Quasiconformal mapping class groups having common fixed points on the asymptotic Teichmüller spaces

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Abstract. For an analytically infinite Riemann surface R, we consider the action of the quasiconformal mapping class group $\mathrm{MCG}(R)$ on the Teichmüller space $T(R)$, which preserves the fibers of the projection $\alpha : T(R) \to AT(R)$ onto the asymptotic Teichmüller space $AT(R)$. We prove that, if $\mathrm{MCG}(R)$ has a common fixed point $\alpha(p) \in AT(R)$, then it acts discontinuously on the fiber T_p over $\alpha(p)$, which is a separable subspace of $T(R)$. This in particular implies that $\mathrm{MCG}(R)$ is a countable group. This is a generalization of a fact that $\mathrm{MCG}(R)$ acts discontinuously on $T_o = T(R)$ for an analytically finite Riemann surface R.

§1. Introduction

The asymptotic Teichmüller space $AT(R)$ of a Riemann surface R is a certain quotient space of the Teichmüller space $T(R)$. A quasiconformal homeomorphism f of R is called *asymptotically conformal* if, for every $\varepsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f)$ of f is less than $1+\varepsilon$ on $R-V$. Two quasiconformal homeomorphisms f_1 and f_2 of R are said to be *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1 : R \to R$ is homotopic to the identity relative to the ideal boundary at infinity of R. The *asymptotic Teichmüller space* $AT(R)$ is the set of all asymptotic equivalence classes $[f]_{@}$ of quasiconformal homeomorphisms f of R. Since a conformal homeomorphism is asymptotically conformal, the Teichmüller equivalence, which is defined by "conformal" instead of "asymptotically conformal", is stronger than the asymptotic equivalence. Hence there exists a natural projection $\alpha : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $p = [f] \in T(R)$ to the asymptotic equivalence class $\alpha(p) = [f]_{@} \in AT(R)$.

It is proved in [4] that the asymptotic Teichmüller space $AT(R)$ has a complex Banach manifold structure. This is the unique complex structure on $AT(R)$ such that the projection $\alpha : T(R) \to AT(R)$ is holomorphic with respect to the complex structure on $T(R)$. It is also proved that the fiber T_o over the base point $\alpha(o) \in AT(R)$ is a closed submanifold of $T(R)$, which is separable in the sense that it has a countable dense subset. These properties are also valid for the fiber T_p over any $\alpha(p) \in AT(R)$.

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For a quasiconformal homeomorphism f of R, the *boundary dilatation* $H(f)$ is defined by the infimum of the maximal dilatations $K(f|_{R-V})$ taken over all compact subsets V of R. For a Teichmüller class $[f] \in T(R)$, we set

$$K([f]) = \inf_{f \in [f]} K(f); \quad H([f]) = \inf_{f \in [f]} H(f).$$

Then $H([f]) = 1$ if and only if $[f] \in T_o$, which is a consequence from the fact that T_o is closed. We say that $[f]$ is asymptotically conformal if $[f]$ belongs to T_o. Moreover, $H([f]_@) := \inf_{f \in [f]_@} H(f)$ is equal to $H([f])$. A distance between $[f] \in T(R)$ and the base point $o = [id]$ is defined by $d([f], o) = \frac{1}{2} \log K([f])$, and a distance between $[f]_@ \in AT(R)$ and the base point $\alpha(o) = [id]_@$ is defined by $d_@([f]_@, \alpha(o)) = \frac{1}{2} \log H([f]_@)$. They canonically extend to definitions of the distances between any two points, which become the Teichmüller distance d on $T(R)$ and the asymptotic Teichmüller distance $d_@$ on $AT(R)$.

A *quasiconformal mapping class* is a homotopy equivalence class $[\gamma]$ of quasiconformal automorphisms γ of R and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ is the group of all these mapping classes. Here the homotopy is also meant to be relative to the ideal boundary at infinity. Every element $[\gamma] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism of $T(R)$ by $[f] \mapsto [f \circ \gamma^{-1}]$, which is also isometric with respect to the Teichmüller distance d. In this way, $\mathrm{MCG}(R)$ acts on $T(R)$ and a representation $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ is given. Similarly, $[\gamma] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism of $AT(R)$ by $[f]_@ \mapsto [f \circ \gamma^{-1}]_@$ (see [5, §6.2]), which is also isometric with respect to the asymptotic Teichmüller distance $d_@$. Thus we have a representation $\iota_@ : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$. We denote both $\iota([\gamma])$ and $\iota_@([\gamma])$ by the same symbol $[\gamma]_*$. The action of $\mathrm{MCG}(R)$ preserves the fibers, which means that $[\gamma]_* \circ \alpha(p) = \alpha \circ [\gamma]_*(p)$ for every $p \in T(R)$ and for every $[\gamma] \in \mathrm{MCG}(R)$. For $[\gamma] \in \mathrm{MCG}(R)$, set

$$K([\gamma]) = \inf_{\gamma \in [\gamma]} K(\gamma); \quad H([\gamma]) = \inf_{\gamma \in [\gamma]} H(\gamma),$$

namely, $d([\gamma]_*(o), o) = \frac{1}{2} \log K([\gamma])$ and $d_@([\gamma]_*(\alpha(o)), \alpha(o)) = \frac{1}{2} \log H([\gamma])$. We say that $[\gamma]$ is asymptotically conformal if $[\gamma]$ fixes $\alpha(o)$, that is, $H([\gamma]) = 1$.

In our previous papers [12] and [13], we first construct a Riemann surface R such that every element of $\mathrm{MCG}(R)$ fixes the base point of $AT(R)$ and then make another Riemann surface such that every element of $\mathrm{MCG}(R)$ fixes every point of $AT(R)$. In these examples, the Riemann surfaces R have the properties that R satisfies the divergent geometry condition defined just below and that $\mathrm{MCG}(R)$ consists only of a countable number of elements.

Definition. We say that a hyperbolic Riemann surface R satisfies the *divergent geometry condition* if R has no ideal boundary at infinity and if, for every $L > 0$, the number of closed geodesics in R whose lengths are less than L is finite.

In this present paper, we consider the relationship between the following three conditions mentioned above.

(1) $\mathrm{MCG}(R)$ has a common fixed point on $AT(R)$;
(2) $\mathrm{MCG}(R)$ is countable;
(3) R satisfies the divergent geometry condition.

The implication (2) $\Rightarrow$ (3) was given in [12] and (1) $\Rightarrow$ (3) was proved in [7]. In Corollary 5.2 of this paper, we prove the implication (1) $\Rightarrow$ (2). Actually, we prove in Theorem 5.1 that, if $\mathrm{MCG}(R)$ has a common fixed point $\alpha(p) \in AT(R)$, then $\mathrm{MCG}(R)$ acts on the fiber T_p discontinuously. The separability of T_p then implies that $\mathrm{MCG}(R)$ is countable.

For an analytically finite Riemann surface R, the asymptotic Teichmüller space $AT(R)$ is trivial and hence the fiber T_o over the base point is coincident with $T(R)$. It is known that the mapping class group $\mathrm{MCG}(R)$ is finitely generated (and hence countable) and it acts discontinuously on $T(R)$. Therefore the above mentioned result in this paper may be regarded as a generalization of these facts for analytically finite Riemann surfaces.

Next we consider the stabilizer subgroup $\mathrm{MCG}_{\alpha(p)}(R)$ of $\alpha(p) \in AT(R)$ in the quasiconformal mapping class group $\mathrm{MCG}(R)$ in general. We will see in Theorem 6.1 that $\mathrm{MCG}_{\alpha(p)}(R)$ is countable if the Riemann surface R satisfies a certain bounded geometry condition opposite to the divergent geometry condition.

In Theorem 7.1, we show that the implication (2) $\Rightarrow$ (1) is not always true by giving an example of a Riemann surface that satisfies (2) but not (1). In Theorem 8.1, we show that the implication (3) $\Rightarrow$ (2) is not always true by giving an example of a Riemann surface that satisfies (3) but not (2).

The next three sections (Sections 2–4) are devoted to preliminary results for the proofs of the theorems mentioned above. They are concerning the classification of quasiconformal mapping classes (Section 2), an estimate of the dilatations of twists along simple closed geodesics (Section 3) and an estimate of the norms of the Schwarzian derivatives by the Bers projection (Section 4). Although they are prepared for later purpose, the results themselves seem to be applicable widely.

§2. Mapping classes of divergence type

In this section, we consider several properties of a quasiconformal mapping class of divergence type. For an analytically finite Riemann surface R, this is nothing but a mapping class of infinite order.

Definition. We say that a quasiconformal mapping class $[\gamma] \in \mathrm{MCG}(R)$ is of *divergence type* if the orbit $\{[\gamma]_*^n(p)\}$ for some, hence for all, $p \in T(R)$ diverges to the point at infinity of $T(R)$, that is, $d([\gamma]_*^n(p), p) \to \infty$ as $n \to \infty$.

In particular, if $[\gamma]$ is of divergence type, then it has no fixed point in $T(R)$, or equivalently, the mapping class $[\gamma]$ has no realization as a conformal automorphism. For an analytically infinite Riemann surface R, there exists an example of a mapping class $[\gamma] \in \mathrm{MCG}(R)$ of infinite order but not of divergence type. Concerning the classification of the quasiconformal mapping classes, see [14].

Before stating a result in this section, we prepare notations and basic facts on representations of mapping classes. For any quasiconformal mapping class $[\gamma] \in \mathrm{MCG}(R)$ in general, we choose a representative $\gamma : R \to R$, lift it to a quasiconformal automorphism $\tilde{\gamma}$ of the unit disk Δ against the universal cover $\pi_R : \Delta \to R = \Delta/H$ and extend it to a quasisymmetric automorphism $\bar{\gamma}$ of the unit circle $\partial\Delta$. Note that $\bar{\gamma}$ is determined up to conjugation of the elements of the

Fuchsian group H (regarded as acting on $\partial\Delta$) and is independent of the choice of the representative γ.

Every mapping class $[\gamma] \in \mathrm{MCG}(R)$ induces an outer automorphism of the fundamental group $\pi_1(R) \cong H$ by the correspondence of $\beta \in \pi_i(R, a)$ to $\gamma(\beta) \in \pi_i(R, \gamma(a))$. The outer automorphism is determined independently of the choices of a representative $\gamma \in [\gamma]$ or an arc connecting a and $g(a)$ in R. This gives a representation $\iota_\# : \mathrm{MCG}(R) \to \mathrm{Out}(\pi_1(R))$ to the group of all outer automorphisms and we denote $\iota_\#([\gamma])$ by $[\gamma]_\#$. A sequence of outer automorphisms $[\gamma_n]_\#$ converges to id by definition if, for every finitely generated subgroup H' of $\pi_1(R)$, there exists n_0 such that the restriction $[\gamma_n]_\#|_{H'}$ to H' is the identity modulo inner automorphisms of $\pi_1(R)$ for every $n \geq n_0$.

Proposition 2.1. *Let $R = \Delta/H$ be a Riemann surface having no ideal boundary at infinity, namely, the limit set $\Lambda(H)$ of the Fuchsian group H is the whole $\partial\Delta$. Then the following conditions are equivalent for a sequence of quasiconformal mapping classes $[\gamma_n] \in \mathrm{MCG}(R)$.*

1. *There exist quasiconformal automorphisms $\gamma_n \in [\gamma_n]$ of R such that γ_n converge locally uniformly to id;*
2. *The outer automorphisms $[\gamma_n]_\#$ converge to id;*
3. *There exist quasisymmetric automorphisms $\bar{\gamma}_n$ of $\partial\Delta$ corresponding to $[\gamma_n]$ such that $\bar{\gamma}_n$ converge uniformly to id.*

Proof. It is easy to see that (1) implies (2). Take any subgroup H' of the Fuchsian group $H \cong \pi_1(R)$ and its limit set $\Lambda(H') \subset \partial\Delta$. Then the condition that $[\gamma_n]_\#|_{H'}$ is the identity modulo inner automorphisms of $\pi_1(R)$ is equivalent to that there is some $\bar{\gamma}_n$ satisfying $\bar{\gamma}_n|_{\Lambda(H')} = id$. Since an exhaustion of H by a sequence of finitely generated subgroups $\{H'\}$ gives an exhaustion of $\Lambda(H) = \partial\Delta$ by $\{\Lambda(H')\}$, we see that (2) implies (3). As is given in [2], the conformally barycentric extensions of $\bar{\gamma}_n$ yield quasiconformal automorphisms γ_n of R converging locally uniformly to id. Hence (3) implies (1). □

Now we show the following theorem concerning a quasiconformal mapping class of divergence type.

Theorem 2.2. *Let $[\gamma] \in \mathrm{MCG}(R)$ be a quasiconformal mapping class that is not of divergence type. Assume that there exists a non-cyclic subgroup H' of $\pi_1(R)$ such that the restriction $[\gamma]_\#|_{H'}$ is the identity modulo inner automorphisms of $\pi_1(R)$. Then $[\gamma] = [id]$.*

Proof. We can regard H' as a subgroup of the Fuchsian group $H \cong \pi_1(R)$. The limit set $\Lambda(H')$ is a closed subset of $\partial\Delta$ having more than two points. From the assumption, we see that there exists a corresponding quasisymmetric automorphism $\bar{\gamma}$ of $\partial\Delta$ that fixes every point of $\Lambda(H')$.

To prove that $\bar{\gamma} = id$, which is equivalent to saying that $[\gamma] = [id]$, suppose to the contrary that $\bar{\gamma} \neq id$. Then the set L of points that are fixed by $\bar{\gamma}$ is a proper closed subset of $\partial\Delta$ containing $\Lambda(H')$. Take any interval J in $\partial\Delta - L$, which is invariant under $\bar{\gamma}$. Since $\bar{\gamma}|_J$ is monotonous, for every point $x \in J$, $\bar{\gamma}^{\pm n}(x)$ converge to $j_\pm$ respectively as $n \to \infty$, where $j_\pm$ are the end points of J. Take any $y \in L$ other than $j_\pm$. Then the ratio of the moduli of the quadrilaterals $(\partial\Delta; \bar{\gamma}^{\pm n}(x), j_+, y, j_-)$ to $(\partial\Delta; x, j_+, y, j_-)$ converges to 0 or ∞ as $n \to \infty$. This implies that the maximal dilatations $K([\gamma^n])$ tend to ∞ and hence the orbit $\{[\gamma]_*^n(o)\}$ of the base point $o \in T(R)$ diverges to the point at infinity. This contradicts the assumption that $[\gamma]$ is not of divergence type. □

Corollary 2.3. *Let R be a Riemann surface whose fundamental group $\pi_1(R)$ is non-cyclic. If a sequence of distinct mapping classes $[\gamma_n] \in \mathrm{MCG}(R)$ satisfies $[\gamma_n]_\# \to id$, then $[\gamma_n]$ are of divergence type for all sufficiently large n.*

We also note here another corollary to Theorem 2.2, which is not used later in this paper but is of independent interest.

Corollary 2.4. *Let $[\gamma] \in \mathrm{MCG}(R)$ be a quasiconformal mapping class that is not of divergence type.*

(1) *Assume that there exists an essential subsurface $W \subset R$ with a non-cyclic fundamental group such that $\gamma(c)$ is freely homotopic to c in R for each closed curve c in W. Then $[\gamma] = [id]$.*
(2) *Assume that there exists a non-trivial closed curve c such that $\gamma(c)$ is freely homotopic to c in R. Then $[\gamma]$ is of finite order.*

Proof. Statement (1) immediately follows from Theorem 2.2 if we take $\pi_1(W) \subset \pi_1(R)$ as the subgroup H'.

To prove statement (2), suppose to the contrary that $[\gamma]$ is of infinite order. Since $[\gamma]$ is not of divergence type, there is some bounded subset $B \subset T(R)$ and an infinite sequence $\{n_k\} \subset \mathbb{Z}$ such that $[\gamma]_*^{n_k}(o) \in B$ for every k. Then there are representatives $\gamma_k \in [\gamma]^{n_k}$ whose maximal dilatations are uniformly bounded and which fix the homotopy class of c. Hence, taking a subsequence, we may assume that γ_k converge locally uniformly to a quasiconformal automorphism of R.

This implies that, for a sufficiently large k, the quasiconformal mapping class $[\gamma_k^{-1} \circ \gamma_{k+1}] \in \langle[\gamma]\rangle$ satisfies the assumption in statement (1). Note that, if $[\gamma]$ is not of divergence type then either is $[\gamma]^n$ for $n \neq 0$, and vice versa. Applying (1), we conclude that $[\gamma]^{n_k} = [\gamma]^{n_{k+1}}$. However, this contradicts that $[\gamma]$ is of infinite order. □

§3. Maximal dilatation of twist

In this section, we give an estimate of the maximal dilatations of twists along simple closed geodesics as in the following Theorem 3.1. In the former work [11], we gave a similar estimate in the case where the twists are n-times full Dehn twists. Although the essential idea for the proofs are the same, the assertion in the present version is more general and in particular includes the old one. The estimate consists of two parts: from below and from above. As we can see easily, if the simple closed geodesic has a collar of large width, then the difference between the upper and lower bounds becomes small.

Theorem 3.1. *Let $\{c_i\}_{i=1}^\infty$ be a family of mutually disjoint simple closed geodesics in a hyperbolic Riemann surface R. Assume that c_i has a collar $A(c_i, \omega_i)$ of width*

$\omega_i > 0$ such that $A(c_i,\omega_i) \cap A(c_j,\omega_j) = \emptyset$ for $i \neq j$. Let $[f]$ be a Teichmüller class of quasiconformal homeomorphisms $f : R \to R'$ determined by the twists of hyperbolic lengths t_i along c_i for all i. Then the maximal dilatation $K([f])$ satisfies

$$K([f]) \geq \sup_{i\in\mathbb{N}} \left[1 + \left(\frac{\max\{t_i - 2b(\omega_i), t_i - \ell(c_i), 0\}}{\pi}\right)^2\right];$$

$$K([f]) \leq \sup_{i\in\mathbb{N}} \left[\left\{1 + \left(\frac{t_i}{4\arctan(\sinh\omega_i)}\right)^2\right\}^{1/2} + \frac{t_i}{4\arctan(\sinh\omega_i)}\right]^2$$

$$\leq \sup_{i\in\mathbb{N}} \left[1 + \frac{t_i}{2\arctan(\sinh\omega_i)}\right]^2,$$

where

$$b(x) = \log\frac{1+\cos\{\arctan(\sinh x)\}}{1-\cos\{\arctan(\sinh x)\}}.$$

The proof of Theorem 3.1 begins here. We represent $R = \mathbf{H}/H$ by a Fuchsian group H acting on the upper half-plane $\mathbf{H}$ and consider the universal cover $\pi_R : \mathbf{H} \to R$. Choose an arbitrary simple closed geodesic c_i from $\{c_i\}_{i=1}^{\infty}$ and denote it by c. Set $\ell = \ell(c_i)$, $\omega = \omega_i$ and $t = t_i$. We may assume that a connected component $\tilde{c}$ of the inverse image $\pi_R^{-1}(c)$ is the imaginary axis and the corresponding hyperbolic element of H is $h(z) = e^{\ell}z$.

Let $A = \mathbf{H}/\langle h\rangle$ and consider the universal cover $\pi_A : \mathbf{H} \to A$ and the annular cover $\pi' : A \to R$ such that $\pi_R = \pi' \circ \pi_A$. The ideal boundary at infinity of A with respect to the hyperbolic metric consists of two components, $\partial_0 A = \mathbb{R}_{>0}/\langle h\rangle$ and $\partial_1 A = \mathbb{R}_{<0}/\langle h\rangle$, where $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$ are the positive and the negative real axes respectively. A fundamental domain $\{z \in \mathbf{H} \mid 1 \leq |z| \leq e^{\ell}\}$ of $\langle h\rangle$ is mapped onto a rectangle in the complex plane by a conformal map $\zeta = \log(z)$. Then the annulus A has the conformal coordinate $\zeta = (\xi, \eta)$, where $\xi \in [0, \ell)$ and $\eta \in [0, \pi]$. Hereafter, we refer to the euclidean metric with respect to this conformal coordinate as the euclidean metric on A. In this coordinate, $\partial_0 A = [0, \ell) \times \{0\}$ and $\partial_1 A = [0, \ell) \times \{\pi\}$.

Consider the inverse image $L \subset \mathbf{H}$ of all the simple closed geodesics $\{c_i\}_{i=1}^{\infty}$ under π_R. The complement of L in $\mathbf{H}$ consists of simply connected components, each of which is bounded by complete geodesic lines and non-empty ideal boundary on $\partial\mathbf{H} = \mathbb{R} \cup \{\infty\}$. Let E_0 and E_1 be the adjacent components of $\mathbf{H} - L$ facing to each other along $\tilde{c}$. Both of them are invariant under the cyclic group $\langle h\rangle$.

We make a lift $\tilde{f} : \mathbf{H} \to \mathbf{H}$ of some representative $f \in [f]$ so that $\tilde{f}$ is the identity on $\partial E_0 \cap \partial\mathbf{H}$. It is basically obtained by shifting the components of $\mathbf{H} - L$ other than E_0 along L by hyperbolic translation. The boundary value of $\tilde{f}$ on $\partial\mathbf{H}$ is determined only by this shifting operation and it does not depend on the choice of a representative of $[f]$. The component E_1 is invariant under $\langle h\rangle$ and it moves along the imaginary axis $\tilde{c}$ by hyperbolic length t. Although E_0 and E_1 have geodesic boundaries other than $\tilde{c}$ and the shifting operations are also carried out along them, they do not affect the value of $\tilde{f}$ on $\partial E_0 \cap \partial\mathbf{H}$ or $\partial E_1 \cap \partial\mathbf{H}$.

Let $f_A : A \to A$ be the lift of f that is the projection of $\tilde{f}$ onto A. Its maximal dilatation $K(f_A)$ is equal to $K(f)$.

Here we calculate the euclidean lengths of the intervals in ∂A spanned with the hyperbolic geodesic lines $\pi_A(L)$. They are bounded from above by the length of $\underline{\beta_\omega}$ given in the following proposition.

Proposition 3.2. *Let $A = \{(\xi, \eta) \mid \xi \in [0, \ell),\ \eta \in [0, \pi]\}$ be an annulus with the core geodesic c of hyperbolic length ℓ. Let β_ω be a hyperbolic geodesic line in A whose hyperbolic distance from c is $\omega > 0$ and let $\underline{\beta_\omega}$ be the interval in ∂A spanned with β_ω. Then the euclidean length of $\underline{\beta_\omega}$ with respect to the conformal coordinate (ξ, η) is*

$$b(\omega) = \log \frac{1 + \cos\{\arctan(\sinh \omega)\}}{1 - \cos\{\arctan(\sinh \omega)\}},$$

which tends to 0 as $\omega \to \infty$.

Proof. The euclidean distance between c and β_ω is $\theta = \arctan(\sinh \omega)$. Then elementary geometry yields the assertion. □

In the annulus A, consider the segments $\alpha_\xi = \{\xi\} \times [0, \pi]$ for every $\xi \in [0, \ell)$. For a quasiconformal automorphism g of A preserving each component of ∂A, let $\tau_\xi \geq 0$ be the difference between the ξ-coordinates of $g(\xi, 0)$ and $g(\xi, \pi)$ taking account of a multiple of ℓ by the winding number of $g(\alpha_\xi)$. The minimum of τ_ξ taken over all $\xi \in [0, \ell)$ is defined to be $\tau(g)$. For the quasiconformal automorphism f_A of the annular cover A, it is easy to see that

$$\tau(f_A) \geq \max\{t - 2b(\omega), t - \ell, 0\}.$$

For a curve family $\mathcal{V}$ in a domain D in general, the extremal length $\lambda(\mathcal{V})$ is defined by

$$\lambda(\mathcal{V}) = \sup_{\rho} \frac{\{\inf_{\alpha \in \mathcal{V}} \int_\alpha \rho(\zeta)|d\zeta|\}^2}{\iint_D \rho(\zeta)^2 d\xi d\eta},$$

where the supremum is taken over all Borel measurable non-negative functions $\rho(\zeta)$ on D. Under a K-quasiconformal homeomorphism $g : D \to D'$, the extremal lengths of the curve families $\mathcal{V}$ and $g(\mathcal{V})$ satisfy

$$K^{-1}\lambda(\mathcal{V}) \leq \lambda(g(\mathcal{V})) \leq K\lambda(\mathcal{V}).$$

Using extremal lengths, we can estimate maximal dilatations of quasiconformal maps.

Lemma 3.3. *Let g be a quasiconformal automorphism of A preserving each component of ∂A and having a positive $\tau(g) > 0$. For every $\varepsilon > 0$ less than $\tau(g)$, there exists a rectangle $Q = [\xi_0, \xi_1] \times [0, \pi]$ in A such that a family $\mathcal{V}_Q$ of all rectifiable curves in Q connecting the sides $[\xi_0, \xi_1] \times \{0\}$ and $[\xi_0, \xi_1] \times \{\pi\}$ satisfies*

$$\frac{\lambda(g(\mathcal{V}_Q))}{\lambda(\mathcal{V}_Q)} \geq 1 + \left(\frac{\tau(g) - \varepsilon}{\pi}\right)^2.$$

Proof. Take a positive integer k so that $\ell/k \le \varepsilon < \tau(g)$ and divide the annulus A into k rectangles $Q_1, \dots Q_k$ in parallel, where $Q_i = [(i-1)\ell/k, i\ell/k] \times [0,\pi]$. Consider the images $\{g(Q_i)\}$ in $g(A) = A$. Then at least one rectangle $Q = [\xi_0, \xi_1] \times [0,\pi]$ among $\{Q_i\}$ has a property that the euclidean area of $g(Q)$ is not greater than that of Q, namely, $\mathrm{area}(g(Q)) \le \pi\ell/k$.

It is known that the euclidean metric restricted to Q is the extremal metric for $\mathcal{V}_Q$ and hence $\lambda(\mathcal{V}_Q) = \pi k/\ell$. On the other hand, the euclidean length $\int_{g(\alpha)} |d\zeta|$ for any $\alpha \in \mathcal{V}_Q$ is greater than or equal to $\{\pi^2 + (\tau(g) - \ell/k)^2\}^{1/2}$. Therefore the extremal length $\lambda(g(\mathcal{V}_Q))$ can be estimated by

$$\lambda(g(\mathcal{V}_Q)) \ge \frac{\{\inf_{\alpha\in\mathcal{V}_Q} \int_{g(\alpha)} |d\zeta|\}^2}{\mathrm{area}(g(Q))} \ge \frac{\pi^2 + (\tau(g)-\varepsilon)^2}{\pi\ell/k}.$$

Hence we have the estimate for $\lambda(g(\mathcal{V}_Q))/\lambda(\mathcal{V}_Q)$ as in the statement. □

Corollary 3.4. *A quasiconformal automorphism g of A preserving each boundary component satisfies $K(g) \ge 1 + (\tau(g)/\pi)^2$.*

Proof. The maximal dilatation $K(g)$ satisfies $K(g) \ge \lambda(g(\mathcal{V}_Q))/\lambda(\mathcal{V}_Q)$ for any rectangle Q in A. By taking the rectangle Q as in Lemma 3.3 for an arbitrarily small $\varepsilon > 0$, we have the lower estimate of $K(g)$. □

Remark. Let $\mathcal{V}_\# = \{\alpha_\xi\}_{\xi\in[0,\ell)}$ be a curve family in A consisting of all the segments $\alpha_\xi = \{\xi\} \times [0,\pi]$. Instead of considering the curve family $\mathcal{V}_Q$ in the rectangle Q, we can use $\mathcal{V}_\#$ to obtain the same estimate as in Corollary 3.4, which is the original method given in [11]. In fact, we may regard each segment α_ξ as a limit of the rectangles Q of the side lengths ℓ/k when $k \to \infty$.

Since $\tau(f_A) \ge \max\{t - 2b(\omega), t-\ell, 0\}$, Corollary 3.4 implies that

$$K(f) = K(f_A) \ge 1 + \left(\frac{\max\{t-2b(\omega), t-\ell, 0\}}{\pi}\right)^2.$$

This gives the lower bound in Theorem 3.1.

Next we estimate the upper bound in Theorem 3.1. We consider a quasiconformal homeomorphism $f_0 \in [f]$ of R whose restriction to each collar $A(c_i,\omega_i) \subset R$ is a canonical quasiconformal map h_i representing the twist of length t_i and that is conformal outside of $\bigcup A(c_i,\omega_i)$. Then the estimates on all $A(c_i,\omega_i)$ give $K(f_0)$ and hence the required upper bound. Fix an index i and set $c = c_i$, $\ell = \ell(c_i)$, $\omega = \omega_i$ and $t = t_i$ as before. Note that the collar $A(c,\omega)$ can be embedded in the annular cover A with respect to c. In the following Proposition 3.5, we calculate the maximal dilatation of the canonical quasiconformal automorphism h of the annulus $A_\theta = A(c,\omega)$, where $\theta = \arctan(\sinh\omega)$ stands for the width of the collar measured by the euclidean metric on A.

Proposition 3.5. *Let $h(\xi,\eta) = (\xi + t\eta/(2\theta), \eta)$ be a quasiconformal automorphism of a sub-annulus*

$$A_\theta = \{(\xi,\eta) \in A \mid |\eta - \pi/2| \le \theta\} \quad (0 < \theta \le \pi/2)$$

for a given constant $t \geq 0$. Then its maximal dilatation is

$$K(h) = \left[\left\{ 1 + \left(\frac{t}{4\theta} \right)^2 \right\}^{1/2} + \frac{t}{4\theta} \right]^2 .$$

Proof. Since $h_\xi = 1$ and $h_\eta = t/(2\theta) + i$, we have $\partial h(\zeta) = 1 - it/(4\theta)$ and $\bar{\partial} h(\zeta) = it/(4\theta)$. Substituting them for

$$K(h) = \frac{|\partial h(\zeta)| + |\bar{\partial} h(\zeta)|}{|\partial h(\zeta)| - |\bar{\partial} h(\zeta)|},$$

we have the assertion. □

That completes the proof of Theorem 3.1. □

In the remainder of this section, we estimate the boundary dilatation $H([f])$ of f given by the same twists as before.

Theorem 3.6. *To the same circumstances as in Theorem 3.1, we add an extra assumption that, for any compact subset $V \subset R$, the number of the collars $A(c_i, \omega_i)$ having the intersection with V is finite. Then the boundary dilatation $H([f])$ satisfies*

$$H([f]) \geq \limsup_{i \to \infty} \left\{ 1 + \left(\frac{\max\{t_i - 2b(\omega_i), t_i - \ell(c_i), 0\}}{\pi} \right)^2 \right\}.$$

Proof. Denote by K_i the value before taken the limit in the required inequality above. Suppose to the contrary that $\limsup_{i\to\infty} K_i - H([f])$ is positive and set this value by δ. For this $\delta > 0$, there exist a compact subset $V \subset R$ and a quasiconformal homeomorphism $f \in [f]$ such that $K(f|_{R-V}) < H([f]) + \delta$. Hence $K(f|_{R-V}) < \limsup_{i\to\infty} K_i$. Take a constant $\theta \in (0, \pi/2)$ so that

$$K' := \frac{2\theta}{\pi} K(f|_{R-V}) + \frac{\pi - 2\theta}{\pi} K(f) < \limsup_{i \to \infty} K_i.$$

For this θ, there exists a number i_0 such that the hyperbolic distance between c_i and V is greater than $\operatorname{arcsinh}(\tan\theta)$ for every $i \geq i_0$. Then, fix some $i \geq i_0$ such that $K' < K_i$. Set $\varepsilon := K_i - K' > 0$.

Let A be the annular cover of R with respect to this c_i and let f_A be the lift of f to A. It satisfies $\tau(f_A) \geq \max\{t_i - 2b(\omega_i), t_i - \ell(c_i), 0\}$. For the ε above, we choose $\varepsilon' > 0$ such that

$$1 + \left(\frac{\tau(f_A) - \varepsilon'}{\pi} \right)^2 > K_i - \varepsilon.$$

Then by Lemma 3.3 applied to this ε', we have a rectangle $Q = [\xi_0, \xi_1] \times [0, \pi]$ in A such that

$$\frac{\lambda(f_A(\mathcal{V}_Q))}{\lambda(\mathcal{V}_Q)} > K_i - \varepsilon = K'.$$

However, the following Proposition 3.7, which has been used in [7], asserts that the ratio of the extremal lengths satisfies $\lambda(f_A(\mathcal{V}_Q))/\lambda(\mathcal{V}_Q) \leq K'$. This contradiction proves the assertion. □

Proposition 3.7. *Let $g : Q \to g(Q)$ be a K-quasiconformal homeomorphism of a rectangle $Q = [\xi_0, \xi_1] \times [0, \pi]$ and let $\mathcal{V}_Q$ be a family of all rectifiable curves in Q connecting the sides $[\xi_0, \xi_1] \times \{0\}$ and $[\xi_0, \xi_1] \times \{\pi\}$. If the restriction $g|_{Q_\theta}$ to $Q_\theta = [\xi_0, \xi_1] \times [\pi/2 - \theta, \pi/2 + \theta]$ is H-quasiconformal $(1 \leq H \leq K)$, then the ratio of the extremal lengths satisfies*

$$\frac{\lambda(g(\mathcal{V}_Q))}{\lambda(\mathcal{V}_Q)} \leq \frac{2\theta}{\pi} H + \frac{\pi - 2\theta}{\pi} K.$$

Proof. Let $\mathcal{V}_Q^*$ be the dual family to $\mathcal{V}_Q$ that consists of all rectifiable curves in Q connecting the other sides $\{\xi_0\} \times [0, \pi]$ and $\{\xi_1\} \times [0, \pi]$. Since $\lambda(\mathcal{V}_Q)\lambda(\mathcal{V}_Q^*) = 1$ and $\lambda(g(\mathcal{V}_Q))\lambda(g(\mathcal{V}_Q^*)) = 1$, we estimate the ratio $\lambda(g(\mathcal{V}_Q^*))/\lambda(\mathcal{V}_Q^*)$ from below.

Set a conformal metric $\rho(z)|dz|$ on $g(Q)$ by

$$\rho(z) := \frac{1}{|\partial g(g^{-1}(z))| - |\bar\partial g(g^{-1}(z))|}.$$

Since $|dz| \geq (|\partial g(\zeta)| - |\bar\partial g(\zeta)|)|d\zeta|$, we have

$$\int_{g(\beta)} \rho(z)|dz| \geq \int_\beta |d\zeta|$$

for every $\beta \in \mathcal{V}_Q^*$. On the other hand,

$$\begin{aligned}\iint_{g(Q)} \rho(z)^2 dxdy &= \iint_Q \frac{|\partial g(\zeta)| + |\bar\partial g(\zeta)|}{|\partial g(\zeta)| - |\bar\partial g(\zeta)|} d\xi d\eta \\ &\leq H \,\mathrm{area}(Q_\theta) + K \,\mathrm{area}(Q - Q_\theta) \\ &= \left(\frac{2\theta}{\pi} H + \frac{\pi - 2\theta}{\pi} K\right) \mathrm{area}(Q).\end{aligned}$$

Thus we have

$$\lambda(g(\mathcal{V}_Q^*)) \geq \left(\frac{2\theta}{\pi} H + \frac{\pi - 2\theta}{\pi} K\right)^{-1} \lambda(\mathcal{V}_Q^*),$$

which yields the required inequality. □

§4. Bers projection and conformally barycentric extension

Let $\mathrm{Bel}(R)$ denote the Banach space of all measurable Beltrami differentials μ on a Riemann surface R with the L^∞-norm $\|\mu\|_\infty$ finite, and let $\mathrm{Bel}_1(R)$ be the open unit ball of $\mathrm{Bel}(R)$. By the measurable Riemann mapping theorem, the quasiconformal homeomorphisms f of R and the Beltrami differentials μ in $\mathrm{Bel}_1(R)$ correspond bijectively up to post-composition of conformal homeomorphisms, through the complex dilatation $\mu_f = \bar\partial f/\partial f$ of f. Hence a Teichmüller class $[f] \in T(R)$ can be also denoted by $[\mu]$ for $\mu = \mu_f \in \mathrm{Bel}_1(R)$. In this manner, we have the Teichmüller projection $\pi : \mathrm{Bel}_1(R) \to T(R)$ by $\mu \mapsto [\mu]$, which is a holomorphic split submersion.

Two Riemann surfaces R and R^* are complex conjugate to each other if there exists an anti-conformal homeomorphism $j : R \to R^*$. For a subset $V \subset R$, the complex conjugate $V^* \subset R^*$ is defined by $V^* = j(V)$.

Let $B(R^*)$ be the Banach space of all bounded holomorphic quadratic differentials φ on R^* having the hyperbolic L^∞-norm

$$\|\varphi\| = \sup_{z \in R^*} \rho_{R^*}(z)^{-2}|\varphi(z)| < \infty,$$

where ρ_{R^*} is the hyperbolic density on R^*. The norm restricted to a subset $V^* \subset R^*$ is denoted by $\|\varphi\|_{V^*} = \sup_{z \in V^*} \rho_{R^*}(z)^{-2}|\varphi(z)|$. The *boundary semi-norm* $\|\varphi\|_0$ is defined to be the infimum of $\|\varphi\|_{R^*-V^*}$ taken over all compact subsets $V^* \subset R^*$. We say that a bounded holomorphic quadratic differentials φ on R^* *vanishes at infinity* if $\|\varphi\|_0 = 0$. Let $B_0(R^*) \subset B(R^*)$ be the Banach subspace consisting of all bounded holomorphic quadratic differentials vanishing at infinity.

The Bers projection $\mathcal{B} : \mathrm{Bel}_1(R) \to B(R^*)$ is defined as follows. For any $\mu \in \mathrm{Bel}_1(R)$, consider its lift $\tilde{\mu}$ to the unit disk Δ and take the quasiconformal homeomorphism F of the Riemann sphere $\hat{\mathbb{C}}$ such that $\mu_F = \tilde{\mu}$ on Δ and $\mu_F = 0$ on $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$. Then take the Schwarzian derivative $\mathcal{S}_{F|_{\Delta^*}}$ on Δ^* and project it on R^* as a holomorphic quadratic differential φ. This belongs to $B(R^*)$ and defines $\mathcal{B}(\mu)$. The Bers projection $\mathcal{B}$ is decomposed by the Teichmüller projection $\pi : \mathrm{Bel}_1(R) \to T(R)$ into the Bers embedding $\beta : T(R) \to B(R^*)$ such that $\mathcal{B} = \beta \circ \pi$. The image of $\mathcal{B}$ is a bounded domain in $B(R^*)$, which is biholomorphic to $T(R)$ under β.

The Bers projection above is defined with respect to the Fuchsian projective structure on the Riemann surface $R^* = \Delta^*/H$, and it is possible to generalize it to any quasifuchsian projective structure on R^*. For a Teichmüller class $[g] = [\nu] \in T(R)$, consider its lift $\tilde{\nu}$ to Δ and take the quasiconformal homeomorphism G of $\hat{\mathbb{C}}$ such that $\mu_G = \tilde{\nu}$ on Δ and $\mu_G = 0$ on Δ^*. Next, for any $[\mu] \in T(g(R))$, consider its lift $\tilde{\mu}$ to $G(\Delta)$ and take the quasiconformal homeomorphism F of $\hat{\mathbb{C}}$ such that $\mu_F = \tilde{\mu}$ on $G(\Delta)$ and $\mu_F = 0$ on $G(\Delta^*)$. Then take the Schwarzian derivative $\mathcal{S}_{F|_{G(\Delta^*)}}$ and project it on R^* as a holomorphic quadratic differential φ. This belongs to $B(R^*)$ and defines $\mathcal{B}_g(\mu)$. We call this map $\mathcal{B}_g : \mathrm{Bel}_1(g(R)) \to B(R^*)$ the generalized Bers projection with respect to $[g] \in T(R)$.

The norm of a bounded holomorphic quadratic differential obtained by the generalized Bers projection is estimated as follows. See [10, III.4.2].

Proposition 4.1. *Let $\mathcal{B}_g : \mathrm{Bel}_1(g(R)) \to B(R^*)$ be the generalized Bers projection with respect to $[g] \in T(R)$. Then there exist positive constants $c(K)$ and $C(K)$ depending only on $K = K([g]) \geq 1$ such that*

$$c(K)\|\mu\|_\infty \leq \|\mathcal{B}_g(\mu)\| \leq C(K)\|\mu\|_\infty$$

for any extremal $\mu \in \mathrm{Bel}_1(g(R))$ in the Teichmüller class $[\mu] \in T(g(R))$. Moreover, $c(K)$ and $C(K)$ are monotone increasing and decreasing respectively as $K \to 1$ with $c(1) = 1/2$ and $C(1) = 3/2$.

This proposition gives a global relationship between the norms of Beltrami differentials and quadratic differentials under the correspondence of the Bers projection. On the other hand, there is a local relationship which can be expressed as follows. If a Beltrami differential $\mu \in \mathrm{Bel}_1(R)$ is small on a subset $E \subset R$, then the holomorphic quadratic differential $\mathcal{B}(\mu) \in B(R^*)$ is small on the corresponding subset $E^* \subset R^*$. See [15, Lemma 3.1].

The conformally barycentric extension [2] gives a systematic way of taking a representative f from each Teichmüller class $[f] \in T(R)$. In other words, it defines a global section $s : T(R) \to \mathrm{Bel}_1(R)$ for the projection $\pi : \mathrm{Bel}_1(R) \to T(R)$. This section has a good property as in the following proposition, which was proved partially in [4] and completed in [6]. We can understand it as another formulation of the above mentioned principle on the local relationship between Beltrami differentials and quadratic differentials.

Proposition 4.2. *The following conditions are equivalent for any $p \in T(R)$:*

1. *The Teichmüller class $p \in T(R)$ is asymptotically conformal, that is, $p \in T_o$;*
2. *The quasiconformal homeomorphism of R that is determined by the Beltrami differential $s(p) \in \mathrm{Bel}_1(R)$ is asymptotically conformal;*
3. *The bounded holomorphic quadratic differential $\beta(p) \in B(R^*)$ vanishes at infinity.*

In virtue of this proposition, we see that the Bers embedding $\beta : T(R) \to B(R^*)$ of the Teichmüller space $T(R)$ is projected to the asymptotic Bers embedding $\hat{\beta} : AT(R) \to B(R^*)/B_0(R^*)$ in a commutative way with the projection $\alpha : T(R) \to AT(R)$ and the quotient map $B(R^*) \to B(R^*)/B_0(R^*)$. Here $B(R^*)/B_0(R^*)$ is regarded as the quotient Banach space with the quotient norm.

Let $A(R)$ be the Banach space of all integrable holomorphic quadratic differentials on R. Since the Poincaré series operator $\Theta : A(\Delta) \to A(R)$ is surjective bounded linear and since polynomials are dense in $A(\Delta)$, we see that $A(R)$ is separable (cf. [9, §3.2]). In [3], the predual Banach space $Z_0(R)$ to $A(R)$ is given, which is a closed subspace of the tangent space $Z(R)$ of $T(R)$ at the base point. Since a Banach space X is separable in general if so is the dual space X^*, we see that $Z_0(R)$ is separable. Also there is a bijective bounded linear map between $Z(R)$ and $B(R^*)$ that maps $Z_0(R)$ onto $B_0(R^*)$ (see [4]). Hence $B_0(R^*)$ is also separable. Since the fiber T_o over the base point $\alpha(o) \in AT(R)$ is embedded into $B_0(R^*)$ by β, we see that T_o is separable. This is also valid for the fiber T_p over any $\alpha(p) \in AT(R)$.

The following lemma is a direct consequence from a well-known formula for Schwarzian derivatives and Proposition 4.2. This will be used later.

Lemma 4.3. *Let $\mathcal{B}_g : \mathrm{Bel}_1(g(R)) \to B(R^*)$ be the generalized Bers projection with respect to $[g] \in T(R)$. Then any $[f] \in T(g(R))$ satisfies*

$$\mathcal{B}(\mu_{f \circ g}) = \mathcal{B}(\mu_g) + \mathcal{B}_g(\mu_f).$$

If both $[g]$ and $[f]$ are asymptotically conformal, then $\mathcal{B}_g(\mu_f)$ vanishes at infinity.

Proof. Let G be the quasiconformal automorphism of $\hat{\mathbb{C}}$ determined by μ_g on Δ and let F be the quasiconformal automorphism of $\hat{\mathbb{C}}$ determined by μ_f on $G(\Delta)$.

Then the Cayley identity for Schwarzian derivatives

$$\mathcal{S}_{F\circ G|_{\Delta^*}}(z) = \mathcal{S}_{G|_{\Delta^*}}(z) + \mathcal{S}_{F|_{G(\Delta^*)}}(G(z))G'(z)^2 \quad (z \in \Delta^*)$$

implies the first assertion. If g and f are asymptotically conformal, then so is $f \circ g$. This implies that $\mathcal{B}(\mu_{f\circ g})$ vanishes at infinity by Proposition 4.2 as well as $\mathcal{B}(\mu_g)$ does. Hence so does $\mathcal{B}_g(\mu_f)$. □

§5. A common fixed point on the asymptotic Teichmüller space

There is an example of a Riemann surface R such that the quasiconformal mapping class group MCG(R) acts on the asymptotic Teichmüller space $AT(R)$ having a common fixed point and that MCG(R) is a countable group [12]. In this section, we prove that MCG(R) is always countable whenever MCG(R) acts on $AT(R)$ having a common fixed point. This is a consequence from the following theorem.

Theorem 5.1. *Suppose that* MCG(R) *has a common fixed point* $\alpha(p)$ *on* $AT(R)$ *and hence* MCG(R) *acts on the fiber* $T_p \subset T(R)$. *Then* MCG(R) *acts on* T_p *discontinuously, namely, for every* $q \in T_p$, *there exists a neighborhood* $U_q \subset T_p$ *of* q, *such that the number of elements* $[\gamma] \in$ MCG(R) *satisfying* $[\gamma]_*(U_q) \cap U_q \neq \emptyset$ *is finite.*

Corollary 5.2. *If* MCG(R) *has a common fixed point* $\alpha(p)$ *on* $AT(R)$, *then it is a countable group.*

Proof. If MCG(R) is uncountable, then the orbit of any point in T_p under MCG(R) has an accumulation point because T_p is separable. However, this contradicts the consequence of Theorem 5.1 that MCG(R) acts on T_p discontinuously. □

Corollary 5.2 is a generalization and a factorization of the following Lemma 5.3 proved in [7]. We use this result in the course of proving Theorem 5.1. Note that Lemma 5.3 can be proved by using Theorem 3.6, which is basically the same argument as in the proof of [7].

Lemma 5.3. *If* MCG(R) *has a common fixed point on* $AT(R)$, *then* R *satisfies the divergent geometry condition.*

We begin proving Theorem 5.1 here. We may assume that the common fixed point is the base point $\alpha(o)$ of $AT(R)$. Suppose that MCG(R) acts on T_o but not discontinuously. Then there exist a sequence of distinct elements $[\gamma_n] \in$ MCG(R) and a point $q \in T_o$ such that $[\gamma_n]_*(q)$ converge to q as $n \to \infty$. Without loss of generality, we may assume that q is the base point o of $T(R)$. We choose an extremal quasiconformal representative γ_n in each asymptotically conformal mapping class $[\gamma_n]$.

By Lemma 5.3, the Riemann surface R satisfies the divergent geometry condition. Then, for any closed geodesic $c \subset R$, each image $\gamma_n(c)$ is freely homotopic to one of a finite number of closed geodesics in R because the geodesic lengths of $\gamma_n(c)$ are close to the length of c. This implies that a subsequence of $\{\gamma_n\}$ converges locally uniformly to a conformal automorphism h of R. Then $\gamma_n \circ h^{-1}$ converge to id. Renaming $\gamma_n \circ h^{-1}$ by γ_n, we see that $[\gamma_n]_*(o) \to o$ and $[\gamma_n]_\# \to id$ as $n \to \infty$. See

Proposition 2.1. Then, by Corollary 2.3, we see that $[\gamma_n]$ is of divergence type for every sufficiently large n. Hence, for any constant $r > 0$, there exists an integer n_0 such that $[\gamma_n]$ is of divergence type and satisfies $0 < d_T([\gamma_n]_*(o), o) \leq r$ for every $n \geq n_0$. Since the orbit under $\langle [\gamma_n] \rangle$ is unbounded, there exists the least integer $k(n) \geq 1$ satisfying $r \leq d([\gamma_n]_*^{k(n)}(o), o)$. Since $d([\gamma_n]_*^{k(n)-1}(o), o) < r$, this implies $d([\gamma_n]_*^{k(n)}(o), o) \leq 2r$ as well.

Summing up, we have a sequence of asymptotically conformal mapping classes $[\gamma_n]$ and positive integers $k(n)$ such that $[\gamma_n]_\# \to id$ as $n \to \infty$ and that

$$0 < r \leq d([\gamma_n]_*^{k(n)}(o), o) \leq 2r$$

for each n. Here we set $[g_n] := [\gamma_n]^{k(n)}$, which is also an asymptotically conformal mapping class. No matter how the power $k(n)$ is dependent on n, we have $[g_n]_\# \to id$ as $n \to \infty$. This is because the condition $[\gamma_n]_\#|_{H'} = id$ for a subgroup $H' \subset \pi_1(R)$ implies $[g_n]_\#|_{H'} = id$. Then by Proposition 2.1, we have a sequence of quasisymmetric automorphisms $\bar{g}_n : \partial\Delta \to \partial\Delta$ corresponding to $[g_n]$ so that $\bar{g}_n$ converge to id uniformly on $\partial\Delta$.

Let $\tilde{g}_n$ be a quasiconformal automorphism of Δ obtained by the conformally barycentric extension of $\bar{g}_n$ and let $\mu_{\tilde{g}_n}$ be the complex dilatation of $\tilde{g}_n$. By a property of the conformally barycentric extension [2, Prop.2], $\tilde{g}_n$ converge to id uniformly, $\partial\tilde{g}_n$ converge to 1 locally uniformly and $\bar{\partial}\tilde{g}_n$ converge to 0 locally uniformly on Δ. Hence the complex dilatations $\mu_{\tilde{g}_n}$ in particular converge to 0 locally uniformly on Δ. On the other hand, as in [2, Prop.7], the maximal dilatation $K(\tilde{g}_n)$ of $\tilde{g}_n$ is estimated by

$$K(\tilde{g}_n) \leq A \exp\left\{Be^{4r}\right\} =: K_0,$$

where A and B are certain positive constants and r is the constant satisfying $d([g_n]_*(o), o) \leq 2r$.

Since $\tilde{g}_n$ are compatible with the Fuchsian group H, they project on R to be quasiconformal automorphisms g_n, which converge to id locally uniformly on R. Also the complex dilatations μ_{g_n} of g_n converge to 0 locally uniformly on R and the maximal dilatations $K(g_n)$ are bounded by the constant $K_0 \geq 1$. Since $[g_n]$ is an asymptotically conformal mapping class, the representative g_n defined by the conformally barycentric extension is asymptotically conformal by Proposition 4.2.

Lemma 5.4. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of asymptotically conformal automorphisms of R such that $g_n \to id$ and $\mu_{g_n} \to 0$ locally uniformly and that $K(g_n) \leq K_0$ for every n. Then, for any $\varepsilon > 0$, there exists a subsequence $\{g_{n_i}\}_{i\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that any composition of a finite number of elements in $\{g_{n_i}\}$ respecting the order is (K_0e^{ε})-quasiconformal.*

Proof. Fix a constant $\delta > 0$ arbitrarily. For a subset $V \subset R$ and a map $g : R \to R$ in general, we denote by $g(V)^\delta$ the open δ-neighborhood of $g(V)$ with respect to the hyperbolic distance d_H on R. Take any element g_{n_1} out of $\{g_n\}$. Since g_{n_1} is asymptotically conformal, there is a compact subset $V_1 \subset R$ such that $\log K(g_{n_1}|_{R-V_1}) \leq \varepsilon/2$.

Since $g_n \to id$ and $\mu_{g_n} \to 0$ uniformly on $g_{n_1}(V_1)^\delta$, there is $n_2 > n_1$ such that

$$\sup\{d_H(g_n(z), z) \mid z \in g_{n_1}(V_1)^\delta,\ n \geq n_2\} \leq \delta/4; \quad \log K(g_{n_2}|_{g_{n_1}(V_1)^\delta}) \leq \varepsilon/4.$$

Since g_{n_2} is asymptotically conformal, there is a compact subset $V_2 \subset R$ with $g_{n_1}(V_1)^\delta \cap V_2 = \emptyset$ such that $\log K(g_{n_2}|_{R-V_2}) \leq \varepsilon/4$.

Suppose that we have taken $g_{n_1}, \ldots, g_{n_i}$ and compact subsets $V_1, \ldots, V_i$ in R satisfying

$$\{g_{n_1}(V_1)^\delta \cup \cdots \cup g_{n_{i-1}}(V_{i-1})^\delta\} \cap V_i = \emptyset;\ \log K(g_{n_i}|_{R-V_i}) \leq \varepsilon/2^i.$$

Then there is $n_{i+1} > n_i$ such that

$$\begin{aligned} &\sup\{d_H(g_n(z), z) \mid z \in g_{n_1}(V_1)^\delta \cup \cdots \cup g_{n_i}(V_i)^\delta,\ n \geq n_{i+1}\} \leq \delta/2^{i+1}; \\ &\log K(g_{n_{i+1}}|_{g_{n_1}(V_1)^\delta \cup \cdots \cup g_{n_i}(V_i)^\delta}) \leq \varepsilon/2^{i+1}. \end{aligned}$$

Also there is a compact subset $V_{i+1} \subset R$ such that

$$\{g_{n_1}(V_1)^\delta \cup \cdots \cup g_{n_i}(V_i)^\delta\} \cap V_{i+1} = \emptyset;\ \log K(g_{n_{i+1}}|_{R-V_{i+1}}) \leq \varepsilon/2^{i+1}.$$

In this manner, we choose a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ inductively.

We take a finite number of elements out of $\{g_{n_i}\}$ and, for the sake of simplicity, assume them to be $g_{n_1}, g_{n_2}, \ldots, g_{n_k}$. Consider the composition $h = g_{n_k} \circ \cdots \circ g_{n_1}$. For any i and j $(1 \leq i < j \leq k)$, the compact subset V_j is disjoint from $g_{n_i}(V_i)^\delta$. Also $(g_{n_{j-1}} \cdots g_{n_i})(V_i)$ is contained in $g_{n_i}(V_i)^\delta$ because $\sum_{i=1}^\infty \delta/2^i = \delta$. Hence V_j and $(g_{n_{j-1}} \cdots g_{n_i})(V_i)$ are disjoint, from which we see that the k compact subsets

$$V_1, g_{n_1}^{-1}(V_2), \ldots, (g_{n_{i-1}} \cdots g_{n_1})^{-1}(V_i), \ldots, (g_{n_{k-1}} \cdots g_{n_1})^{-1}(V_k)$$

are mutually disjoint. For each factor g_{n_i}, its maximal dilatation is bounded by K_0 on V_i and by $\exp(\varepsilon/2^i)$ elsewhere. This contributes to the maximal dilatation $K(h)$ of the composition h by at most K_0 on $(g_{n_{i-1}} \cdots g_{n_1})^{-1}(V_i)$ and at most $\exp(\varepsilon/2^i)$ elsewhere. Hence $K(h)$ is bounded by $K_0 e^\varepsilon$ because $\sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$. □

The subsequence $\{g_{n_i}\}$ obtained by this lemma is renamed as $\{g_n\}$ hereafter. Remember that each g_n satisfies $d([g_n]_*(o), o) \geq r > 0$. We will further choose an infinite subsequence of $\{g_n\}$ so that their composition converges locally uniformly to a quasiconformal automorphism of R whose mapping class is not asymptotically conformal. To show that the limit is not asymptotically conformal, we use bounded holomorphic quadratic differentials obtained by the generalized Bers embedding.

Lemma 5.5. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of asymptotically conformal automorphisms of R with $d([g_n]_*(o), o) \geq r > 0$ such that $g_n \to id$ and $\mu_{g_n} \to 0$ locally uniformly and that any composition of a finite number of elements in $\{g_n\}$ respecting the order is uniformly K_1-quasiconformal for some $K_1 \geq 1$. Then there exists a subsequence $\{g_{n_j}\}$ of $\{g_n\}$ such that the composition $h_k := g_{n_k} \circ \cdots \circ g_{n_1}$ converges* *locally uniformly to a quasiconformal automorphism* h_∞ *of* R *as* $k \to \infty$ *such that the mapping class* $[h_\infty] \in \mathrm{MCG}(R)$ *is not asymptotically conformal.*

Proof. Since $g_n \to id$ locally uniformly and since any finite composition is uniformly K_1-quasiconformal, by passing to a subsequence of $\{g_n\}$ if necessary, we may assume that, for any subsequence $\{g_{n_j}\}$ of $\{g_n\}$, the composition $h_k = g_{n_k} \circ \cdots \circ g_{n_1}$ converges locally uniformly to a K_1-quasiconformal automorphism h_∞ of R. Hence we have only to choose the subsequence $\{g_{n_j}\}$ so that the mapping class $[h_\infty]$ is not asymptotically conformal for this limit h_∞.

Take arbitrarily g_{n_1} and consider $\varphi_1 = \mathcal{B}(\mu_{g_{n_1}})$. Since $d([g_{n_1}]_*(o), o) \geq r$, Proposition 4.1 asserts that $\|\varphi_1\| \geq c(1)b \geq c(K_1)b$, where $b := (e^{2r}-1)(e^{2r}+1)^{-1} > 0$. Also, since $h_1 := g_{n_1}$ is asymptotically conformal, Proposition 4.2 asserts that φ_1 vanishes at infinity. Set

$$W_1 = \left\{z \in R^* \mid \rho_{R^*}(z)^{-2}|\varphi_1(z)| \geq c(K_1)b/2^2\right\}.$$

Then this is a non-empty compact subset of R^*.

Consider the generalized Bers projection $\mathcal{B}_{h_1} : \mathrm{Bel}_1(h_1(R)) \to B(R^*)$ with respect to $[h_1]$. Since $\mu_{g_n} \to 0$ locally uniformly, $\mathcal{B}_{h_1}(\mu_{g_n})$ converge to 0 uniformly on W_1. Then there exists g_{n_2} with $n_2 > n_1$ such that $\varphi_2 := \mathcal{B}_{h_1}(\mu_{g_{n_2}})$ satisfies $\|\varphi_2\|_{W_1} < c(K_1)b/2^3$. It also satisfies $\|\varphi_2\| \geq c(K_1)b$ by Proposition 4.1. Since both h_1 and g_{n_2} are asymptotically conformal, Proposition 4.2 and Lemma 4.3 imply that φ_2 vanishes at infinity. Then

$$W_2 := \left\{z \in R^* \mid \rho_{R^*}(z)^{-2}|\varphi_2(z)| \geq c(K_1)b/2^3\right\}$$

is a non-empty compact subset of R^* disjoint from W_1. The composition $h_2 := g_{n_2} \circ g_{n_1}$ holds $\mathcal{B}(\mu_{h_2}) = \varphi_1 + \varphi_2$.

Similarly, there exists g_{n_3} with $n_3 > n_2$ such that $\varphi_3 := \mathcal{B}_{h_2}(\mu_{g_{n_3}})$ satisfies $\|\varphi_3\|_{W_1 \cup W_2} < c(K_1)b/2^4$ in addition to that $\|\varphi_3\| \geq c(K_1)b$ and $\varphi_3 \in B_0(R^*)$. Then

$$W_3 := \left\{z \in R^* \mid \rho_{R^*}(z)^{-2}|\varphi_3(z)| \geq c(K_1)b/2^4\right\}$$

is a non-empty compact subset of R^* disjoint from $W_1 \cup W_2$. The composition $h_3 := g_{n_3} \circ h_2 = g_{n_3} \circ g_{n_2} \circ g_{n_1}$ holds $\mathcal{B}(\mu_{h_3}) = \varphi_1 + \varphi_2 + \varphi_3$.

Inductively, assume that we have taken $h_{k-1} = g_{n_{k-1}} \circ \cdots \circ g_{n_1}$, $\varphi_1, \ldots, \varphi_{k-1} \in B_0(R^*)$ and $W_1, \ldots, W_{k-1} \subset R^*$. Then we choose g_{n_k} with $n_k > n_{k-1}$ such that $\varphi_k := \mathcal{B}_{h_{k-1}}(\mu_{g_{n_k}}) \in B_0(R^*)$ satisfies

$$\|\varphi_k\|_{W_1 \cup \cdots \cup W_{k-1}} < c(K_1)b/2^{k+1}; \quad \|\varphi_k\| \geq c(K_1)b.$$

We also set

$$W_k := \left\{z \in R^* \mid \rho_{R^*}(z)^{-2}|\varphi_k(z)| \geq c(K_1)b/2^{k+1}\right\},$$

which is a non-empty compact subset of R^* disjoint from $W_1 \cup \cdots \cup W_{k-1}$, and set

$$h_k := g_{n_k} \circ h_{k-1} = g_{n_k} \circ \cdots \circ g_{n_1},$$

which holds $\mathcal{B}(\mu_{h_k}) = \varphi_1 + \cdots + \varphi_k$.

The sequence $\{h_k\}$ converges locally uniformly to a quasiconformal automorphism h_∞ of R. Hence $\Phi_k := \sum_{j=1}^k \varphi_j = \mathcal{B}(\mu_{h_k})$ converge locally uniformly to $\Phi := \mathcal{B}(\mu_{h_\infty})$. In particular, φ_k converge locally uniformly to 0 as $k \to \infty$. This implies that the compact subsets $\{W_k\}$ exit to infinity, that is, for every compact subset $V \subset R^*$, there exists k_0 such that $W_k \cap V = \emptyset$ for every $k \geq k_0$.

We will show that Φ does not vanish at infinity, which implies that $[h_\infty]$ is not asymptotically conformal by Proposition 4.2. Each holomorphic quadratic differential φ_k satisfies $\|\varphi_k\|_{W_k} = \|\varphi_k\| \geq c(K_1)b$. On the other hand, since W_k is disjoint from any other W_m $(m \neq k)$, we see that $\|\varphi_m\|_{W_k} < c(K_1)b/2^{m+1}$. Hence Φ satisfies

$$\|\Phi\|_{W_k} > c(K_1)b\left(1 - \sum_{m=1}^{\infty} 1/2^{m+1}\right) = \frac{c(K_1)b}{2}$$

for every $k \in \mathbb{N}$. This implies that Φ does not vanish at infinity. □

Summing up all the above arguments, we complete the proof.

Proof of Theorem 5.1. We are assuming that all the elements of MCG(R) fix the base point of $AT(R)$. The conclusion of Lemma 5.5 that $[h_\infty] \in \mathrm{MCG}(R)$ is not asymptotically conformal has been derived from the assumption that MCG(R) does not act on T_o discontinuously. However, this is a contradiction, for such $[h_\infty]$ does not fix the base point. Thus we have proved that MCG(R) acts on T_o discontinuously. □

§6. Countability of the stabilizer subgroups

In the previous section, we have dealt with a particular quasiconformal mapping class group MCG(R) having a common fixed point on $AT(R)$ and investigated its properties. In this section, however, we will consider the stabilizer subgroup $\mathrm{MCG}_{\alpha(p)}(R)$ of MCG(R) in general, where

$$\mathrm{MCG}_{\alpha(p)}(R) = \{[\gamma] \in \mathrm{MCG}(R) \mid [\gamma]_* \circ \alpha(p) = \alpha(p)\}$$

for $\alpha(p) \in AT(R)$. We prove that $\mathrm{MCG}_{\alpha(p)}(R)$ is countable under the following assumption on hyperbolic geometry of R.

Definition. We say that a hyperbolic Riemann surface R satisfies *the lower bound condition* if the injectivity radius at every point of R is uniformly bounded away from zero except in cusp neighborhoods. We say that R satisfies *the upper bound condition* if there exists a subdomain $\check{R}$ of R such that the injectivity radius at every point of $\check{R}$ is uniformly bounded from above and such that the homomorphism $\pi_1(\check{R}) \to \pi_1(R)$ induced by the inclusion map $\check{R} \hookrightarrow R$ is surjective. We say that R satisfies *the bounded geometry condition* if R has no ideal boundary at infinity and if both the lower and upper bound conditions are satisfied.

For example, a non-universal normal cover R of a compact Riemann surface always satisfies the bounded geometry condition. More generally, if R admits a pants decomposition such that the lengths of geodesic boundary components of all pairs of pants in the decomposition are uniformly bounded from above and below, then R satisfies the bounded geometry condition. The bounded geometry condition and the divergent geometry condition are mutually exclusive except for analytically finite Riemann surfaces.

Theorem 6.1. *Assume that a Riemann surface R satisfies the bounded geometry condition. Then the stabilizer subgroup $\mathrm{MCG}_{\alpha(p)}(R)$ for any point $\alpha(p) \in AT(R)$ is countable.*

Proof. Suppose that $\mathrm{MCG}_{\alpha(p)}(R)$ is uncountable. Take a closed geodesic c in R. Since the closed geodesics in R are countably many, there exists a closed geodesic c' such that $\gamma(c)$ is freely homotopic to c' for uncountably many elements $[\gamma]$ in $\mathrm{MCG}_{\alpha(p)}(R)$. Since the subspace $T_p \subset T(R)$ is separable, there exists a sequence of distinct elements $[\gamma_n] \in \mathrm{MCG}_{\alpha(p)}(R)$ such that $\gamma_n(c)$ is freely homotopic to c' and that $[\gamma_n]_*(p)$ converge to some point $p' \in T_p$ as $n \to \infty$. However, when R satisfies the bounded geometry condition, there is no such sequence as is proved in [8]. Thus we see that $\mathrm{MCG}_{\alpha(p)}(R)$ is countable. □

In Section 5, we first show the discontinuity of the action and then prove the countability from it. However there is no converse way. In fact, even if $\mathrm{MCG}_{\alpha(p)}(R)$ is countable, it does not necessarily act discontinuously on T_p. A simple example can be seen if we assume R to satisfy the bounded geometry condition and choose p as a fixed point of some $[\gamma] \in \mathrm{MCG}(R)$ of infinite order, which is represented by a conformal automorphism γ of the Riemann surface corresponding to p.

§7. Countable but no common fixed point

In this section, we will show that the converse of Corollary 5.2 does not necessarily hold.

Theorem 7.1. *There exists a Riemann surface $\hat{R}$ such that $\mathrm{MCG}(\hat{R})$ is countable but $\mathrm{MCG}(\hat{R})$ does not have a common fixed point on $AT(\hat{R})$.*

In [12], we have obtained an analytically infinite Riemann surface R such that the quasiconformal mapping class group $\mathrm{MCG}(R)$ consists only of a countable number of elements and the base point $\alpha(o) \in AT(R)$ is a common fixed point of $\mathrm{MCG}(R)$. The Riemann surface $\hat{R}$ as in Theorem 7.1 is constructed from R by giving a certain amount of twist along infinitely many simple closed geodesics. Then $\mathrm{MCG}(\hat{R})$ remains countable but there exists an element of $\mathrm{MCG}(\hat{R})$ that moves $\alpha(o) \in AT(\hat{R})$ arbitrarily far away.

A pair of pants P is a hyperbolic surface with three geodesic boundary components that is homeomorphic to a three-punctured sphere. Every pair of pants admits the canonical orientation-reversing isometric involution. The fixed point loci of this involution consist of three geodesic segments, which we call the symmetry axes. Cutting along the symmetry axes, we have two congruent right-angled hexagons D. Let P_0 be a pair of pants the lengths of whose geodesic boundary components are $0!$ and $1!$ and $1!$. Let P_1 be a pair of pants with the lengths $1!$ and $2!$ and $2!$. In the same way, for every non-negative integer n, let P_n be a pair of pants with the lengths $n!$ and $(n+1)!$ and $(n+1)!$. The three symmetry axes σ_n divide P_n into two congruent right-angled hexagons D_n. The geodesic boundary components of length $n!$ and $(n+1)!$ in P_n are denoted by c_n and c_{n+1} respectively.

The Riemann surface R is made of 2^{n+1} copies of P_n for all $n \geq 0$ as follows. We take 2 copies of P_0 and glue the geodesic boundary component c_0 of each P_0 together. The resulting hyperbolic surface with 4 geodesic boundary components c_1 is denoted by R_1. Next take 4 copies of P_1 and glue the geodesic boundary component c_1 of each P_1 with the 4 boundary components of R_1. The resulting hyperbolic surface with 8 geodesic boundary components c_2 is denoted by R_2. Continuing this process, we obtain, for every $n \geq 1$, a hyperbolic surface R_n with 2^{n+1} geodesic boundary components c_n made of R_{n-1} and 2^n copies of P_{n-1}. Then we take the exhaustion of these surfaces R_n, which is $R = \bigcup_{n=1}^{\infty} R_n$. Each connected component of $R - \overline{R_n}$ is called an end neighborhood and is denoted by E_n. At each step of gluing, we give an appropriate amount of twist along c_n so that R is a complete hyperbolic surface without ideal boundary at infinity.

We choose an end neighborhood E_n for every $n \geq 1$ so that the family $\{E_n\}_{n\in\mathbb{N}}$ are mutually disjoint and denote it by E_n^*. Each E_n^* contains 2^i simple closed geodesics c_{n+i}. For $i \geq 2$, take the $(2^{i-1}-1)$-th one from the right and denote it by c'_{n+i}. The end neighborhood bounded by c'_{n+i} is denoted by E'_{n+i}. In E'_{n+i}, we take a simple closed geodesic c_m $(m > n+i)$ that has at least hyperbolic distance i away from c'_{n+i}. We denote this c_m by $c^*_{m(n,i)}$. Then, for each $n \geq 1$ and each $i \geq 2$, we give a twist of hyperbolic length n along $c^*_{m(n,i)}$ to the right. The resulting surface is our required $\hat{R}$. See Figure.

Proposition 7.2. *In the pair of pants P_n, let $\delta_n (\not\subset c_n)$ be the shortest geodesic arc that connect its boundary component c_n with itself and let $\delta'_n (\not\subset c_{n+1})$ be the shortest geodesic arc that connect its boundary component c_{n+1} with itself. Then their hyperbolic lengths $\ell(\delta_n)$ and $\ell(\delta'_n)$ satisfy*

$$\ell(\delta_n) > n! \times n; \quad \ell(\delta'_n) > \frac{n!}{2} + 1.$$

Proof. The estimate for $\ell(\delta_n)$ is given in [12, Prop.5] by using trigonometry on the right-angled hexagon D_n. We will prove the estimate for $\ell(\delta'_n)$ in a similar way. It is easy to see that the half of δ'_n in D_n is the geodesic segment connecting c_{n+1} and σ_n perpendicularly. Then by trigonometry on the right-angled pentagon (cf. [1, p.37]), we have

$$\begin{aligned}\cosh\frac{\ell(\delta'_n)}{2} &= \sinh\frac{n!}{2}\cdot\sinh\ell(\sigma_n)\\ &> \sinh\frac{n!}{2}\cdot\frac{1}{\sinh(n!/4)} = 2\cosh\frac{n!}{4} > \cosh\frac{n!+2}{4}.\end{aligned}$$

Here we have used an estimate of $\sinh\ell(\sigma_n)$ from below, which was given by [12, Prop.3]. Hence we have $\ell(\delta'_n) > (n!/2) + 1$. □

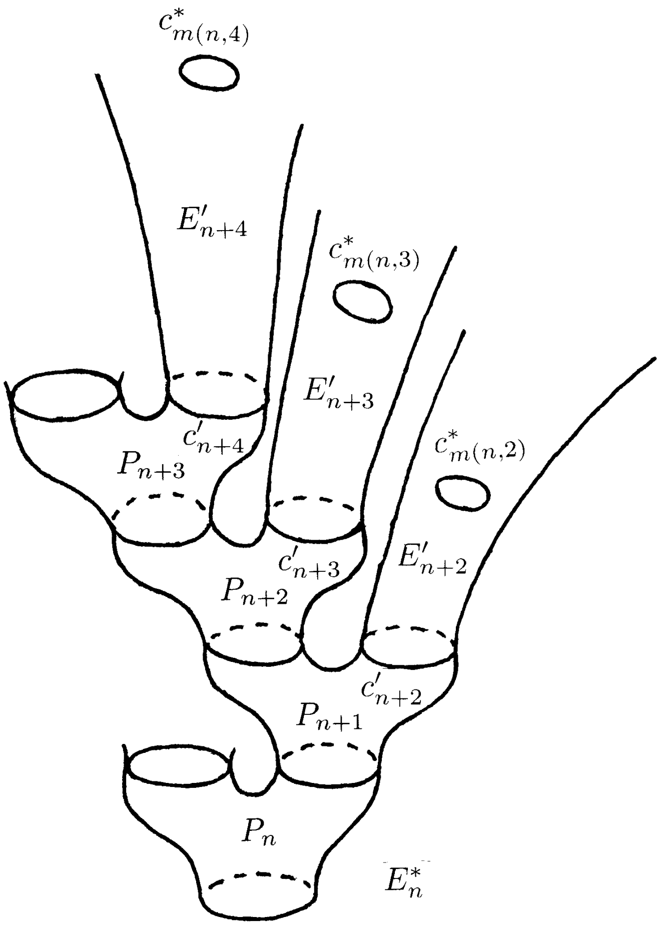

FIGURE. The end neighborhood E^*_n in $\hat{R}$

Proposition 7.3. *The simple closed geodesic $c^*_{m(n,i)}$ in R, as well as in $\hat{R}$, has a collar $A^*_{m(n,i)}$ of width i that is contained in $E'_{n+i} \subset E^*_n$. Moreover $A^*_{m(n,i)} \cap A^*_{m(n,j)} = \emptyset$ for $i \neq j$.*

Proof. Consider the shortest geodesic arc $\delta(\not\subset c^*_{m(n,i)})$ connecting $c^*_{m(n,i)}$ with itself. We will show that $\ell(\delta) > 2i$, which implies that $c^*_{m(n,i)}$ has a collar neighborhood $A^*_{m(n,i)}$ of width i in both directions.

If δ lies outside $c^*_{m(n,i)}$, namely in $E_{m(n,i)}$, then δ contains a subarc in P_k for some $k \geq m(n,i) \geq n+i$ whose endpoints are on c_k. By Proposition 7.2, $\ell(\delta) > k! \times k \geq 2i$. If δ lies in the other side of $c^*_{m(n,i)}$ and if $\ell(\delta) \leq 2i$, then δ is in E'_{n+i} and δ contains a subarc either in $P_{k'}$ for some $k' \geq n+i$ (where $m(n,i)-1 \geq k'$) whose endpoints are on $c_{k'+1}$ or in P_k for some $k \geq n+i+1$ whose endpoints are on c_k. The estimate for the latter case is the same as before. In the former case, $\ell(\delta) > (k'!/2)+1 \geq 2i$ for $i \geq 2$ again by Proposition 7.2.

Since the distance between c'_{n+i} and $c^*_{m(n,i)}$ is more than i, the collar $A^*_{m(n,i)}$ is contained in $E'_{n+i} \subset E^*_n$. In particular, $E'_{n+i} \cap E'_{n+j} = \emptyset$ implies $A^*_{m(n,i)} \cap A^*_{m(n,j)} = \emptyset$ for $i \neq j$. $\square$

The Riemann surface $\hat{R}$ is obtained from R by giving the twist of hyperbolic length n along all $c^*_{m(n,i)}$ ($n \geq 1,\ i \geq 2$). By (the proof of) Theorem 3.1, there exists a locally quasiconformal homeomorphism $f : R \to \hat{R}$ such that f is K_n-quasiconformal on each E^*_n and is conformal outside $\bigcup E^*_n$, where

$$K_n = \left\{1 + \frac{n}{2\arctan(\sinh 2)}\right\}^2.$$

Lemma 7.4. *The quasiconformal mapping class group* $\mathrm{MCG}(\hat{R})$ *for the Riemann surface* $\hat{R}$ *is countable.*

Proof. The Riemann surfaces R and $\hat{R}$ consist of the same pairs of pants P_n as building blocks with the same combination. The argument in [12] for proving that $\mathrm{MCG}(R)$ is countable is also applicable to seeing that the quasiconformal automorphisms of $\hat{R}$ modulo free homotopy, which is also known as the reduced mapping class group, is countable. If $\hat{R}$ has no ideal boundary at infinity, then the free homotopy and the homotopy relative to the ideal boundary at infinity are coincident, which means that the reduced mapping class group is nothing but $\mathrm{MCG}(\hat{R})$.

To show that $\hat{R}$ does not have ideal boundary at infinity, suppose to the contrary that it does have. Then there is a geodesic line β that bounds a simply connected domain $\hat{B}$ together with ideal boundary at infinity. Since $\hat{B}$ has no intersection with the closed geodesics in $\hat{R}$, it should be contained in one of $\{E^*_n\}$ or in some end neighborhood E_m outside $\bigcup E^*_n$. Since f^{-1} is quasiconformal on this end neighborhood, the corresponding $f^{-1}(E^*_n)$ or $f^{-1}(E_m)$ in R must contain a simply connected domain $f^{-1}(\hat{B})$ facing to ideal boundary at infinity. However, this contradicts the fact that R has no ideal boundary at infinity. □

For a half Dehn twist $[g_n] \in \mathrm{MCG}(R)$ along ∂E^*_n, let $c^{**}_{m(n,i)}$ be the simple closed geodesic in R freely homotopic to $g_n(c^*_{m(n,i)})$ and $A^{**}_{m(n,i)}$ the collar of $c^{**}_{m(n,i)}$ with the width i. Let A_n be a collar of ∂E^*_n disjoint from all $A^*_{m(n,i)}$ and $A^{**}_{m(n,i)}$. The corresponding simple closed geodesics and their collars in $\hat{R}$ under $f : R \to \hat{R}$ are written by the same notations. Consider $f \circ g_n \circ f^{-1}$ of $\hat{R}$. Actually, its homotopy class can be represented by a quasiconformal automorphism whose support is on

$$A_n \sqcup \bigcup_{i \geq 2} (A^*_{m(n,i)} \sqcup A^{**}_{m(n,i)}),$$

which we denote by the same letter g_n. Hence we have $[g_n] \in \mathrm{MCG}(\hat{R})$ for every $n \geq 1$.

Lemma 7.5. *For the half Dehn twist* g_n *along* ∂E^*_n *in* $\hat{R}$*, the asymptotic Teichmüller distance between the base point* $\alpha(o) \in AT(\hat{R})$ *and its image* $[g_n]_*(\alpha(o))$ *satisfies*

$$d_{@}([g_n]_*(\alpha(o)), \alpha(o)) \geq \frac{1}{2} \log\{1 + (n/\pi)^2\}.$$

Proof. By Theorem 3.6, we see that the boundary dilatation $H([g_n])$ for $[g_n] \in \mathrm{MCG}(\hat{R})$ satisfies

$$H([g_n]) \geq \limsup_{i\to\infty} \left\{ 1 + \left(\frac{n - 2b(i)}{\pi} \right)^2 \right\}.$$

This yields the assertion. □

Proof of Theorem 7.1. By Lemma 7.4, the quasiconformal mapping class group $\mathrm{MCG}(\hat{R})$ is countable. Suppose that $\mathrm{MCG}(\hat{R})$ has a common fixed point $\alpha(p) \in AT(\hat{R})$. This in particular implies that $d_{@}([g_n]_*(\alpha(p)), \alpha(p)) = 0$ for the half Dehn twist $[g_n]$ for every $n \geq 1$. Then we have

$$\begin{aligned} & d_{@}([g_n]_*(\alpha(o)), \alpha(o)) \\ \leq & d_{@}([g_n]_*(\alpha(o)), [g_n]_*(\alpha(p))) + d_{@}([g_n]_*(\alpha(p)), \alpha(p)) + d_{@}(\alpha(p), \alpha(o)) \\ = & 2 d_{@}(\alpha(p), \alpha(o)). \end{aligned}$$

However, this contradicts Lemma 7.5 when $n \to \infty$. Hence $\mathrm{MCG}(\hat{R})$ has no common fixed point on $AT(\hat{R})$. □

§8. Divergent geometry but not countable

We construct a Riemann surface S satisfying the properties mentioned in the following theorem by gluing certain pieces along their geodesic boundaries.

Theorem 8.1. *There exists a Riemann surface S such that S satisfies the divergent geometry condition but* $\mathrm{MCG}(S)$ *is not countable.*

First, we prepare hyperbolic surfaces with geodesic boundaries as in the following proposition. Since we can actually make them in various manners, for instance, by a similar construction as in [12], we omit a proof for their existence.

Proposition 8.2. (1) *For every integer $n \geq 2$, there exists a hyperbolic surface X_n with one boundary component x_n but without ideal boundary at infinity such that the number of closed geodesics in X_n whose lengths are less than L is finite for every $L > 0$ and such that x_n is a closed geodesic of the shortest length n in X_n with an inward half-collar of width n.* (2) *There exists a hyperbolic surface Y with infinitely many boundary components $\{y_n\}_{n\geq 2}$ but without ideal boundary at infinity such that the number of closed geodesics in Y whose lengths are less than L is finite for every $L > 0$ and such that each y_n is a closed geodesics of length n with a mutually disjoint inward half-collar of width n.*

For every integer $n \geq 2$, take a compact hyperbolic cone surface Z'_n of no genus, one cone point z'_n of branch number n, two geodesic boundary components x'_n of length n and y'_n of length 1. By the collar lemma (cf. [1, Chap.4]), y'_n has an inward collar of width $\omega := \operatorname{arcsinh}\{1/\sinh(1/2)\} \fallingdotseq 1.4$. Then taking an n-sheeted branched cover of Z'_n, we have a compact hyperbolic surface Z_n with geodesic boundary components $\{x^i_n\}_{1\leq i\leq n}$ of length n and y_n of length n. It admits an isometric automorphism h_n of Z_n corresponding to the branched cover that fixes the branch point z_n, preserves y_n and permutes $\{x^i_n\}$ cyclically. Note that the geodesic boundary y_n also has an inward collar of the width ω in Z_n.

Proposition 8.3. *The length of the shortest closed geodesic in Z_n is n.*

Proof. The cone surface Z'_n admits the canonical isometric involution and its symmetric half is a hyperbolic pentagon Q_n with four right angles and an angle of π/n. Then by trigonometry on Q_n (cf. [1, p.37]), length b of the side of Q_n opposite to the angle of π/n satisfies

$$\cosh b \sinh \frac{n}{2} \sinh \frac{1}{2} = \cosh \frac{n}{2} \cosh \frac{1}{2} + \cos \frac{\pi}{n}.$$

Let $\tilde{Q}_n$ be the double of Q_n with respect to the side between z'_n and y'_n, which is a hyperbolic polygon of seven sides. Consider the shortest geodesic segment of length a in $\tilde{Q}_n$ that connects the side of length b and the side between z'_n and x'_n. Then we have a right-angled hexagon in $\tilde{Q}_n$, which has a pair of opposite sides of lengths a and b. Again by trigonometry on the right-angled hexagon (cf. [1, Th.2.4.2]), we have

$$\cosh b = \frac{\cosh a + \cosh(n/2)\cosh 1}{\sinh(n/2)\sinh 1}.$$

These two equalities yield

$$\cosh a = \cosh \frac{n}{2} + 2\cosh \frac{1}{2} \cos \frac{\pi}{n},$$

which implies $a > n/2$. By elementary geometric consideration, this is enough to see that any closed geodesic in Z_n other than x_n^i and y_n has length greater than n. □

Along each geodesic boundary y_n of Y $(n \geq 2)$, we glue Z_n identifying y_n. Then along all geodesic boundary component x_n^i of Z_n $(1 \leq i \leq n)$ for each $n \geq 2$, we glue n copies of X_n identifying x_n with x_n^i. The resulting hyperbolic Riemann surface is our required S.

Lemma 8.4. *The Riemann surface S satisfies the divergent geometry condition.*

Proof. By Propositions 8.2 and 8.3, for any $L > 0$, the number of closed geodesics contained in X_n, Y or Z_n $(n \geq 2)$ whose lengths are less than L is finite. Since the boundary components x_n in X_n and y_n in Y have inward half-collars of width n, closed geodesics of length less than L that intersect these boundary components are also finitely many. Thus the assertion follows. □

Proof of Theorem 8.1. By Lemma 8.4, the Riemann surface S satisfies the divergent geometry condition. Hence we have only to show that $\mathrm{MCG}(S)$ is not countable. Each simple closed geodesic y_n has a collar $A(y_n, 1) \subset S$ of width 1 such that $A(y_n, 1) \cap A(y_m, 1) = \emptyset$ for $n \neq m$. For any subset $I \subset \{n \geq 2\}$, let g_I be a locally quasiconformal automorphism of S obtained by twists of length 1 along all y_n for $n \in I$. Then by Theorem 3.1, the maximal dilatation of $[g_I]$ satisfies

$$K([g_I]) \leq \left\{1 + \frac{1}{2\arctan(\sinh 1)}\right\}^2,$$

and in particular $[g_I]$ belongs to $\mathrm{MCG}(S)$. Since such subsets I exist uncountably many, so do the elements $[g_I] \in \mathrm{MCG}(S)$. □

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