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Bounded and integrable quadratic differentials: hyperbolic and extremal lengths on Riemann surfaces

KATSUHIKO MATSUZAKI

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro, Tokyo 152
Japan

On a hyperbolic Riemann surface R of finite area, if a holomorphic quadratic differential has finite L^1-norm, then it has finite hyperbolic L^∞-norm, and vice versa. Thus the identity map from the finite dimensional Banach space of the integrable quadratic differentials on R to that of the bounded differentials is a bounded linear operator. In this note, we survey a relation between this operator norm and a geometric property of R. It indicates a kind of distortion of the Riemann surface and relates to the ratio of extremal to hyperbolic length. Moreover this observation is applicable to Riemann surfaces of infinite type.

We begin with the definition of quadratic differential on a Riemann surface R. A holomorphic quadratic differential φ is a differential expressed as $\varphi(z)dz^2$ in a local holomorphic coordinate system such that $\varphi(z)$ is holomorphic in z. More precisely, if a local chart U_1 in z_1 and another chart U_2 in z_2 have the intersection, then the expressions $\varphi_1(z_1)$ in U_1 and $\varphi_2(z_2)$ in U_2 satisfy $\varphi_1(z_1)(dz_1/dz_2)^2 = \varphi_2(z_2)$ there.

In this note, a Riemann surface R is always assumed to be hyperbolic and complete; it is uniformized as $R = \mathbb{H}/G$ by a torsion-free Fuchsian group G acting on the upper half plane model of the hyperbolic plane

$$\mathbb{H} = \{\, z = (x, y) \,|\, y > 0 \,\} \,, \quad ds = \frac{|dz|}{y} \,.$$

We consider a pull-back of a holomorphic quadratic differential on R to the universal cover $\mathbb{H}$; by analytic continuations of the local expressions of the differential, we have a holomorphic function $\varphi(z)$ on $\mathbb{H}$ that satisfies the automorphic condition

$$\varphi(g(z))g'(z)^2 = \varphi(z) \quad \text{for } z \in \mathbb{H},\ g \in G.$$

In this manner, we may identify such an automorphic function with a differential on R and call it a holomorphic quadratic differential for G. Note that it can be defined also for a Fuchsian group with torsion.

For a holomorphic quadratic differential on R, we introduce two distinct norms. One is L^1-norm, namely, the integral of the invariant form $|\varphi(z)||dz|^2$ over R:

$$\|\varphi\|_1 = \iint_R |\varphi(z)||dz|^2 \ .$$

A holomorphic quadratic differential of finite L^1-norm is called integrable and the vector subspace of integrable holomorphic quadratic differentials is denoted by $A(R)$. By L^1-norm, $A(R)$ is a Banach space. The other norm we introduce is hyperbolic L^∞-norm, namely, the supremum of the function $\rho^{-2}(z)|\varphi(z)|$ over R, where $\rho(z)|dz|$ is the hyperbolic metric on R:

$$\|\varphi\|_\infty = \sup_R \rho^{-2}(z)|\varphi(z)| \ .$$

A holomorphic quadratic differential of finite L^∞-norm is called bounded and the vector subspace of bounded holomorphic quadratic differentials is denoted by $B(R)$. By L^∞-norm, $B(R)$ is a Banach space. Importance of $A(R)$ is, for example, in the fact that it is regarded as the cotangent space at R of the Teichmüller space. Importance of $B(R)$ is due to the Bers embedding of the Teichmüller space; it is realized as a bounded domain of $B(R)$ and thus complex structure is provided for it.

The vector space of holomorphic quadratic differentials itself should have certain informations about the geometry and topology of Riemann surfaces. In this note, we consider the ratio of the two norms of holomorphic quadratic differentials . Let $R = R_\rho$ be a Riemann surface (possibly of infinite topological type) with hyperbolic metric ρ. If $A(R_\rho) \subset B(R_\rho)$, we see by the closed graph theorem that the identity map $\varphi \mapsto \varphi$ is a bounded linear operator from $A(R_\rho)$ to $B(R_\rho)$ (cf. [2]) and define the finite operator norm as

$$\kappa(\rho) = \sup\{ \|\varphi\|_\infty \mid \varphi \in A(R_\rho),\ \|\varphi\|_1 = 1 \} \ .$$

If $A(R_\rho) \not\subset B(R_\rho)$, then we set $\kappa(\rho) = \infty$.

The $\kappa(\rho)$ relates to the ratio of extremal length to hyperbolic length. Let $[\alpha]$ be a free homotopy class in R of a simple closed curve α not contractible to a point nor a puncture (which is definite after a metric is given). We denote the set of all such classes $\{[\alpha]\}$ by $\mathcal{S}_R$. Providing a hyperbolic metric ρ with R, we define the hyperbolic length $l_\rho(\alpha)$ of the homotopy class of α by the infimum of lengths of curves in $[\alpha]$ with respect to the hyperbolic metric ρ. On the other hand, the extremal length of the homotopy class of α is by definition

$$E_\rho(\alpha) = \sup_\sigma \frac{\left(\inf_{\alpha\in[\alpha]} \int_\alpha \sigma(z)|dz|\right)^2}{\iint_{R_\rho} \sigma(z)^2|dz|^2} \ ,$$

where the supremum is taken over all Borel measurable conformal metrics $\sigma(z)|dz|$ on R_ρ.

We consider the ratio $E_\rho(\alpha)/l_\rho(\alpha)^2$. By taking the hyperbolic metric $\rho(z)|dz|$ as a conformal metric in the definition of the extremal length, we immediately see that for any $[\alpha] \in \mathcal{S}_R$,

$$\frac{E_\rho(\alpha)}{l_\rho(\alpha)^2} \geq \mathrm{Area}(R_\rho)^{-1} \ .$$

However, the value we are interested in is the upper bound, namely,

$$\nu(\rho) = \sup\{ \frac{E_\rho(\alpha)}{l_\rho(\alpha)^2} \mid [\alpha] \in \mathcal{S}_R \} \ .$$

Moreover, we utilize

$$\lambda(\rho) = \inf_{[\alpha]\in\mathcal{S}_R} l_\rho(\alpha) \ .$$

We assume that if $\mathcal{S}_R = \emptyset$ then $\lambda(\rho) = \infty$.

The following theorem reveals the relation of the index $\kappa(\rho)$ of holomorphic quadratic differentials and the geometric values $\nu(\rho)$ and $\lambda(\rho)$ of a Riemann surface R_ρ.

Theorem. *There exist universal constants r_0 and r_1 such that for an arbitrary hyperbolic Riemann surface R_ρ,*

$$\frac{1}{\pi\lambda(\rho)} \leq \nu(\rho) \leq \kappa(\rho) \ \leq \max\{ \frac{r_0}{\lambda(\rho)}, r_1 \} \ .$$

A proof of the theorem is done by combination of the following three results.

The first one is a famous theorem proved by Jenkins and Strebel, which says that the extremal length is attained by a conformal metric induced by some holomorphic quadratic differential . In more detail, for a Riemann surface R and a homotopy class $[\alpha] \in \mathcal{S}_R$, there is a holomorphic quadratic differential $\varphi(z)dz^2$ such that the natural coordinate $\zeta(p) = \int^{z(p)} \sqrt{\varphi(z)}dz$ $(p \in R)$ maps $R - \{$critical trajectories$\}$ onto an annulus A, and the pull-back of the Euclidean metric $|d\zeta|$ on the annulus is the conformal metric $|\varphi(z)|^{1/2}|dz|$ with singularities on R which attains the extremal length. By this metric, the area of R is $\iint_A |d\zeta|^2$ and the length of $[\alpha]$ is $\inf \int |d\zeta|$.

Lemma 1. *For an arbitrary Riemann surface R_ρ and a homotopy class $[\alpha] \in \mathcal{S}_R$, there is a holomorphic quadratic differential $\varphi(z)dz^2$ on R_ρ such that*

$$E_\rho(\alpha) = \frac{\left(\inf_{\alpha\in[\alpha]} \int_\alpha |\varphi|^{\frac{1}{2}}|dz|\right)^2}{\iint_{R_\rho} |\varphi||dz|^2} \ .$$

Note that this is true not only for closed Riemann surfaces but for arbitrary Riemann surfaces (cf. [9] Chapter VI).

The next one is an estimate based on the mean value theorem of analytic functions. We can find it in Lehner [4], however we refine the multiple constant here. Moreover we may admit parabolic and elliptic elements.

Lemma 2. *There exist universal constants r_0 and r_1 such that any holomorphic quadratic differential φ for an arbitrary Fuchsian group G satisfies,*

$$\|\varphi\|_\infty \le \max\{\frac{r_0}{\inf l_g}, r_1\}\|\varphi\|_1 \ ,$$

where l_g is the translation length of g and the infimum is taken over all the hyperbolic elements in G.

The above constants r_0 and r_1 come from the Marden-Margulis constant $\mu_0 = 0.131467\ldots$ which is the supremum of constants $\mu > 0$ with the following property: for an arbitrary Fuchsian group G and a hyperbolic open disk $D(z, \mu')$ with center $z \in \mathbb{H}$ and radius $\mu' < \mu$, a subgroup $I(D(z, \mu'))$ of G generated by the elements g satisfying $g(D(z, \mu')) \cap D(z, \mu') \neq \emptyset$ is either cyclic or infinite dihedral. See Yamada [10].

Proof of Lemma 2. For $z = (x, y) \in \mathbb{H}$, we take an Euclidean open disk $U(z,t)$ with the center z and the radius $t = y(1 - \exp(-\mu_0))$. It is contained in $D(z, \mu_0)$. By the mean value theorem for a holomorphic function f on $\mathbb{H}$, we have

$$f(z) = \frac{1}{\pi t^2} \iint_{U(z,t)} f(\zeta) d\xi d\eta \ .$$

If $I(D(z, \mu_0))$ is either a hyperbolic cyclic group $\langle g \rangle$ or its $\mathbf{Z}_2$-extension, then at most $8\mu_0/l_g$ points in $U(z,t)$ are equivalent under G. Thus for a holomorphic quadratic differential φ for G, we see

$$\iint_{U(z,t)} |\varphi(\zeta)| d\xi d\eta \ \le \frac{8\mu_0}{l_g} \|\varphi\|_1 \ ,$$

and

$$y^2 |\varphi(z)| \le \frac{8\mu_0}{l_g \pi (1 - \exp(-\mu_0))^2} \|\varphi\|_1 \ .$$

In case $I(D(z, \mu_0))$ is a parabolic cyclic group $\langle g \rangle$, we assume $g(z) = z + 1$, and then find $y = \operatorname{Im} z \gg 2$ because the distance between z and $z + 1$ is less than $2\mu_0$. A half plane $\{ z \mid y > 1 \}$ is precisely invariant under $\langle g \rangle$, and by $w = \exp(2\pi i z)$, it is mapped onto a punctured disk $B = \{ w \mid 0 < |w| < \exp(-2\pi) \}$ which we may regard as a subdomain of $\mathbb{H}/G$. By the local coordinate w, the hyperbolic metric is represented as $\rho(w)|dw| = -|dw|/|w| \log |w|$. We see that an integrable holomorphic quadratic differential $\varphi(w)dw^2$ admits at most single pole at 0. Letting $R = \exp(-2\pi)$, we know $|w| < R/2$ for z with $\operatorname{Im} z > 2$. Thus

$$\begin{aligned}
\rho(w)^{-2} |\varphi(w)| &= |w| (\log |w|)^2 |w \varphi(w)| \\
&\le \frac{C}{\pi (R/2)^2} \iint_B |\zeta \varphi(\zeta)| d\xi d\eta \\
&\le \frac{4C}{\pi R} \|\varphi\|_1 \ ,
\end{aligned}$$

where C is the maximal value of $|w|(\log |w|)^2$ in $|w| < R/2$.

In case $I(D(z, \mu_0))$ is an elliptic cyclic group of order n, we can consider an associated disk B in $\mathbb{H}/G$ as above, with the local coordinate w which represents the hyperbolic metric as $\rho(w)|dw| = 2|dw|/\{n(|w|^{1-\frac{1}{n}} - |w|^{1+\frac{1}{n}})\}$ (the origin of B is a singular point of the metric). An integrable $\varphi(w)dw^2$

admits at most single pole at 0; we estimate $\rho(w)^{-2}|\varphi(w)|$ in the same way as above and then we obtain that it is bounded by a constant times $\|\varphi\|_1$ for w close to the origin.

Finally, if $g(U(z,t)) \cap U(z,t) = \emptyset$, we simply have

$$y^2|\varphi(z)| \leq \frac{1}{\pi(1-\exp(-\mu_0))^2} \iint_{U(z,t)} |\varphi(\zeta)| d\xi d\eta \leq \frac{1}{\pi(1-\exp(-\mu_0))^2} \|\varphi\|_1 \ .$$

Now we can determine the constants r_0 and r_1 from the estimates we have obtained. □

The last one is the sharp estimate of the lower bound of the ratio of extremal to hyperbolic length. Note that this is also true for arbitrary Riemann surfaces (cf. Maskit [6]).

Lemma 3. *For any $[\alpha] \in \mathcal{S}_R$ of an arbitrary hyperbolic Riemann surface R_ρ, we have*

$$\frac{1}{\pi} \leq \frac{E_\rho(\alpha)}{l_\rho(\alpha)} \ .$$

Proof. Let G be a Fuchsian group such that $R_\rho = \mathbb{H}/G$. By conjugation, we may assume that G has a hyperbolic element $\gamma(z) = cz$ $(c = \exp(l_\rho(\alpha)))$ which corresponds to $[\alpha]$. We consider a holomorphic quadratic differential $\varphi(z)dz^2 = z^{-2}dz^2$ for the cyclic group $\Gamma = \langle\gamma\rangle$. The area of the annulus $\mathbb{H}/\Gamma$ by the metric $|\varphi(z)|^{1/2}|dz|$ is $\pi \log c$. By the relative Poincaré series operator $\Theta_{\Gamma\backslash G}$ applied to $|\varphi(z)|^{1/2}|dz|$, we get a conformal metric on $\mathbb{H}/G = R_\rho$ which is

$$\sigma(z)|dz| = \Theta_{\Gamma\backslash G}(|\varphi(z)|^{1/2}|dz|) := \sum_{[h]\in\Gamma\backslash G} |\varphi(h(z))|^{1/2}|h'(z)||dz| \ .$$

Then the area of R_ρ by this metric is $\pi \log c$ and the length of $\alpha \in [\alpha]$ is at least $\int_1^c |\varphi(z)|^{1/2}|dz| = \log c$. By the definition of the extremal length, we see

$$E_\rho(\alpha) \geq \frac{(\log c)^2}{\pi \log c} = \frac{l_\rho(\alpha)}{\pi} \ .$$

Thus we obtain the above estimate. □

Proof of Theorem. The first inequality is known from Lemma 3, and the third is from Lemma 2. Now we have only to show the second.

Let $\varphi \in A(R_\rho)$ be the holomorphic quadratic differential with $\|\varphi\|_1 = 1$ which attains the extremal length $E_\rho(\alpha)$ as in Lemma 1. Let α_0 be the hyperbolic geodesic in $[\alpha]$. Then we see

$$\begin{aligned} E_\rho(\alpha)^{1/2} &\leq \int_{\alpha_0} |\varphi(z)|^{1/2}|dz| = \int_{\alpha_0} (\rho^{-1}(z)|\varphi(z)|^{1/2})\rho(z)|dz| \\ &\leq \|\varphi\|_\infty^{1/2} \int_{\alpha_0} \rho(z)|dz| \leq \kappa(\rho)^{1/2} l_\rho(\alpha) \ . \end{aligned}$$

This means that $\nu(\rho) \leq \kappa(\rho)$. □

There are several direct consequences from Theorem.

Corollary 1 (Neibur-Sheingorn [7])**.** *For a hyperbolic Riemann surface R_ρ, the condition $A(R_\rho) \subset B(R_\rho)$ ($\Leftrightarrow \kappa(\rho) < \infty$) is equivalent to that $\lambda(\rho)$ is positive.*

Corollary 2. *For a homotopy class $[\alpha] \in \mathcal{S}_R$ of an arbitrary hyperbolic Riemann surface R_ρ,*

$$E_\rho(\alpha) \leq \kappa(\rho) l_\rho(\alpha)^2 \ .$$

Compare with Maskit [6]. Even when $l_\rho(\alpha) \to \infty$, the ratio is ruled by the shortest geodesic length. This is the hyperbolic geometry.

Corollary 3. *For a Riemann surface R_ρ of finite area, there is a constant r depending only on the Euler characteristic of R such that*

$$\nu(\rho) \leq \kappa(\rho) \leq \frac{r}{\lambda(\rho)} \leq r\pi\nu(\rho) \ .$$

Note that the length of the shortest geodesic is bounded from above by a constant depending only on the Euler characteristic for a Riemann surface of finite area (Bers [1]).

As an application of Corollary 3, we know a relation between the Teichmüller and Sorvali distances on the finite dimensional Teichmüller space. This result was originally proved by Li Zhong [5].

Let R be a topological surface of genus $g \geq 2$, and σ and τ complex structures on R. Then for the minimizing problem of dilatations of quasiconformal mappings from R_σ to R_τ in the homotopy class of the identity, there is a unique extremal solution f. Denoting the maximal dilatation of f by K_f, we define the Teichmüller distance by $d_T(\sigma,\tau) = \log K_f$. It is known that the extremal one is a stretch map along the foliation of a Teichmüller quadratic differential. Further Kerchhoff [3] showed that this foliation λ maximize the ratio of the extremal lengths $E_\tau(\lambda)/E_\sigma(\lambda)$ among all the measured foliations and

$$d_T(\sigma,\tau) = \log \sup_{[\alpha]\in\mathcal{S}_R} \left\{ \frac{E_\tau(\alpha)}{E_\sigma(\alpha)}, \frac{E_\sigma(\alpha)}{E_\tau(\alpha)} \right\} .$$

On the other hand, Thurston's stretch map is obtained by considering hyperbolic structures σ and τ on R. It is a solution for the minimizing problem of the Lipschitz constants of quasi-isometric mappings from R_σ to R_τ in the homotopy class of the identity. It exists and there is another measured foliation λ such that the ratio $l_\tau(\lambda)^2/l_\sigma(\lambda)^2$ is maximal; we define another distance on the Teichmüller space as

$$d_S(\sigma,\tau) = \log \sup_{[\alpha]\in\mathcal{S}_R} \left\{ \frac{l_\tau(\alpha)^2}{l_\sigma(\alpha)^2}, \frac{l_\sigma(\alpha)^2}{l_\tau(\alpha)^2} \right\} .$$

Li Zhong called this the Sorvali distance.

By Corollary 3, we can estimate the difference of these two distances.

Another application is about the energy of a harmonic map. Let σ be a conformal structure and τ a hyperbolic structure on R. For the homotopy class of the identity $R_\sigma \to R_\tau$, there is a unique harmonic map f which takes a stationary point of the energy functional

$$\mathcal{E}(f) = \int_{R_\sigma} \left\{ \left| \frac{\partial f}{\partial x}(z) \right|_\tau^2 + \left| \frac{\partial f}{\partial y}(z) \right|_\tau^2 \right\} \sigma(z)dxdy .$$

Minsky [7] proved that the energy of the harmonic map f is essentially equal to $\sup_{[\alpha]\in\mathcal{S}_R} l_\tau(\alpha)^2/E_\sigma(\alpha)$ (up to an additive constant). This value can be estimated by combination of above results, which gives a relation between the energy of the harmonic map and the maximal dilatation of the extremal quasiconformal map.

Corollary 4. *Let σ and τ be hyperbolic structures on a surface R of finite area. Then*

$$\frac{\lambda(\tau)}{r} \exp(d_T(\sigma,\tau)) \le \sup_{[\alpha]\in\mathcal{S}_R} \frac{l_\tau(\alpha)^2}{E_\sigma(\alpha)} \le \mathrm{Area}(R) \exp(d_T(\sigma,\tau)) ,$$

where r depends only on the Euler characteristic of R.

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