the two ends of  $\eta$  lie on different sides of  $c$ , we have a frame  $Q = (c, c^{-1}, \eta^{-1})$ . Then we obtain the thetaframe  $X_1^\#$  by taking  $\theta(Q)$ . If  $X_0$  and  $X_1$  are  $D$ -thetaframes, then  $Q$  is a  $5D$ -frame since the length of  $\eta$  is bounded by  $5D$ . Hence  $X_1^\#$  is a  $15D$ -thetaframe by Proposition 4.3 and  $\hat{X}(X_0, X_1^\#)$  is a  $15D$ -hexapod. Moreover, if a quasiconformal automorphism  $g$  of  $R$  satisfies  $g(X_0)_* = X_0$  and  $g(X_1)_* \neq X_1$ , then  $g(Q)_* \neq Q$  and hence  $g(X_1^\#)_* \neq X_1^\#$  again by Proposition 4.3.

Now we deal with quasiconformal deformations of hexapods in a Riemann surface  $R$ . Let  $\hat{X} = (c, \eta_0, \eta_1, \eta_2)$  be a hexapod based on a simple closed geodesic  $c$  on  $R$  and  $H(\hat{X}) = (\eta_i, \delta_i)_{i=0,1,2}$  the regular right-angled hexagon associated with  $\hat{X}$ . For a quasiconformal homeomorphism  $f$  of  $R$  onto another Riemann surface  $R'$ , we define the corresponding hexapod in  $R'$  by

$$f(\hat{X})_* = (f(c)_*, f(\eta_0)_*, f(\eta_1)_*, f(\eta_2)_*).$$

Hence we have the regular right-angled hexagon  $H(f(\hat{X})_*)$  in  $R'$ . We denote its sides by  $\{\eta'_i\}$  and  $\{\delta'_i\}$  and their lengths by  $\{a'_i\}$  and  $\{b'_i\}$  respectively. Set

$$A(\hat{X}; f) = \max \left\{ \frac{a_0}{a'_0}, \frac{a'_0}{a_0}, \frac{a_1}{a'_1}, \frac{a'_1}{a_1}, \frac{a_2}{a'_2}, \frac{a'_2}{a_2} \right\} \quad (\geq 1)$$

and

$$B(\hat{X}; f) = \max \left\{ \frac{b_0}{b'_0}, \frac{b'_0}{b_0}, \frac{b_1}{b'_1}, \frac{b'_1}{b_1}, \frac{b_2}{b'_2}, \frac{b'_2}{b_2} \right\} \quad (\geq 1).$$



**Proposition 5.4.** *For constants  $K \geq 1$  and  $D > 0$ , there exists a constant  $\kappa > 0$  depending only on  $K$  and  $D$  such that*

$$B(\hat{X}; f)^\kappa \leq A(\hat{X}; f)$$

*is satisfied for every regular right-angled hexagon  $H(\hat{X})$  associated with a  $D$ -hexapod  $\hat{X}$  in a Riemann surface  $R$  and for every  $K$ -quasiconformal homeomorphism  $f$  of  $R$ .*

*Proof.* For an orientation-preserving diffeomorphism  $F$  of  $\mathbb{R}^3$ , set

$$L_F(x) = \max_v \|(dF)_x(v)\|$$

for  $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ , where the maximum is taken over all tangent vectors  $v \in T_x(\mathbb{R}^3)$  with  $\|v\| = 1$ . Since  $L_F$  is a continuous function on  $\mathbb{R}^3$ , the maximum  $L = \max L_F(x) < \infty$  exists on a convex compact subset  $V$  of  $\mathbb{R}^3$ , and hence an inequality

$$\|F(x) - F(x')\| \leq L\|x - x'\|$$

is satisfied for  $x, x' \in V$ . For the distance  $d_\infty(x, x') = \max_{i=0,1,2} \{|x_i - x'_i|\}$ , this yields

$$d_\infty(F(x), F(x')) \leq \sqrt{3}L d_\infty(x, x').$$

For the moduli triples  $(a_0, a_1, a_2)$  and  $(b_0, b_1, b_2)$  of a regular right-angled hexagon  $H(\hat{X})$ , we set  $\alpha_i = \log a_i$  and  $\beta_i = \log b_i$  ( $i = 0, 1, 2$ ). Then the formula for regular right-angled hexagons asserts

$$\beta_i = \log \operatorname{arccosh} \left( \frac{\cosh e^{\alpha_i} + \cosh e^{\alpha_j} \cosh e^{\alpha_k}}{\sinh e^{\alpha_j} \sinh e^{\alpha_k}} \right)$$

for any distinct  $i, j, k \in \{0, 1, 2\}$ . By this relation, we have an orientation-preserving diffeomorphism  $F$  of  $\mathbb{R}^3$  defined by  $(\alpha_0, \alpha_1, \alpha_2) \mapsto (\beta_0, \beta_1, \beta_2)$ .

Let  $\{a'_i\}$  and  $\{b'_i\}$  be the lengths of the sides of  $H(f(\hat{X})_*)$  and set  $\alpha'_i = \log a'_i$  and  $\beta'_i = \log b'_i$ . By Proposition 4.5, all the  $a'_i$  and  $b'_i$  are bounded by a constant depending on  $K$  and  $D$ . Then, by Lemma 5.1, we have an appropriate convex compact subset  $V \subset \mathbb{R}^3$  to consider the function  $F$ . By the above observation, we see that the inequality

$$\frac{1}{\sqrt{3}L} d_\infty(y, y') \leq d_\infty(x, x')$$

holds for  $x^{(l)} = (\alpha_0, \alpha_1, \alpha_2)^{(l)}$  and  $y^{(l)} = (\beta_0, \beta_1, \beta_2)^{(l)}$ , where  $L$  is the constant depending on  $V$ , and hence depending on  $K$  and  $D$ . By taking  $\kappa = (\sqrt{3}L)^{-1}$ , we have the assertion.  $\square$

The moduli triples of right-angled hexagons are conformally invariant. In the next theorem, we obtain an estimate of the variation of the moduli triple under a quasiconformal automorphism by the movement of frames.

**Theorem 5.5.** *There exists a constant  $A_0 = A_0(K, D) > 1$  depending only on  $K \geq 1$  and  $D > 0$  that satisfies the following: for a  $K$ -quasiconformal automorphism  $g$  of a Riemann surface  $R$ , if there exist thetaframes  $X_0 = (c, c, \eta_0)$  and  $X_1 = (c, c, \eta_1)$  such that  $\hat{X} = \hat{X}(X_0, X_1)$  is a  $D$ -hexapod and that the conditions  $g(X_0)_* = X_0$  and  $g(X_1)_* \neq X_1$  are satisfied, then  $A(\hat{X}; g) \geq A_0$ .*

*Proof.* Set  $X'_1 = g(X_1)_* = (c, c, \eta'_1) \neq X_1$ . Starting from the terminal point  $x_0^+$  of  $\eta_0$  for the orientation of  $c$ , we choose one of  $\eta_1$  or  $\eta'_1$  whose initial point comes first. We may assume that it is  $\eta_1$ . Then consider a thetaframe  $X_1^{-1} = (c, c, \eta_1^{-1})$  and construct a hexapod

$$\hat{X}' = \hat{X}(X_1^{-1}, X'_1) = (c, \eta_1^{-1}, \eta'_1, \eta)$$

out of  $X_1^{-1}$  and  $X'_1$  by taking a bridge  $\eta = \eta(c, \eta_1^{-1}, \eta'_1)$ . The regular right-angled hexagon  $H(\hat{X}')$  consists of the six sides denoted by  $\eta_1^{-1}, \lambda, \eta'_1, \lambda_1^{-1}, \eta, \lambda_1'$  in order. As before, we denote the lengths of the  $\delta$ -sides of  $H(\hat{X})$  by  $\{b_i\}$  and the lengths of the  $\delta$ -sides of  $H(g(\hat{X})_*)$  by  $\{b'_i\}$ . Then  $\ell(\lambda) = b'_2 - b_2$ . See Figure 5.

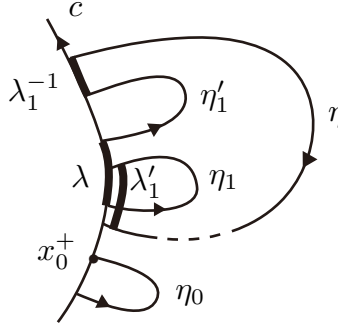


FIGURE 5. Movement of hexapods

By the formula on regular right-angled hexagons, we have

$$\begin{aligned} \cosh \ell(\lambda) &= \frac{\cosh \ell(\eta) + \cosh \ell(\eta_1) \cosh \ell(\eta'_1)}{\sinh \ell(\eta_1) \sinh \ell(\eta'_1)} \\ &> \frac{\cosh \ell(\eta_1) \sinh \ell(\eta'_1)}{\sinh \ell(\eta_1) \sinh \ell(\eta'_1)} = \frac{1}{\tanh \ell(\eta_1)} \\ &\geq \frac{1}{\tanh D} > 1. \end{aligned}$$

Then, by  $b'_2/b_2 = \ell(\lambda)/b_2 + 1 \geq \ell(\lambda)/D + 1$ , we see that  $B(\hat{X}, g) \geq B_0$  for some constant  $B_0 > 1$  depending on  $D$ . Proposition 5.4 says that the assertion of the theorem follows by taking  $A_0 = B_0^*$ .  $\square$

## 6. A SUFFICIENT CONDITION FOR ASYMPTOTIC PERIODICITY

In this section, we prove Theorem 2.9 by using the results in Sections 4 and 5. Theorem 2.9 is a consequence of the following theorem. Recall that  $[g] \in \text{MCG}(R)$  is an asymptotically conformal mapping class if and only if  $[g]_{**} \in \text{Mod}_{AT}(R)$  is elliptic, that is,  $[g]_{**}$  has a fixed point on  $AT(R)$ .

**Theorem 6.1.** *Let  $R$  be a Riemann surface satisfying  $(m, M)$ -bounded geometry condition. Let  $[g] \in \text{MCG}(R)$  be an asymptotically conformal mapping class. Suppose that, for some constant  $\ell > 0$  and in any topologically infinite neighborhood of each topological end of  $R$ , there exists a simple closed geodesic  $c$  with  $\ell(c) \leq \ell$  such that  $g(c)$  is freely homotopic to  $c$ . Then there exists an integer  $t \geq 1$  depending only on  $m$ ,  $M$  and  $\ell$  such that the quasiconformal mapping class  $[g^t]$  is essentially trivial, that is,  $[g^t] \in G_\infty(R)$ .*

*Proof.* Let  $\{R_n\}_{n=1}^\infty$  be a regular exhaustion of  $R$ . Namely,  $\{R_n\}_{n=1}^\infty$  is an increasing sequence of compact subsurfaces  $R_n$  satisfying  $R = \bigcup_{n=1}^\infty R_n$  and each connected component of the complement  $R - R_n$  is not relatively compact. Consider the sequence of geodesic subsurfaces  $\{(R_n)_*\}_{n=1}^\infty$  instead, though  $(R_n)_*$  is not compact if  $R - R_n$  has a cuspidal component. Since  $R$  has no ideal boundary at infinity,  $\{(R_n)_*\}_{n=1}^\infty$  also gives an exhaustion of  $R$ . For each  $n \geq 1$ , let  $(U_n)_*$  denote any topologically infinite connected component of  $R - (R_n)_*$ .

By changing the base Riemann surface of the Teichmüller space, we may assume that there is an asymptotically conformal automorphism  $g$  of  $R$  representing the mapping class  $[g] \in \text{MCG}(R)$ . We fix this representative  $g$  throughout.

Let  $D = D(m, M, \ell) > 0$  be the constant obtained in Proposition 4.7. Define the greatest integer that does not exceed  $-4D/\log(\tanh D)$  by

$$N = \left\lfloor -\frac{4D}{\log(\tanh D)} \right\rfloor (\geq 1),$$

which gives an upper bound of the number of  $2D$ -thetaframes based on a fixed simple closed geodesic by Proposition 4.8. Let  $\tilde{D} = \tilde{D}(m, M, D) (> D)$  and  $\tilde{B} = \tilde{B}(m, M) > 0$  be the constants obtained in Lemma 4.9. For the maximal dilatation  $K = K(g)$  of the representative  $g$ , let  $\beta = \beta(K^{N!}, 45\tilde{D}) > 0$  be the constant obtained in Lemma 4.6, and let  $A_0 = A_0(K^{N!}, 45\tilde{D}) > 1$  be the constant in Theorem 5.5. We take a positive constant  $\epsilon > 0$  so that

$$\max \left\{ 1 + 2\epsilon\beta, \frac{1}{1 - 2\epsilon\beta} \right\} \leq \min \left\{ 2^{\frac{1}{N}}, A_0 \right\}.$$

Since  $g$  is an asymptotically conformal automorphism of  $R$ , there exists a compact subsurface  $V \subset R$  such that  $K(g|_{R-V}) < 1 + \epsilon$ . Set  $d_n = d((U_n)_*, V)$  for any topologically infinite connected components  $(U_n)_*$  of  $R - (R_n)_*$ . Note

that  $d_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Since the constant  $\alpha$  in Proposition 4.4 satisfies  $\alpha(K, \epsilon, d) \rightarrow 1 + \epsilon$  as  $d \rightarrow \infty$ , there is an integer  $n_1$  such that  $V \cap (U_n)_* = \emptyset$  and  $\alpha(K, \epsilon, d_n) < 1 + 2\epsilon$  for every  $n \geq n_1$ .

By the assumption, we can take a simple closed geodesic  $c_n$  on each  $(U_n)_*$  such that  $g(c_n)_* = c_n$  and  $\ell(c_n) \leq \ell$ . We may assume that  $d(c_n, \partial(U_n)_*) \geq 2D$ . By Proposition 4.7, we have a  $D$ -thetaframe  $X_n = (c_n, c_n, \eta_n)$ , which is contained in  $(U_n)_*$ .

**Lemma 6.2.** *For the  $D$ -thetaframe  $X_n = (c_n, c_n, \eta_n)$  in  $(U_n)_*$  for  $n \geq n_1$ , there exists an integer  $s(n)$  ( $1 \leq s(n) \leq N$ ) such that  $g^{s(n)}(X_n)_* = X_n$ .*

*Proof.* Since the constant  $\beta$  in Lemma 4.6 can be taken as

$$\beta(K, 2D) \leq \beta(K^{N!}, 45\tilde{D}) = \beta,$$

we have

$$1 + (\alpha(K, \epsilon, d_n) - 1)\beta(K, 2D) < 1 + 2\epsilon\beta \leq 2^{\frac{1}{N}}$$

for every  $n \geq n_1$ . Then Lemma 4.6 shows that

$$\ell(g(\eta_n)_*) < (1 + 2\epsilon\beta)\ell(\eta_n) \leq 2^{\frac{1}{N}}D.$$

This in particular implies that  $g(X_n)_* = (c_n, c_n, g(\eta_n)_*)$  is a  $2D$ -thetaframe contained in  $(U_n)_*$ . Inductively, for all integers  $k$  ( $0 \leq k \leq N$ ), we see that

$$\ell(g^k(\eta_n)_*) < (1 + 2\epsilon\beta)^k \ell(\eta_n) \leq 2^{\frac{k}{N}}D.$$

Then  $g^k(X_n)_* = (c_n, c_n, g^k(\eta_n)_*)$  are  $2D$ -thetaframes for all such  $k$ . However, since the number of  $2D$ -thetaframes  $(c_n, c_n, \cdot)$  is at most  $N$ , there must be at least two distinct integers, say  $k_1$  and  $k_2$  ( $0 \leq k_1 < k_2 \leq N$ ), such that  $g^{k_1}(X_n)_* = g^{k_2}(X_n)_*$ . Thus, by setting  $s(n) = k_2 - k_1$ , we have  $g^{s(n)}(X_n)_* = X_n$ .  $\square$

*Proof of Theorem 6.1 continued.* Since  $s(n) \leq N$ , we have  $g^{N!}(X_n)_* = X_n$  for all  $n \geq n_1$ . We define a quasiconformal automorphism  $\hat{g} = g^{N!}$  whose maximal dilatation  $K(\hat{g})$  is bounded by  $K^{N!}$ . Although  $\hat{g}$  is still asymptotically conformal,  $K(\hat{g})$  can be larger than  $K$  on a larger part of  $R$ . Here we again choose a compact subsurface  $\hat{V} \subset R$  such that  $K(\hat{g}|_{R-\hat{V}}) < 1 + \epsilon$ . Let  $(\hat{U}_n)_*$  be the smallest geodesic subsurface that contains the  $\tilde{B}$ -neighborhood of  $(U_n)_*$ . Set  $\hat{d}_n = d((\hat{U}_n)_*, \hat{V})$ . Then, there exists an integer  $\hat{n}$  ( $\geq n_1$ ) such that  $\hat{V} \cap (\hat{U}_n)_* = \emptyset$  and  $\alpha(K^{N!}, \epsilon, \hat{d}_n) < 1 + 2\epsilon$  for every  $n \geq \hat{n}$ .

We will prove that  $[\hat{g}]$  is an essentially trivial mapping class. Suppose to the contrary that  $[\hat{g}] \notin G_\infty(R)$ . Then there is a topologically infinite connected component  $(U_n)_*$  for some  $n \geq \hat{n}$  such that  $\hat{g}|_{(U_n)_*}$  is not homotopic to the inclusion map of  $(U_n)_*$ . We fix this  $(U_n)_*$  hereafter. The  $D$ -frame  $X = X_n$  in  $(U_n)_*$  satisfies  $\hat{g}(X)_* = X$  as in Lemma 6.2. Then by Lemma 4.9, there exist

$\tilde{D}$ -frames  $Y = (c_Y, c'_Y, \eta_Y)$  and  $Z = (c_Z, c'_Z, \eta_Z)$  in  $(\hat{U}_n)_*$  either with  $c_Y = c_Z$  or with  $c_Y = (c_Z)^{-1}$  that satisfy  $\hat{g}(Y)_* = Y$  and  $\hat{g}(Z)_* \neq Z$ .

From these  $\tilde{D}$ -frames  $Y$  and  $Z$ , we first make the  $3\tilde{D}$ -thetaframes  $X_0 = \theta(Y)$  and  $X_1 = \theta(Z)$ . Then construct the hexapod  $\hat{X} = \hat{X}(X_0, X_1)$  or  $\hat{X} = \hat{X}(X_0, X_1^\#)$  as in Section 5 according to the coincidence of the orientation. Thus we obtain the  $45\tilde{D}$ -hexapod  $\hat{X}$  in  $(\hat{U}_n)_*$  and the regular right-angled hexagon  $H(\hat{X})$  associated with  $\hat{X}$ . Note that we have  $\hat{g}(X_0)_* = X_0$  in both cases and  $\hat{g}(X_1)_* \neq X_1$  or  $\hat{g}(X_1^\#)_* \neq X_1^\#$  in each case. Consider the moduli triples  $(a_0, a_1, a_2)$  of  $H(\hat{X})$  and  $(a'_0, a'_1, a'_2)$  of  $H(\hat{g}(\hat{X})_*)$ . Then Theorem 5.5 can be applied to show that

$$A(\hat{X}; \hat{g}) = \max \left\{ \frac{a_0}{a'_0}, \frac{a'_0}{a_0}, \frac{a_1}{a'_1}, \frac{a'_1}{a_1}, \frac{a_2}{a'_2}, \frac{a'_2}{a_2} \right\} \geq A_0$$

for the constant  $A_0 = A_0(K^{N!}, 45\tilde{D}) > 1$ .

On the other hand, we can estimate  $A(\hat{X}; \hat{g})$  from above. For each bridge  $\eta_i$  ( $i = 0, 1, 2$ ) in the hexapod  $\hat{X}$ , Lemma 4.6 yields that

$$\{1 - (\alpha(K^{N!}, \epsilon, \hat{d}_n) - 1)\beta\}\ell(\eta_i) \leq \ell(\hat{g}(\eta_i)_*) \leq \{1 + (\alpha(K^{N!}, \epsilon, \hat{d}_n) - 1)\beta\}\ell(\eta_i),$$

where  $\ell(\eta_i) = a_i$  and  $\ell(\hat{g}(\eta_i)_*) = a'_i$ . Then by  $\alpha(K^{N!}, \epsilon, \hat{d}_n) < 1 + 2\epsilon$ , we have

$$A(\hat{X}; \hat{g}) < \max \left\{ 1 + 2\epsilon\beta, \frac{1}{1 - 2\epsilon\beta} \right\} \leq A_0.$$

However, this is a contradiction. Thus we complete the proof.  $\square$

As the following example shows, Theorem 6.1 is not true if  $R$  does not satisfy the bounded geometry condition.

**Example 6.3.** Let  $R$  be a Riemann surface of one topological end that does not satisfy the lower bound condition. Namely, there exists a sequence of mutually disjoint simple closed geodesics  $\{c_n\}_{n=1}^\infty$  such that their hyperbolic lengths tend to 0 as  $n \rightarrow \infty$ . Let  $[g] \in \text{MCG}(R)$  be a quasiconformal mapping class that is caused by infinitely many Dehn twists with respect to all  $c_n$ . Then  $[g]_{**} = \text{id}$  and  $g(c_n)_* = c_n$  for every  $n$ . However, there is no integer  $t \geq 1$  such that  $[g^t]$  is essentially trivial.

Also, we cannot replace the conclusion of Theorem 6.1 with the statement that the mapping class  $[g]$  itself is essentially trivial.

**Example 6.4.** We consider a topologically finite geodesic subsurface  $S$  of two punctures and two boundary geodesics of the same length on which there exists a conformal involution  $h$  fixing the two boundary geodesics and changing the two punctures. We prepare infinitely many copies of  $S$  and glue the copies

isometrically along the boundary geodesics one after the other. Then a topologically infinite Riemann surface  $R$  is obtained and the involution  $h$  induces a conformal automorphism  $g$  of  $R$  of order 2, which satisfies the assumptions of Theorem 6.1. However,  $[g] \in \text{MCG}(R)$  is not essentially trivial.

## 7. PINCHING DEFORMATION ON MUTUALLY DISJOINT COLLARS

In this section, we prove the existence of a quasiconformal deformation satisfying a certain condition on the hyperbolic lengths of simple closed geodesics. We recall the collar lemma (Lemma 5.2) again. Keeping in mind the correspondence of the width  $\omega$  of the collar  $A(c)$  to the hyperbolic length  $l = \ell(c)$  of a simple closed geodesic  $c$ , we define a function of  $l \in (0, \infty)$  by

$$\theta(l) = \arctan \left( \frac{1}{\sinh(l/2)} \right)$$

and denote its inverse function of  $\theta \in (0, \pi/2)$  by  $l(\theta)$ . Here  $\theta$  indicates the half angle of the ideal 2-gon that is the inverse image of  $A(c)$  under the universal covering projection  $\mathbb{D} \rightarrow R$ .

We consider the annular cover  $\hat{A}(c)$  of  $R$  with respect to  $c$ , which is endowed with a conformal coordinate  $(s, t)$ . Here  $s \in (-\pi/2, \pi/2)$  represents the signed angle from the core simple closed geodesic  $c$ , and  $t \in [0, \ell(c)) \pmod{\ell(c)}$  represents the point of the orthogonal projection to  $c$ . The conformal modulus  $\lambda(\hat{A}(c))$  of  $\hat{A}(c)$  is given by  $2\pi^2/\ell(c)$ . The collar  $A(c)$  can be embedded in  $\hat{A}(c)$  as

$$A(c) = \{(s, t) \in \hat{A}(c) \mid s \in (-\theta(\ell(c)), \theta(\ell(c)))\}.$$

The conformal modulus  $\lambda(A(c))$  of  $A(c)$  is  $4\pi\theta(\ell(c))/\ell(c)$ .

A pinching deformation  $f$  of  $R$  with respect to  $c$  is induced by a canonical stretching map on the collar  $A(c)$  defined by  $(s, t) \mapsto (Ks, t)$  for a constant  $K \geq 1$ . Outside  $A(c)$ , we extend  $f$  to be conformal. This is a  $K$ -quasiconformal homeomorphism of  $R$  supported on  $A(c)$ . Similarly, for a family of mutually disjoint simple closed geodesics  $\{c_n\}_{n=1}^\infty$ , we can define the canonical stretching map on  $\{A(c_n)\}_{n=1}^\infty$  and the  $K$ -quasiconformal homeomorphism of  $R$  supported on  $\bigcup_{n=1}^\infty A(c_n)$ . The conformal modulus  $\lambda(f(A(c)))$  is equal to  $K\lambda(A(c)) = 4\pi K\theta(\ell(c))/\ell(c)$ , which is not greater than the conformal modulus  $\lambda(\hat{A}(f(c)_*)) = 2\pi^2/\ell(f(c)_*)$  of the annular cover  $\hat{A}(f(c)_*)$  of the Riemann surface  $f(R)$ . Hence we have a facile estimate

$$\ell(f(c)_*) \leq \frac{\ell(c)}{K} \cdot \frac{\pi/2}{\theta(\ell(c))}$$

for the pinching deformation.

**Lemma 7.1.** *Let  $R$  be a Riemann surface satisfying the lower bound condition for a constant  $m > 0$ , and let  $M (\geq m)$  be another constant. Then there exists a constant  $K \geq 1$  depending only on  $m$  and  $M$  that satisfies the following: for any family of mutually disjoint simple closed geodesics  $\{c_n\}_{n=1}^\infty$  on  $R$  with  $\ell(c_n) \leq M$ , there exists a  $K$ -quasiconformal homeomorphism  $f$  of  $R$  such that  $2\ell(f(c_n)_*) < \ell(f(c)_*)$  for every  $n$  and for every simple closed geodesic  $c$  other than  $\{c_n\}_{n=1}^\infty$ .*

*Proof.* Set  $r = 2m\theta(M)/(M\pi) < 1$  and  $l_0 = l(\frac{\pi}{2}(1 - r/3)) < l(\pi/3)$ . By choosing a suitable constant  $K_0$ , we consider a  $K_0$ -quasiconformal homeomorphism  $f_0 : R \rightarrow S_0$  induced by the canonical stretching maps on the collars  $\{A(c_n)\}_{n=1}^\infty$  so that  $\ell(f_0(c_n)_*) \leq l_0/3$  for every  $n$ . Since  $\ell(c_n) \leq M$ , the width of  $A(c_n)$  measured by the angle from  $c_n$  is not less than  $\theta(M)$ . Hence it is appropriate to choose the constant  $K_0 = 3M\pi/(2l_0\theta(M))$  for that purpose. On the other hand, every simple closed geodesic  $c$  satisfies  $\ell(c) \geq m$  and hence  $\ell(f_0(c)_*) \geq m/K_0$ . From the inequalities above, we obtain

$$\frac{\ell(f_0(c)_*)}{\ell(f_0(c_n)_*)} \geq \frac{m}{K_0} \cdot \frac{3}{l_0} = \frac{m}{M} \cdot \frac{\theta(M)}{\pi/2} = r$$

for every  $n$ .

Next we perform further pinching deformation for  $S_0$ . Let  $f_1 : S_0 \rightarrow S$  be a  $K_1$ -quasiconformal homeomorphism induced by the canonical stretching maps on the collars  $\{A(f_0(c_n)_*)\}_{n=1}^\infty$ . Denote the composition  $f_1 \circ f_0$  by  $f : R \rightarrow S$ . The maximal dilatation  $K$  of  $f$  is bounded by  $K_0K_1$ . Fixing a simple closed geodesic  $c$  different from  $\{c_n\}_{n=1}^\infty$ , we divide the arguments into three cases according to the intersection of  $c$  with  $c_n$ :

- (1)  $c$  is disjoint from all  $c_n$  and  $\ell(f_0(c)_*) \geq l_0$ ;
- (2)  $c$  is disjoint from all  $c_n$  and  $\ell(f_0(c)_*) < l_0$ ;
- (3)  $c$  intersects at least one of  $c_n$ .

We will prove that  $\ell(f(c_n)_*) < \ell(f(c)_*)/2$  for every  $n$  by choosing the constant  $K_1 = 2(3/r - 1)/(1 - r)$ .

First of all, the facile estimate of the pinching deformation with an inequality  $\theta(\ell(f_0(c_n)_*)) > \theta(l_0) = \frac{\pi}{2}(1 - r/3)$  gives that

$$\ell(f(c_n)_*) \leq \frac{\ell(f_0(c_n)_*)}{K_1} \cdot \frac{\pi/2}{\theta(\ell(f_0(c_n)_*))} < \frac{\ell(f_0(c_n)_*)}{K_1(1 - r/3)}.$$

Then in case (1), by applying  $\ell(f_0(c_n)_*) \leq l_0/3$  and  $1 - r/3 > 2/3$  to the previous inequality, we have

$$\ell(f(c_n)_*) < \frac{l_0}{2K_1} \leq \frac{\ell(f_0(c)_*)}{2K_1} \leq \frac{\ell(f(c)_*)}{2}.$$

In case (2), the assumption also yields  $\theta(\ell(f_0(c)_*)) > \theta(l_0) = \frac{\pi}{2}(1 - r/3)$  for  $c$ . Then the collar lemma (Lemma 5.2) gives that the width  $\omega$  of the collar

$A(f_0(c)_*)$  is greater than  $\operatorname{arcsinh}\{\tan \frac{\pi}{2}(1 - r/3)\}$  as well as that the distance  $d$  from the support of  $f_1$  to  $f_0(c)_*$  is not less than  $\omega$ . Hence Proposition 4.4 implies that

$$\frac{\ell(f(c)_*)}{\ell(f_0(c)_*)} \geq \frac{1}{\alpha(K_1, 0, d)} > \frac{1}{K_1(r/3) + (1 - r/3)}.$$

This inequality combined with other ones obtained above concludes

$$\begin{aligned} \frac{\ell(f(c_n)_*)}{\ell(f(c)_*)} &= \frac{\ell(f(c_n)_*)}{\ell(f_0(c_n)_*)} \cdot \frac{\ell(f_0(c_n)_*)}{\ell(f_0(c)_*)} \cdot \frac{\ell(f_0(c)_*)}{\ell(f(c)_*)} \\ &< \frac{1}{K_1(1 - r/3)} \cdot \frac{1}{r} \cdot \{K_1(r/3) + 1 - r/3\} = \frac{K_1 + 3/r - 1}{K_1(3 - r)}. \end{aligned}$$

By the definition of  $K_1$ , the last term of the above inequality is equal to  $1/2$ . Hence we have  $\ell(f(c_n)_*) < \ell(f(c)_*)/2$ .

In case (3), the hyperbolic length  $\ell(f(c)_*)$  is not less than twice the width of the collar  $A(f(c_n)_*)$ . We see that this is greater than  $2\ell(f(c_n)_*)$ . Indeed, we have already made  $\ell(f_0(c_n)_*)$  so small that  $\ell(f_0(c_n)_*) < l_0/2$  for  $l_0 < l(\pi/4)$ . Note by the collar lemma that, when the hyperbolic length of a simple closed geodesic is  $l(\pi/4)$ , the width of the collar is exactly the half of this length. Further pinching by  $f_1$  does not make the situation worse.  $\square$

The combination of Lemma 7.1 and the following lemma yields a crucial consequence. It can be summarized as Theorem 7.3 below.

**Lemma 7.2** ([12]). *Let  $g$  be a quasiconformal automorphism of a Riemann surface  $R$ . Suppose there exists a constant  $\delta > 1$  such that, for every compact subsurface  $V$  of  $R$ , there is a simple closed geodesic  $c$  on  $R$  outside of  $V$  satisfying either  $\ell(g(c)_*)/\ell(c) \geq \delta$  or  $\ell(g(c)_*)/\ell(c) \leq 1/\delta$ . Then  $g$  is not homotopic to any asymptotically conformal automorphism of  $R$ , in particular,  $[g] \notin \operatorname{Ker} \iota_{AT}$ .*

**Theorem 7.3.** *Let  $R$  be a Riemann surface satisfying the lower bound condition, and  $[g] \in \operatorname{MCG}(R)$  a quasiconformal mapping class of  $R$ . Suppose that there exists a sequence of mutually disjoint simple closed geodesics  $\{c_n\}_{n=1}^\infty$  such that the hyperbolic lengths of  $c_n$  are uniformly bounded and  $g(c_n)_* \neq c_{n'}$  for any  $n$  and  $n'$ . Then  $[g]$  is not asymptotically trivial, namely,  $[g] \notin \operatorname{Ker} \iota_{AT}$ .*

*Proof.* By Lemma 7.1, for the sequence  $\{c_n\}_{n=1}^\infty$ , there exists a quasiconformal homeomorphism  $f$  of  $R$  such that  $2\ell(f(c_n)_*) < \ell(f(c)_*)$  for every  $n$  and for every simple closed geodesic  $c$  other than  $\{c_n\}_{n=1}^\infty$ . Set  $\tilde{g} = f \circ g \circ f^{-1}$ . By the assumption  $g(c_n)_* \neq c_{n'}$ , we see that

$$\frac{\ell(\tilde{g}(f(c_n))_*)}{\ell(f(c_n)_*)} = \frac{\ell(f(g(c_n))_*)}{\ell(f(c_n)_*)} > 2$$



for every  $n$ . Since the sequence  $\{f(c_n)_*\}_{n=1}^\infty$  exits from any compact subsurface in the Riemann surface  $f(R)$  (see [23, Proposition 1]), we can apply Lemma 7.2. Then we conclude that  $\tilde{g}$  is not homotopic to any asymptotically conformal automorphism of  $f(R)$ . This implies that  $[g] \notin \text{Ker } \iota_{AT}$ .  $\square$

## 8. PROPERTIES OF PERIODIC MAPPING CLASSES

In this section, we first show that every non-trivial periodic element of the quasiconformal mapping class group acts on the asymptotic Teichmüller space non-trivially. Then, by using this result, we prove a certain relationship between the stable quasiconformal mapping class group  $G_\infty(R)$  and the asymptotically trivial mapping class group  $\text{Ker } \iota_{AT}$ .

Recall that a Teichmüller modular transformation  $[g]_* \in \text{Mod}(R)$  is elliptic if it has a fixed point on  $T(R)$ . This is equivalent to saying that the mapping class  $[g] \in \text{MCG}(R)$  can be realized as a conformal automorphism, namely,  $[g]$  is a conformal mapping class. The following result proved in [21] is a generalization of the Nielsen theorem to the Teichmüller space of an arbitrary Riemann surface.

**Proposition 8.1.** *For a quasiconformal mapping class  $[g] \in \text{MCG}(R)$ , if the orbit  $\{[g]_*^n(p)\}_{n \in \mathbb{Z}}$  of any point  $p \in T(R)$  is bounded in the Teichmüller space  $T(R)$ , then  $[g]_* \in \text{Mod}(R)$  is elliptic, or equivalently,  $[g]$  is a conformal mapping class. In particular, if  $[g]$  is a periodic element, then  $[g]$  is a conformal mapping class.*

We expect that no conformal mapping class belongs to  $\text{Ker } \iota_{AT}$  and prove this statement under the bounded geometry condition on  $R$ . Although we do not assume that the order of a conformal mapping class is finite in the following theorem, the arguments are essential only for this case.

**Theorem 8.2.** *Let  $R$  be a topologically infinite Riemann surface satisfying the bounded geometry condition. Then no non-trivial conformal mapping class  $[g] \in \text{MCG}(R)$  belongs to  $\text{Ker } \iota_{AT}$ .*

*Proof.* Let  $[g] \in \text{MCG}(R)$  be a non-trivial conformal mapping class. By changing the base Riemann surface of the Teichmüller space, we may assume that there is a conformal automorphism  $g$  of  $R$  representing  $[g]$ . We also assume that  $R$  satisfies  $(m, M)$ -bounded geometry condition.

We first take a simple closed geodesic  $c_1$  on  $R$  so that  $\ell(c_1) \leq M$ . We want to make  $c_1$  satisfy  $g(c_1) \neq c_1$  by replacing  $c_1$  with another one if necessary. Suppose that  $g(c_1) = c_1$ . Set  $D = 2B + M$ , where  $B$  is the constant appearing at Claim 2 in the proof of Lemma 4.9. Take a point  $z \in \dot{R}$  away from  $c_1$  by the distance  $B + M$  and find another simple closed geodesic  $c'_1$  with  $d(z, c'_1) \leq B$  and  $\ell(c'_1) \leq M$ . Then  $c'_1$  is disjoint from  $c_1$  and  $d(c_1, c'_1) \leq D$ . If  $g(c'_1) \neq c'_1$ ,

then we have done by taking this  $c'_1$  instead; hence we may also assume that  $g(c'_1) = c'_1$ .

We choose the shortest bridge  $\eta_1$  connecting  $c_1$  with  $c'_1$  and make a  $D$ -frame  $X_1 = (c_1, c'_1, \eta_1)$  by giving an appropriate orientation. If  $g(\eta_1) = \eta_1$ , then  $g(X_1) = X_1$  and the conformal automorphism  $g$  would be the identity in this case. Hence  $g(\eta_1)$  must be different from  $\eta_1$ , which serves as another shortest bridge connecting  $c_1$  with  $c'_1$ . In this situation, we consider a closed curve  $c_1 \cdot \eta_1 \cdot c'_1 \cdot \eta_1^{-1}$  based at the initial point of  $\eta_1$  whose hyperbolic length is bounded by  $2(M + D)$ . We see that this closed curve is freely homotopic to a simple closed geodesic  $\tilde{c}_1$  because the existence of the other shortest bridge  $g(\eta_1)$  prevents  $\tilde{c}_1$  from shrinking to a puncture. Since  $g(\eta_1) \neq \eta_1$ , we have  $g(\tilde{c}_1) \neq \tilde{c}_1$ . By renaming this  $\tilde{c}_1$  as  $c_1$ , we obtain a simple closed geodesic  $c_1$  such that  $g(c_1) \neq c_1$  and its hyperbolic length is bounded by the constant  $2(M + D)$ .

We choose a simple closed geodesic  $c_2$  sufficiently far away from  $c_1$ ,  $g(c_1)$  and  $g^{-1}(c_1)$ . By the same process as above, we can make  $c_2$  satisfy  $g(c_2) \neq c_2$  and  $\ell(c_2) \leq 2(M + D)$ , keeping  $c_2$  disjoint from both  $c_1$  and  $g(c_1)$  and keeping  $g(c_2)$  disjoint from  $c_1$ . Inductively, we have a simple closed geodesic  $c_n$  with  $g(c_n) \neq c_n$  and  $\ell(c_n) \leq 2(M + D)$  such that  $c_n$  is disjoint from both  $c_i$  and  $g(c_i)$  ( $1 \leq i \leq n - 1$ ) and  $g(c_n)$  is disjoint from  $c_i$  ( $1 \leq i \leq n - 1$ ). In this way, we have a sequence  $\{c_n\}_{n=1}^\infty$  of mutually disjoint simple closed geodesics with uniformly bounded hyperbolic lengths satisfying  $g(c_n) \neq c_{n'}$  for any  $n$  and  $n'$ . Then Theorem 7.3 implies that  $[g] \notin \text{Ker } \iota_{AT}$ .  $\square$

**Remark 8.3.** In [25], it has been proved that, no conformal mapping class  $[g] \in \text{MCG}(R)$  of infinite order belongs to  $\text{Ker } \iota_{AT}$  for an arbitrary Riemann surface  $R$ . It is interesting that certain analytic methods work for this problem when the order is infinite but do not work when the order is finite. We need some geometric arguments as in the proof of Theorem 8.2 for the finite order case.

The following lemma makes an important step towards our purpose. This will be used in the next section combined with Theorem 6.1.

**Lemma 8.4.** *Suppose that a quasiconformal mapping class  $[g] \in \text{MCG}(R)$  satisfies  $[g^t] \in G_\infty(R)$  for a positive integer  $t$ . Then we have the following: (i) the mapping class  $[g]$  is asymptotically conformal; (ii) if  $R$  satisfies the bounded geometry condition in addition and if  $[g] \in \text{Ker } \iota_{AT}$ , then  $[g] \in G_\infty(R)$ .*

*Proof.* Since  $[g^t]$  is essentially trivial, there exists a topologically finite subsurface  $V$  of  $R$  such that, for each connected component  $W$  of  $R - V$ , the restriction  $g^t|_W : W \rightarrow R$  is homotopic to the inclusion map  $\text{id}|_W : W \hookrightarrow R$  relative to the ideal boundary at infinity. If  $g(V) \cap V = \emptyset$ , we choose a topologically finite subsurface that contains  $g(V) \cup V$ . In this way, we may assume

that  $V$  satisfies  $g(V) \cap V \neq \emptyset$  for every representative  $g$  of the quasiconformal mapping class  $[g]$ . We may also assume that each component  $W$  is topologically infinite and its relative boundary  $\partial W$  consists of a single simple closed curve.

Take the union

$$\tilde{V} = V_* \cup g(V)_* \cup \dots \cup g^{t-1}(V)_*$$

and consider the topologically finite geodesic subsurface  $\tilde{V}_*$ . This is homotopically  $g$ -invariant, in other words,  $g(\tilde{V})_* = \tilde{V}_*$ . Indeed, as  $\tilde{V}_*$  is determined by the union of  $\{g^i(V)_*\}_{i=0}^{t-1}$ , so is  $g(\tilde{V})_*$  by the union of  $\{g^{i+1}(V)_*\}_{i=0}^{t-1}$ . However, since  $g^t(V)_* = V_*$ , they should be coincident. Hence, by deforming a representative of  $[g]$  in its homotopy class, we may assume that the quasiconformal automorphism  $g$  of  $R$  preserves the geodesic subsurface  $\tilde{V}_*$  invariant as well as its complement  $R - \tilde{V}_*$ .

Let  $\{W_*^{(k)}\}_{1 \leq k \leq m}$  be the family of all connected components of  $R - \tilde{V}_*$ . We make the double  $\widehat{W}^{(k)}$  of each  $W_*^{(k)}$  with respect to the geodesic boundary  $\partial W_*^{(k)}$ . Extend  $g|_{R - \tilde{V}_*}$  to a quasiconformal automorphism  $\hat{g}$  of the union of the Riemann surfaces  $S = \bigcup_{1 \leq k \leq m} \widehat{W}^{(k)}$  by reflection. Then we see that  $\hat{g}^t$  is homotopic to the identity on each component of  $S$  relative to the ideal boundary at infinity, by which we are allowed to say that the mapping class  $[\hat{g}] \in \text{MCG}(S)$  is periodic.

The quasiconformal mapping class group  $\text{MCG}(S)$  acts on the product Teichmüller space  $T(S) = \prod_{1 \leq k \leq m} T(\widehat{W}^{(k)})$  in the same manner as usual. If  $[\hat{g}] \in \text{MCG}(S)$  is periodic, then  $[\hat{g}]_* \in \text{Mod}(S)$  has a fixed point  $(p_1, \dots, p_m)$  in  $T(S)$  by Proposition 8.1. Indeed, for each  $k$ , we can find a minimum positive integer  $t_k$  dividing the period of  $[\hat{g}]$  such that  $\hat{g}^{t_k}$  preserves  $\widehat{W}^{(k)}$ . Then we apply Proposition 8.1 to this periodic mapping class  $[\hat{g}^{t_k}]$  to obtain a fixed point  $p_k$  of  $[\hat{g}^{t_k}]_*$  in  $T(\widehat{W}^{(k)})$ . Giving each  $\widehat{W}^{(k)}$  the complex structure corresponding to  $p_k$ , we can realize the mapping class  $[\hat{g}]$  as a conformal automorphism of  $S$ .

Statement (i) then follows directly from this consequence. Indeed, by restricting the complex structure  $p_k$  to  $W_*^{(k)}$  for each  $k$ , we have a point  $\hat{p}$  of the asymptotic Teichmüller space  $AT(R)$ . Since  $[g]$  has a representative that is conformal outside  $\tilde{V}_*$ , it is an asymptotically conformal mapping class such that  $[g]_{**} \in \text{Mod}_{AT}(R)$  fixes  $\hat{p}$ .

For statement (ii), we further assume that  $[g] \in \text{Ker } \iota_{AT}$ . If there exists some  $k$  such that  $g(W_*^{(k)}) \cap W_*^{(k)} = \emptyset$ , then  $g(c)_* \neq c$  for every simple closed geodesic  $c$  in  $W_*^{(k)}$ . We can choose a sequence of mutually disjoint simple closed geodesics  $\{c_n\}_{n=1}^\infty$  in  $W_*^{(k)}$  whose hyperbolic lengths are uniformly bounded. Then Theorem 7.3 implies that  $[g] \notin \text{Ker } \iota_{AT}$ , however, this contradicts the assumption. Hence we have only to consider the case where  $g(W_*^{(k)}) = W_*^{(k)}$

for every  $k$ . The assumption  $[g] \in \text{Ker } \iota_{AT}$  implies that the mapping class is asymptotically trivial on each  $W_*^{(k)}$  and hence  $[\hat{g}]$  is asymptotically trivial on  $\widehat{W}^{(k)}$ . However, Theorem 8.2 combined with Proposition 8.1 asserts that the periodic mapping class  $[\hat{g}]$  is not asymptotically trivial unless  $[\hat{g}] = \text{id}$ . This implies that  $[\hat{g}] = \text{id}$  and hence  $[g] \in \text{MCG}(R)$  is essentially trivial.  $\square$

## 9. TOPOLOGICAL CHARACTERIZATION OF ASYMPTOTIC TRIVIALITY

In this section, by using the results in Sections 6, 7 and 8, we first prove Theorem 2.5 and then discuss its application. For the proof, we have only to show the following statement.

**Theorem 9.1.** *Let  $R$  be a Riemann surface satisfying the bounded geometry condition. Then  $\text{Ker } \iota_{AT} \subset G_\infty(R)$ .*

*Proof.* Let  $\{R_k\}_{k=1}^\infty$  be a regular exhaustion of  $R$ . For each  $k$ , let  $\{U_k^{(i)}\}_{i=1}^{N(k)}$  be the set of all topologically infinite connected components of the complement of  $R_k$ . We renumber the sequence

$$U_1^{(1)}, \dots, U_1^{(N(1))}, U_2^{(1)}, \dots, U_2^{(N(2))}, \dots, U_k^{(1)}, \dots, U_k^{(N(k))}, \dots$$

as  $\{U_n\}_{n=1}^\infty$ .

We assume that  $R$  satisfies  $(m, M)$ -bounded geometry condition. For the constants  $m$  and  $M$ , we take the quasiconformal constant  $K \geq 1$  obtained by Lemma 7.1. For each  $n$ , we take a pair of disjoint simple closed geodesics  $a_n^i$  ( $i = 1, 2$ ) in the geodesic subsurface  $(U_n)_*$  so that  $\ell(a_n^i) \leq M$  and  $d(a_n^i, \partial(U_n)_*)$  is sufficiently large. This is possible by Claim 2 in the proof of Lemma 4.9. In addition to these conditions, we also require that

$$d(a_k^i, a_n^j) \geq nK^2 d(a_{n-1}^1, a_{n-1}^2) + (n+1)KC(K)$$

for every  $n > 1$ , for every  $k$  with  $1 \leq k \leq n$  and for every  $i, j \in \{1, 2\}$  except for the case of both  $k = n$  and  $i = j$ . Here  $C(K)$  is the constant as in Proposition 4.5. In particular, the union of  $\{a_n^1\}_{n=1}^\infty$  and  $\{a_n^2\}_{n=1}^\infty$  becomes a family of mutually disjoint simple closed geodesics.

By Lemma 7.1, we can take a  $K$ -quasiconformal homeomorphism  $f$  of  $R$  such that  $2\ell(f(a_n^i)_*) < \ell(f(a)_*)$  ( $i = 1, 2$ ) for every  $n \geq 1$  and for every simple closed geodesic  $a$  different from  $\{a_n^1\} \cup \{a_n^2\}_{n=1}^\infty$ . Consider the Riemann surface  $S = f(R)$ , which also satisfies the bounded geometry condition. Set  $W_n = f(U_n)$ .

Take an arbitrary asymptotically trivial mapping class  $[g] \in \text{Ker } \iota_{AT}$ . Then there exists an asymptotically conformal automorphism  $\tilde{g}$  of  $S$  in the homotopy class of  $f \circ g \circ f^{-1}$ . By abuse of notation, we denote this  $\tilde{g}$  also by  $g$ . We take a positive constant  $\epsilon \leq 1/2$ . There exists a compact subsurface  $V \subset S$  such that  $K(g|_{S-V}) < 1 + \epsilon$ . Set  $d_n = d((W_n)_*, V)$ , which tends to  $\infty$  as  $n \rightarrow \infty$ . Since

the constant  $\alpha$  in Proposition 4.4 satisfies  $\alpha(K(g), \epsilon, d) \rightarrow 1 + \epsilon$  as  $d \rightarrow \infty$ , there is an integer  $n_1$  such that  $V \cap (W_n)_* = \emptyset$  and  $\alpha(K(g), \epsilon, d_n) < 1 + 2\epsilon$  for every  $n \geq n_1$ .

For each  $n \geq 1$ , let  $c_n^1$  and  $c_n^2$  be the simple closed geodesics on  $S$  that are freely homotopic to  $f(a_n^1)$  and  $f(a_n^2)$  respectively. They belong to  $(W_n)_*$  and their hyperbolic lengths are bounded by  $KM$ . In addition, our construction imposes the following restraint on them.

**Lemma 9.2.** *There exists an integer  $n_2 (\geq n_1)$  such that  $\{g(c_n^1)_*, g(c_n^2)_*\} = \{c_n^1, c_n^2\}$  for every  $n \geq n_2$ .*

*Proof.* By Proposition 4.4, we have

$$\frac{\ell(g(c_n^i)_*)}{\ell(c_n^i)} \leq \alpha(K(g), \epsilon, d_n) < 1 + 2\epsilon \leq 2$$

for all  $n \geq n_1$ . Since  $\ell(g(c_n^i)_*) < 2\ell(c_n^i)$  and since  $2\ell(c_n^i) < \ell(c)$  for any other simple closed geodesic  $c$  on  $S$ , we see that each  $g(c_n^i)$  must be freely homotopic to either  $c_{n'}^1$  or  $c_{n'}^2$  for some  $n' \geq 1$ .

Next, we consider the hyperbolic distance between the simple closed geodesics, which is attained by the shortest bridge connecting them. By the choice of  $\{a_n^i\}_{n=1}^\infty$  and by Proposition 4.5, we have

$$\begin{aligned} d(c_k^i, c_n^j) &\geq \frac{1}{K} d(a_k^i, a_n^j) - C(K) \\ &\geq \frac{1}{K} \{nK^2 d(a_{n-1}^1, a_{n-1}^2) + (n+1)KC(K)\} - C(K) \\ &= nK d(a_{n-1}^1, a_{n-1}^2) + nC(K) \\ &\geq nK \left\{ \frac{1}{K} d(c_{n-1}^1, c_{n-1}^2) - \frac{1}{K} C(K) \right\} + nC(K) \\ &= n \cdot d(c_{n-1}^1, c_{n-1}^2) > d(c_{n-1}^1, c_{n-1}^2) \end{aligned}$$

for every  $n > 1$ , for every  $k$  with  $1 \leq k \leq n$  and for every  $i, j \in \{1, 2\}$  except for the case of both  $k = n$  and  $i = j$ . In particular, the hyperbolic distance  $d(c_n^1, c_n^2)$  diverges to  $\infty$  as  $n \rightarrow \infty$ .

We take a positive integer  $n_2 (\geq n_1)$  with  $n_2 > K(g) + 1$  satisfying a property that  $d(c_{n-1}^1, c_{n-1}^2) \geq C(K(g))$  for every  $n \geq n_2$ . Then we will show that  $\{g(c_n^1)_*, g(c_n^2)_*\} = \{c_n^1, c_n^2\}$  for every  $n \geq n_2$ . If this is not true, we can always find some positive integers  $n, n'$  and  $k$  with  $n > n'$ ,  $n \geq k$  and  $n \geq n_2$  such that  $\{g(c_n^1)_*, g(c_n^2)_*\} = \{c_k^i, c_n^j\}$  for some  $i, j \in \{1, 2\}$ . Here, by Proposition

4.5 again, we have

$$\begin{aligned}
d(c_k^i, c_n^j) &= d(g(c_{n'}^1)_*, g(c_{n'}^2)_*) \\
&\leq K(g)d(c_{n'}^1, c_{n'}^2) + C(K(g)) \\
&\leq K(g)d(c_{n-1}^1, c_{n-1}^2) + C(K(g)) \\
&\leq (K(g) + 1)d(c_{n-1}^1, c_{n-1}^2) \\
&< n \cdot d(c_{n-1}^1, c_{n-1}^2).
\end{aligned}$$

However, since  $d(c_k^i, c_n^j) \geq n \cdot d(c_{n-1}^1, c_{n-1}^2)$ , this is a contradiction.  $\square$

*Proof of Theorem 9.1 continued.* By Lemma 9.2, we see that  $g(c_n^1)_*$  is either  $c_n^1$  or  $c_n^2$  for all but finitely many  $n$ . If  $g(c_n^1)_* = c_n^1$  for all such  $n$ , then the mapping class  $[g^t]$  is essentially trivial for some integer  $t \geq 1$  by Theorem 6.1. Thus by Lemma 8.4, the mapping class  $[g]$  itself is essentially trivial. If  $g(c_n^1)_* = c_n^2$  for infinitely many  $n$ , then we apply Theorem 7.3 for these mutually disjoint simple closed geodesics of uniformly bounded hyperbolic lengths, which is a subset of  $\{c_n^1\}_{n=1}^\infty$ . Then we have  $[g] \notin \text{Ker}_{AT}$ , but this contradicts the assumption. (Or alternatively, we may conclude that  $[g^{2t}]$  is essentially trivial by Theorem 6.1 since  $g^2(c_n^1)_* = c_n^1$  for all sufficiently large  $n$ , and have the conclusion by the same reason as above.)  $\square$

Finally, we give an application of Theorem 2.5 to the Nielsen theorem on the asymptotic Teichmüller space. More precisely, this refers to Theorem 2.8 in Section 2, which asserts that every asymptotic Teichmüller modular transformation of finite order is elliptic if  $R$  satisfies the bounded geometry condition.

*Proof of Theorem 2.8.* The assumption implies that  $[g^t] \in \text{Ker } \iota_{AT}$  for some integer  $t \geq 1$ . Then  $[g^t] \in G_\infty(R)$  by Theorem 2.5 (Theorem 9.1). Thus we conclude that  $[g]$  is an asymptotically conformal mapping class by Lemma 8.4, which means that  $[g]_{**} \in \text{Mod}_{AT}(R)$  is elliptic.  $\square$

Theorem 2.8 promotes the statement of Theorem 2.9 to the necessary and sufficient condition for an asymptotic Teichmüller modular transformation to be of finite order. We collect our main results to show the last theorem.

**Theorem 9.3.** *Let  $R$  be a Riemann surface satisfying the bounded geometry condition. An asymptotic Teichmüller modular transformation  $[g]_{**} \in \text{Mod}_{AT}(R)$  is of finite order if and only if  $[g]_{**}$  is elliptic and there exist an integer  $s \geq 1$  and a constant  $\ell > 0$  such that, in any topologically infinite neighborhood of each topological end of  $R$ , there exists a simple closed geodesic  $c$  with  $\ell(c) \leq \ell$  such that  $g^s(c)$  is freely homotopic to  $c$ .*

*Proof.* Suppose that  $[g]_{**}$  is of finite order. Then  $[g]_{**}$  is elliptic by Theorem 2.8 and  $[g^s] \in \text{Ker } \iota_{AT}$  for some integer  $s \geq 1$  as well. By Theorem 2.5, this implies that  $[g^s]$  is essentially trivial, namely, there exists a topologically finite

subsurface  $V$  of  $R$  such that, for each connected component  $W$  of  $R - V$ , the restriction  $g^s|_W : W \rightarrow R$  is homotopic to the inclusion map  $\text{id}|_W : W \hookrightarrow R$ . Thus  $g^s(c)$  is freely homotopic to  $c$  for every simple closed geodesic  $c$  in  $W$ . This shows the sufficiency.

Conversely, suppose that  $[g]_{**}$  is elliptic and there exist an integer  $s \geq 1$  and a constant  $\ell > 0$  such that, in any topologically infinite neighborhood of each topological end of  $R$ , there exists a simple closed geodesic  $c$  with  $\ell(c) \leq \ell$  such that  $g^s(c)$  is freely homotopic to  $c$ . Since  $[g^s]_{**}$  is also elliptic, we apply Theorem 2.9 to  $[g^s]$ . Then we conclude that  $[g^s]_{**}$  is of finite order, and hence so is  $[g]_{**}$ . This shows the necessity.  $\square$

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DEPARTMENT OF MATHEMATICS, CHIBA UNIVERSITY, 1-33 YAYOI-CHO, INAGE-KU, CHIBA 263-8522, JAPAN

*E-mail address:* fujikawa@math.s.chiba-u.ac.jp

DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, 3-1-1 TSUSHIMA-NAKA, KITA-KU, OKAYAMA 700-8530, JAPAN

*E-mail address:* matsuzak@math.okayama-u.ac.jp