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Convergence of the Hausdorff dimension of the limit sets of Kleinian groups

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Abstract. We consider convergence of the Hausdorff dimension of limit sets for an algebraically convergent sequence of Kleinian groups, and prove that if the limit sets converge to a closed set Λ with the Hausdorff dimension not less than 1 in the Hausdorff topology then their Hausdorff dimensions converge to that of Λ.

§1. Backgrounds and statement of results

A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, the group of all orientation preserving isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$. It acts on $\mathbb{H}^3$ properly discontinuously. In this paper, we always assume that Γ is *torsion-free*. Then it acts on $\mathbb{H}^3$ freely and the quotient space $N_\Gamma = \mathbb{H}^3/\Gamma$ is a complete hyperbolic manifold.

In the unit ball model B^3 of the hyperbolic space, $\mathrm{Isom}^+(\mathbb{H}^3)$ is nothing but the group of all orientation preserving Möbius transformations that preserve B^3. The boundary S^2 of B^3 is regarded as the ideal boundary of the hyperbolic space and called the sphere at infinity. Since Möbius transformations act also on S^2, the action of a Kleinian group Γ extends to the sphere at infinity in this way. We define the largest open subset $\Omega(\Gamma)$ of S^2 where Γ acts properly discontinuously to be the region of discontinuity. The complement $S^2 - \Omega(\Gamma)$ is denoted by $\Lambda(\Gamma)$, which is called the limit set of Γ. If the cardinality of $\Lambda(\Gamma)$ is not greater than 2 then Γ is called elementary. In the case that Γ is torsion-free (which is always assumed in this paper), this is equivalent to saying that Γ is abelian. Throughout this paper, we assume that a Kleinian group is *non-elementary*.

The convex hull $H(\Lambda)$ of a closed subset Λ of S^2 is the smallest, convex, closed subset in B^3 that contains every geodesic line connecting any distinct two points of Λ. If Λ is invariant under a Kleinian group Γ then so is $H(\Lambda)$ and thus it projects to a closed, convex set in the hyperbolic manifold N_Γ. This is denoted by $C_\Gamma(\Lambda)$. The limit set $\Lambda(\Gamma)$ is the smallest, non-empty, Γ-invariant, closed set in S^2. If we take $\Lambda(\Gamma)$ as a Λ, $C_\Gamma(\Lambda(\Gamma))$ is the smallest, convex, closed set in N_Γ such that the

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inclusion map is a homotopy equivalence. This is called the convex core of N_Γ and denoted by C_Γ.

Each component Σ of the boundary $\partial C_\Gamma(\Lambda)$ can be regarded as a hyperbolic pleated surface. Namely, there is an isometric map $f : S \to N_\Gamma$ from a complete hyperbolic surface S onto Σ such that every point $x \in S$ is in the interior of some geodesic arc that is mapped by f to a geodesic arc in N_Γ. Here "isometric" means that the map f takes each rectifiable path on S to a rectifiable path of the same length. A pleated surface Σ is divided into two parts: the pleating locus is the set of those points of Σ that are in the interior of one and only one geodesic arc contained in Σ; the flat pieces are the complement of the pleating locus. The area $\mathrm{Area}\,(\Sigma)$ of the pleated surface Σ is, by definition, the area of the hyperbolic surface S. This is equal to the total area of the flat pieces of Σ when $\mathrm{Area}\,(\Sigma) < \infty$. For details, consult [6] and [9].

If a Kleinian group Γ is finitely generated and the volume $\mathrm{Vol}\,(C_\Gamma)$ of the convex core is finite, we say that Γ is geometrically finite. Due to the Ahlfors finiteness theorem, which will be explained later, this is equivalent to saying that both $\mathrm{Vol}\,(C_\Gamma)$ and $\mathrm{Area}\,(\partial C_\Gamma)$ are finite. If Γ satisfies the condition $\mathrm{Area}\,(\partial C_\Gamma) < \infty$ alone, we say that Γ is analytically finite.

For a separable metric space X in general, the Hausdorff topology on the space $Cl(X)$ of all closed subsets of X is defined so that F_n converge to F in $Cl(X)$ if the following conditions are satisfied: (a) every neighborhood of a point $x \in F$ intersects all but finitely many F_n; and (b) if every neighborhood of x intersects infinitely many F_n then $x \in F$. It is known that $Cl(X)$ is sequentially compact in the Hausdorff topology. Further suppose that X is compact. Then the Hausdorff distance d between two sets F_1 and F_2 in $Cl(X)$ can be defined as

$$d(F_1, F_2) = \inf\{\delta > 0 \mid F_i \subset \mathcal{N}_\delta(F_{3-i}) \text{ for } i = 1, 2\},$$

where $\mathcal{N}_\delta(F)$ is the δ-neighborhood of F. The topology induced by this distance coincides with the Hausdorff topology and it makes $Cl(X)$ a compact metric space.

We consider sequences of Kleinian groups. Let $\rho_n : \Gamma_0 \to \mathrm{PSL}_2(\mathbb{C})$ be homomorphisms of a finitely generated group Γ_0. We say that the ρ_n converge algebraically to a homomorphism $\rho_\infty : \Gamma_0 \to \mathrm{PSL}_2(\mathbb{C})$ if $\rho_n(\gamma)$ converge to $\rho_\infty(\gamma)$ for every $\gamma \in \Gamma_0$. If $\Gamma_n = \rho_n(\Gamma_0)$ are Kleinian then $\Gamma_\infty = \rho_\infty(\Gamma_0)$ is also Kleinian and it is called the algebraic limit of the sequence. On the other hand, we say that a sequence of subgroups $\Gamma_n \subset \mathrm{PSL}_2(\mathbb{C})$ converges geometrically to a subgroup $G \subset \mathrm{PSL}_2(\mathbb{C})$ if Γ_n converge to G in the Hausdorff topology on the space of all closed subsets of $\mathrm{PSL}_2(\mathbb{C})$. Since $Cl(\mathrm{PSL}_2(\mathbb{C}))$ is sequentially compact, we can draw a geometrically convergent subsequence from any sequence of subgroups. If a sequence of Kleinian groups Γ_n (together with $\rho_n : \Gamma_0 \to \Gamma_n$) has an algebraic limit Γ_∞ and converges geometrically to G, then the geometric limit G contains Γ_∞ and it is also a Kleinian group. See [14, Th.7.12].

Jørgensen discovered the phenomenon of geometric convergence of Kleinian groups and gave some fundamental observations on it, including an example where an algebraic and a geometric limit differ. We may refer to Jørgensen and Marden [12] on these results. Thurston [20] developed the concept of geometric convergence in a more geometric way and used it to study infinite ends of hyperbolic manifolds.

We take the Hausdorff limit of limit sets of Kleinian groups. As is mentioned above, for any sequence of Kleinian groups Γ_n, there exists a subsequence of Γ_n that converges geometrically to a Kleinian group G. Also, by passing to a further subsequence, we may assume that the limit sets $\Lambda(\Gamma_n)$ converge to a closed set Λ of S^2 in the Hausdorff topology. Then we have the following fact about the limit set $\Lambda(G)$ of the geometric limit and the Hausdorff limit Λ of the limit sets. A proof can be found in [19, Lem.7.1].

LEMMA 1. *Λ is invariant under G and Λ contains $\Lambda(G)$.*

In this paper, we consider convergence of the Hausdorff dimensions $\dim \Lambda(\Gamma_n)$ of the limit sets of an algebraically convergent sequence of Kleinian groups Γ_n. Regarding this subject, lower semi-continuity of the Hausdorff dimension was proved by Bishop and Jones [2, Th.1.6].

THEOREM 2. *Let $\rho_n : \Gamma_0 \to \Gamma_n$ be homomorphisms of a finitely generated group Γ_0 onto Kleinian groups Γ_n which converge algebraically to a surjective homomorphism $\rho_\infty : \Gamma_0 \to \Gamma_\infty$. Then*

$$\liminf_{n\to\infty} \dim \Lambda(\Gamma_n) \geq \dim \Lambda(\Gamma_\infty).$$

From this result, we immediately see that if the limit set $\Lambda(\Gamma_\infty)$ has the maximal Hausdorff dimension (= 2) then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda(\Gamma_\infty)$. For example, this is true whenever $\Lambda(\Gamma_\infty) = S^2$. The criterion for the limit set to have the maximal Hausdorff dimension was given also by Bishop and Jones [2, Th.1.2 & Cor.1.3].

THEOREM 3. *If an analytically finite Kleinian group G is not geometrically finite then $\dim \Lambda(G) = 2$. Moreover, for an analytically finite Kleinian group G with $\Omega(G) \neq \emptyset$, $\dim \Lambda(G) < 2$ if and only if G is geometrically finite.*

Therefore the strict inequality of the conclusion in Theorem 2 may occur only if the algebraic limit Γ_∞ is geometrically finite, and this actually occurs as the following well-known example shows:

EXAMPLE 1. On the Bers boundary of the Teichmüller space of a cofinite area Fuchsian group Γ_0, representations $\rho : \Gamma_0 \to \Gamma$ onto totally degenerate Kleinian groups are dense (See Bers [1]. Also cf. [14, §4.3.3]). Hence there is a sequence of isomorphisms $\rho_n : \Gamma_0 \to \Gamma_n$ onto geometrically infinite boundary groups that converges algebraically to an isomorphism $\rho_\infty : \Gamma_0 \to \Gamma_\infty$, where Γ_∞ is a geometrically finite boundary group. In this case, $\dim \Lambda(\Gamma_n) = 2$ but $\dim \Lambda(\Gamma_\infty) < 2$ by Theorem 3.

Canary and Taylor [7] obtained a sufficient condition for continuity of the Hausdorff dimension under algebraic convergence by considering the case that the algebraic limit Γ_∞ is coincident with the geometric limit of the sequence Γ_n. McMullen [15] also studied this problem and obtained complete results on the convergence of the Hausdorff dimension by extending the concept of strong convergence more widely.

Our main theorem of this paper is based on the aforementioned works by Jørgensen and Marden, by Bishop and Jones, by Canary and Taylor and by McMullen, and it asserts the continuity of the Hausdorff dimension by replacing the limit set $\Lambda(\Gamma_\infty)$ of the algebraic limit with a larger set Λ which is the Hausdorff limit of the limit sets $\Lambda(\Gamma_n)$.

Main Theorem. *Let $\rho_n : \Gamma_0 \to \Gamma_n$ be an algebraically convergent sequence of homomorphisms of a finitely generated group Γ_0 onto Kleinian groups Γ_n. Suppose that the limit sets $\Lambda(\Gamma_n)$ converge to a closed subset Λ of S^2 in the Hausdorff topology. Further suppose that $\dim \Lambda \geq 1$. Then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda$.*

Although a proof of this theorem will not be completed until Section 5, we give a brief sketch of it here. We consider a subsequence of Γ_n that converges geometrically to a Kleinian group G and divide our arguments into three cases: (i) G is geometrically finite; (ii) G is analytically finite but not geometrically finite; and (iii) G is not analytically finite.

If G is geometrically finite, then the convergence of the Kleinian groups is strong in the sense of the definition in Section 2 and we apply Theorem 5 due to McMullen [15], which concludes convergence of the Hausdorff dimensions $\dim \Lambda(\Gamma_n)$ to $\dim \Lambda(G)$ as well as convergence of the limit sets $\Lambda(\Gamma_n)$ to $\Lambda(G)$ in the Hausdorff topology. This proves the statement of the main theorem in the first case. The assumption $\dim \Lambda \geq 1$ in the main theorem is essential because Theorem 5 requires the same assumption to conclude the convergence of the Hausdorff dimension.

If G is analytically finite but not geometrically finite, then we rely on Theorem 3. Since $\Lambda \supset \Lambda(G)$ by Lemma 1, this implies that $\dim \Lambda = 2$. We also prove that if G is not geometrically finite then $\dim \Lambda(\Gamma_n)$ should converge to 2 (Lemma 14 in Section 5). Hence we prove the main theorem in the second case.

If G is not analytically finite then the Hausdorff limit Λ should have non-empty interior (Lemma 8 in Section 3). This in particular implies that $\dim \Lambda = 2$ and thus proves the main theorem also in the third case.

§2. Strong convergence

In this section, we review a critical result for proving our main theorem, concerning convergence of the Hausdorff dimension of limit sets under strong convergence. Strong convergence was originally defined for an algebraically convergent sequence of homomorphisms and by the coincidence of geometric limits (of subsequences) with the algebraic limit. In addition, it was usually required that the homomorphisms were injective. The following definition of strong convergence due to McMullen [15] is rather a condition on geometrically convergent sequences and the injectivity is not required any more.

Definition. We say that a sequence of Kleinian groups Γ_n converges strongly to a finitely generated Kleinian group G if Γ_n converge geometrically to G and if there exist surjective homomorphisms $\psi_n : G \to \Gamma_n$ for all sufficiently large n that converge algebraically to the identity isomorphism $id : G \to G$.

The following fact is essentially due to Jørgensen and Marden [12, Prop.3.8], which asserts that any algebraically convergent sequence is strongly convergent in the sense of the definition above if the geometric limit exists and it is finitely generated. See also [14, Prop.7.13].

Proposition 4. *Suppose that a sequence of Kleinian groups Γ_n converges geometrically to a finitely generated Kleinian group G. Then for all sufficiently large n there exist homomorphisms $\psi_n : G \to \Gamma_n$ not necessarily surjective that converge algebraically to the identity isomorphism $id : G \to G$. In addition, if Γ_n are images of homomorphisms ρ_n of a finitely generated group Γ_0 that converge algebraically to*

ρ_∞ then $\rho_n = \psi_n \circ \rho_\infty$ for sufficiently large n, and in particular ψ_n are surjective. Namely Γ_n converge strongly to G.

PROOF. Take a finite generator system $\langle g_1, \dots, g_k \rangle$ of G. Since Γ_n converge geometrically to G, there are elements $\gamma_{i,n} \in \Gamma_n$ for each $i = 1, \dots, k$ such that $\lim_{n\to\infty} \gamma_{i,n} = g_i$. Then we define $\psi_n(g_i) = \gamma_{i,n}$.

Now we want to extend ψ_n to the entire group G. Since G is finitely generated, it is finitely presented [17]. This means that we can choose a finite relation system R with respect to $\{g_1, \dots, g_k\}$. Let r be a word in R, that is $r(g_1, \dots, g_k) = 1$. Then $r(\psi_n(g_1), \dots, \psi_n(g_k)) \in \Gamma_n$ converge to 1. Since no sequence of non-trivial elements $\gamma_n \in \Gamma_n$ can converge to 1, which is essentially due to the Jørgensen inequality (cf. [14, Th.7.1]), there is an integer n_1 such that $r(\psi_n(g_1), \dots, \psi_n(g_k)) = 1$ for any n greater than n_1. Since the number of words in R is finite, there is an integer n_0 such that $r(\psi_n(g_1), \dots, \psi_n(g_k)) = 1$ for any $r \in R$ and for all n greater than n_0. This implies that we can extend ψ_n $(n > n_0)$ to G in such a well-defined way that

$$\psi_n(w(g_1, \dots, g_k)) = w(\psi_n(g_1), \dots, \psi_n(g_k))$$

for any word w. Then clearly ψ_n are homomorphisms into Γ_n and they converge algebraically to id.

Moreover suppose that there are surjective homomorphisms ρ_n of a finitely generated group Γ_0 onto Γ_n that converge algebraically to ρ_∞. Since the algebraic limit $\Gamma_\infty = \rho_\infty(\Gamma_0)$ is contained in G, we can consider homomorphisms $\psi_n \circ \rho_\infty$ of Γ_0 for $n > n_0$. Take a finite generator system $\langle \gamma_1, \dots, \gamma_l \rangle$ of Γ_0. Since $\rho_n(\gamma_j)$ and $\psi_n(\rho_\infty(\gamma_j))$ both converge to $\rho_\infty(\gamma_j)$ $(j = 1, \dots, l)$, they must be eventually coincident by the same reasoning as above. Running j over $1 \le j \le l$, we obtain that $\rho_n = \psi_n \circ \rho_\infty$ for all sufficiently large n. □

Critical consequences of strong convergence in the case that the limit is a geometrically finite Kleinian group are in the following Theorem 5, which has been given by McMullen [15, Th.1.1 & 1.2]. Here we note that the first conclusion in the statement was proved earlier by Jørgensen and Marden [12, Prop.4.7] and similar results to the second conclusion have been proved by Canary and Taylor [7, Main Th.] and by Fan and Jorgenson [10, Th.3.5] independently and by different methods.

THEOREM 5. *Suppose that a sequence of Kleinian groups Γ_n converges strongly to a geometrically finite Kleinian group G. Then $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology. If, in addition, $\dim \Lambda(G) \ge 1$ then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda(G)$.*

In the latter statement of Theorem 5, the assumption $\dim \Lambda(G) \ge 1$ is essential: if we drop it, the statement is not valid any longer. In fact, McMullen [15, Th.1.3 & §8] constructed the following example of discontinuity of the Hausdorff dimension. Actually, the Hausdorff dimension was replaced with the critical exponent in his example, however they are coincident for a geometrically finite (non-elementary) Kleinian group.

EXAMPLE 2. Let Γ_t be a geometrically finite Kleinian group generated by a parabolic transformation $z \mapsto z/(Rz+1)$ $(R > 4)$ and a loxodromic transformation $z \mapsto e^{2\pi i t}z+1$ where t is sufficiently close to 0 in the upper half plane $\mathbb{H}^2$. Let Γ_0 be a geometrically finite Kleinian group generated by $z \mapsto z/(Rz+1)$ and $z \mapsto z+1$.

The limit set $\Lambda(\Gamma_0)$ is a totally disconnected set in the real axis and its Hausdorff dimension $\dim \Lambda(\Gamma_0)$ is less than 1 and greater than 1/2 depending on R. According to another result [15, Th.5.1] in the same paper, Γ_t converge strongly to Γ_0 if and only if $t \to 0$ horocyclically.

On the other hand, as is in Jørgensen and Marden [12, §5], there exists a sequence $t_n \to 0$ such that the cyclic groups $\langle z \mapsto e^{2\pi i t_n} z+1 \rangle$ converge geometrically to a parabolic abelian group J of rank 2. According to [15, Th.5.1] again, for any horocycle $C \subset \mathbb{H}^2$ tangential to the real axis at 0, we can choose such a sequence $\{t_n\}$ on C. Then Γ_{t_n} also converge geometrically to a geometrically finite Kleinian group G which contains J and hence satisfies $\dim \Lambda(G) > 1$.

Similar to the case of algebraic convergence (Theorem 2), the Hausdorff dimension satisfies lower semi-continuity under geometric convergence to a geometrically finite Kleinian group. See [15, Th.7.7].

THEOREM 6. *If a sequence of Kleinian groups Γ_n converges geometrically to a geometrically finite Kleinian group G then*

$$\liminf_{n\to\infty} \dim \Lambda(\Gamma_n) \geq \dim \Lambda(G).$$

Hence we can take a point $t \in C$ such that $\dim \Lambda(\Gamma_t) > 1$. For a sequence $\{C_m\}$ of horocycles tangential at 0 whose diameters tend to 0 as $m \to \infty$, we take such a point t_m on each C_m that $\dim \Lambda(\Gamma_{t_m}) > 1$. Then t_m converge to 0 horocyclically and hence Γ_{t_m} converge strongly to Γ_0. However, $\dim \Lambda(\Gamma_{t_m}) > 1$ and $\dim \Lambda(\Gamma_0) < 1$, which implies discontinuity of the Hausdorff dimension.

Combining Theorem 5 with Proposition 4, we set the following lemma to be used later, saying that the statement of our main theorem is valid if the sequence of Kleinian groups Γ_n also converges geometrically to a geometrically finite Kleinian group.

LEMMA 7. *Let $\rho_n : \Gamma_0 \to \Gamma_n$ be an algebraically convergent sequence of homomorphisms of a finitely generated group Γ_0 onto Kleinian groups Γ_n. Suppose that Γ_n converge geometrically to a geometrically finite Kleinian group G. Then $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology. If, in addition, $\dim \Lambda(G) \geq 1$ then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda(G)$.*

§3. Analytic finiteness of geometric limit

The Ahlfors finiteness theorem asserts that if a Kleinian group Γ is finitely generated then $\Omega(\Gamma)/\Gamma$ consists of a finite number of hyperbolic surfaces of finite area. Later, Bers refined the finiteness theorem quantitatively by giving an upper bound of the area in terms of the minimal number $r(\Gamma)$ of generators of Γ, which is called the area theorem: the total hyperbolic area of $\Omega(\Gamma)/\Gamma$ is bounded by $4\pi(r(\Gamma) - 1)$. See [14, §4.1].

We treat the boundary ∂C_Γ of the convex core in N_Γ instead of $\Omega(\Gamma)/\Gamma$. Let us denote the closed ϵ-neighborhood of the convex core by C_Γ^ϵ. Let $R_\epsilon : N_\Gamma - C_\Gamma^\epsilon \to \partial C_\Gamma^\epsilon$ be a map that takes a point p to the nearest point $R_\epsilon(p) \in \partial C_\Gamma^\epsilon$ to p, called the nearest point retraction. This is a distance-decreasing map. Using coordinates determined by the nearest point and the distance to the nearest point, we can see that $N_\Gamma - C_\Gamma^\epsilon$ is homeomorphic to $\partial C_\Gamma^\epsilon \times (0, \infty)$. If we define the "nearest" point to a point at infinity by using horospheres, we can extend R_ϵ to $\Omega(\Gamma)/\Gamma$. In particular, letting $\epsilon = 0$, we have a homotopy equivalence $R_0 : \Omega(\Gamma)/\Gamma \to \partial C_\Gamma$. See [6] and [9] for details.

We consider the hyperbolic area of the pleated surface ∂C_Γ. Assume that the hyperbolic area of $\Omega(\Gamma)/\Gamma$ is finite. Since R_0 is a homotopy equivalence and it maps cusps of $\Omega(\Gamma)/\Gamma$ to cusps of ∂C_Γ, we can see that the hyperbolic areas of $\Omega(\Gamma)/\Gamma$ and ∂C_Γ are the same. Hence $\text{Area}\,(\partial C_\Gamma) \leq 4\pi(r(\Gamma) - 1)$ by the area theorem. In particular, if Γ is finitely generated then Γ is analytically finite.

Geometric convergence can be interpreted by convergence of the corresponding hyperbolic manifolds. For a Kleinian group Γ in general, let x_Γ denote the image of the origin under the quotient map $\mathbb{H}^3 \to N_\Gamma$ and e_Γ the baseframe of the tangent space at x_Γ that is the projection of a fixed baseframe at the origin. Let $\mathcal{N}_r(x)$ denote a portion of the hyperbolic manifold which is within a distance r of a point x. Then we say that $(N_{\Gamma_n}, e_{\Gamma_n})$ converge to (N_G, e_G) in the sense of Gromov if there exist K_n-biLipschitz diffeomorphisms f_n from $\mathcal{N}_{r_n}(x_{\Gamma_n})$ of N_{Γ_n} into N_G for all sufficiently large n such that $f_n(x_{\Gamma_n}) = x_G$ and $Df_n(e_{\Gamma_n}) = e_G$ where $K_n \to 1$ and $r_n \to \infty$ as $n \to \infty$. Kleinian groups Γ_n converge geometrically to G if and only if $(N_{\Gamma_n}, e_{\Gamma_n})$ converge to (N_G, e_G) in the sense of Gromov. See [6, Chap.3] for details.

When Γ_n converge geometrically to G, it is possible for G to be infinitely generated even if $r(\Gamma_n)$ are uniformly bounded. In fact, such an example was constructed by Jørgensen [11]. However, we can prove that analytic finiteness is still inherited by the geometric limit even in this case if we impose a certain extra assumption.

LEMMA 8. *Suppose that Kleinian groups* Γ_n *satisfying* $\text{Area}\,(\partial C_{\Gamma_n}) \leq M < \infty$ *for all* n *converge geometrically to a Kleinian group* G*. Further assume that, by passing to a subsequence if necessary, the Hausdorff limit* $\Lambda \subset S^2$ *of the limit sets* $\Lambda(\Gamma_n)$ *has empty interior. Then* $\text{Area}\,(\partial C_G) \leq M$*, and in particular* G *is analytically finite.*

PROOF. As the first step, we show that $\text{Area}\,(\partial C_G) \leq \text{Area}\,(\partial C_G(\Lambda))$. We may assume that $\text{Area}\,(\partial C_G(\Lambda)) < \infty$. Take a component Σ of ∂C_G and consider the component E of $N_G - \text{Int}\, C_G$ that has Σ as its frontier. We know that E is homeomorphic to $\Sigma \times [0, \infty)$. Since $C_G(\Lambda)$ is convex and contains C_G, we can see that $E \cap (\text{Int}\, C_G(\Lambda) - \text{Int}\, C_G)$ is homeomorphic to $\Sigma \times [0, \infty)$ unless $C_G(\Lambda) = C_G$. Consider $\Sigma' = E \cap \partial C_G(\Lambda)$. This is not empty, for otherwise Λ would have non-empty interior. Then Σ' must be a pleated surface of the same type as Σ except for finitely many punctures. In particular, $\text{Area}\,(\Sigma) \leq \text{Area}\,(\Sigma')$. Taking the sum over all components of ∂C_G, we obtain the required inequality.

Next assume that $\text{Area}\,(\partial C_G) > M$. Then $\text{Area}\,(\partial C_G(\Lambda)) > M$. Fix a distance r so that $\text{Area}\,(\partial C_G(\Lambda) \cap \mathcal{N}_r(x_G)) > M$. Let $B_r \subset \mathbb{H}^3$ be the closed ball of radius r with center at the origin. By the Hausdorff convergence of the limit sets, $\partial H(\Lambda(\Gamma_n)) \cap B_r$ converge to $\partial H(\Lambda) \cap B_r$ in the Hausdorff topology of $Cl(B_r)$ (See [4, Th.1.4]). Since Γ_n converge to G geometrically, there exist K_n-biLipschitz diffeomorphisms $f_n : \mathcal{N}_r(x_{\Gamma_n}) \to N_G$ such that $f_n(x_{\Gamma_n}) = x_G$ and $Df_n(e_{\Gamma_n}) = e_G$ for all sufficiently large n, where the Lipschitz constants K_n tend to 1 as $n \to \infty$. Let $\tilde{f}_n : B_r \to \mathbb{H}^3$ be a map obtained by lifting f_n. Then $\tilde{f}_n(\partial H(\Lambda(\Gamma_n)) \cap B_r)$ converge to $\partial H(\Lambda) \cap B_r$ in the Hausdorff topology, and thus so do $f_n(\partial C_{\Gamma_n} \cap \mathcal{N}_r(x_{\Gamma_n}))$ to $\partial C_G(\Lambda) \cap \mathcal{N}_r(x_G)$. This implies that $\text{Area}\,(\partial C_{\Gamma_n} \cap \mathcal{N}_r(x_{\Gamma_n})) > M$ for sufficiently large n. However this contradicts the fact that $\text{Area}\,(\partial C_{\Gamma_n}) \leq M$. □

Conjecturally, the statement of Lemma 8 is still valid even if we do not assume anything about the Hausdorff limit of the limit sets. Otherwise at least the following weaker statement is expected to be true.

CONJECTURE. Let $\rho_n : \Gamma_0 \to \Gamma_n$ be an algebraically convergent sequence of surjective homomorphisms and suppose that Kleinian groups Γ_n converge geometrically to G. Then $\mathrm{Area}\,(\partial C_G) \leq 4\pi(r(\Gamma_0) - 1)$.

We can set the following statement as a corollary to Lemma 8.

COROLLARY 9. *Suppose that a sequence of uniformly analytically finite Kleinian groups Γ_n converges geometrically to a Kleinian group G. If $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology then G is analytically finite.*

PROOF. If $\Omega(G) = \emptyset$ then $\partial C_G = \emptyset$ and G is analytically finite by definition. If $\Omega(G) \neq \emptyset$ then $\Lambda(G)$ has no interior and thus Lemma 8 is applicable. □

As a sufficient condition for the limit sets of a geometrically convergent sequence of Kleinian groups to converge to the limit set of the geometric limit in the Hausdorff topology, Kerckhoff and Thurston [13, Prop.2.1 & Cor.2.2] gave the following one (cf. [14, Lem.7.33]).

LEMMA 10. *Suppose that Kleinian groups Γ_n converge geometrically to G. If the injectivity radii of N_{Γ_n} in C_{Γ_n} are uniformly bounded, then $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology. In particular, if all Γ_n are B-groups with $r(\Gamma_n)$ uniformly bounded, then the same conclusion is satisfied.*

Here we show an example of a sequence of Kleinian groups Γ_n with $r(\Gamma_n)$ uniformly bounded that converges geometrically to an analytically finite but infinitely generated Kleinian group. This was given by the infinite volume hyperbolic Dehn surgery construction due to Bonahon and Otal [3]. See also Ohshika [16, Ex.3.5].

EXAMPLE 3. Let Γ_0 be a Fuchsian group of a closed surface S of genus greater than one. Identifying N_{Γ_0} with $S \times \mathbb{R}$, we remove closed tubular neighborhoods of simple closed curves $\{C_i\}$ on $S \times \{i\}$ $(i \in \mathbb{Z})$ so that the resulting manifold has a hyperbolic structure. Let G be an infinitely generated Kleinian group uniformizing this hyperbolic manifold. Then there exist positive integers $\{\nu_i\}_{i \in \mathbb{Z}}$ such that the $1/\nu_i^n$-Dehn filling along C_i for all i provides a manifold that admits a hyperbolic structure for any positive integer n. Let Γ_n be a Kleinian group uniformizing this hyperbolic manifold. We can see that N_{Γ_n} converge to N_G in the sense of Gromov as $n \to \infty$ if we choose suitable baseframes. Hence we may regard that Γ_n converge geometrically to G. All the Γ_n are isomorphic quasifuchsian groups. In fact, we can even take a sequence of isomorphisms $\rho_n : \Gamma_0 \to \Gamma_n$ that converges algebraically. We may use Corollary 9 and Lemma 10 to conclude that G is analytically finite.

§4. Volume of the convex core

For a geometrically finite Kleinian group Γ, the volume of the convex core C_Γ reflects the Hausdorff dimension of the limit set $\Lambda(\Gamma)$. The following result due to Canary [5] is fundamental in this direction, which gives an estimate of the bottom $\lambda_0(N_\Gamma)$ of the spectrum of the Laplacian on the hyperbolic manifold N_Γ in terms of $\mathrm{Vol}\,(C_\Gamma)$ and $\mathrm{Area}\,(\partial C_\Gamma)$.

THEOREM 11. *A geometrically finite Kleinian group* Γ *satisfies*

$$\lambda_0(N_\Gamma) \le \frac{2\,\mathrm{Area}\,(\partial C_\Gamma)}{\mathrm{Vol}\,(C_\Gamma)}.$$

We can paraphrase statements on $\lambda_0(N_\Gamma)$ to those on $\dim \Lambda(\Gamma)$ by the following formula due to Sullivan [18], which generalized earlier results of Elstrodt and Patterson: for a geometrically finite Kleinian group Γ,

$$\lambda_0(N_\Gamma) = \begin{cases} 1 & (\dim \Lambda(\Gamma) \le 1) \\ \dim \Lambda(\Gamma)(2 - \dim \Lambda(\Gamma)) & (\dim \Lambda(\Gamma) \ge 1). \end{cases}$$

Translating Theorem 11 according to this formula and the area theorem, we obtain:

LEMMA 12. *A geometrically finite Kleinian group* Γ *satisfies the following:*

$$\mathrm{Vol}\,(C_\Gamma) \le \begin{cases} 8\pi(r(\Gamma) - 1) & (\dim \Lambda(\Gamma) \le 1) \\ 8\pi(r(\Gamma) - 1)/\{\dim \Lambda(\Gamma)(2 - \dim \Lambda(\Gamma))\} & (\dim \Lambda(\Gamma) \ge 1) \end{cases}$$

Next we consider convergence of volumes of convex cores. Taylor [19, Th.7.2] proved lower semi-continuity of the volume function of the convex core under geometric convergence. This result has a similar flavor to Theorem 6.

THEOREM 13. *Suppose that a sequence of Kleinian groups* Γ_n *converges geometrically to a Kleinian group* G *and* $\Lambda(\Gamma_n)$ *converge to a closed subset* $\Lambda \subset S^2$ *in the Hausdorff topology. Then*

$$\liminf_{n\to\infty} \mathrm{Vol}\,(C_{\Gamma_n}) \ge \mathrm{Vol}\,(C_G(\Lambda)) \ge \mathrm{Vol}\,(C_G).$$

§5. Proof of the main theorem

In this section, we complete the proof of the main theorem. In Lemma 7, we considered the case where the geometric limit is geometrically finite. Now it remains to treat the case that the geometric limit is not geometrically finite.

LEMMA 14. *Suppose that a sequence of Kleinian groups* Γ_n *with* $r(\Gamma_n)$ *uniformly bounded converges geometrically to a Kleinian group* G*. If*

$$\liminf_{n\to\infty} \dim \Lambda(\Gamma_n) < 2$$

then G *is geometrically finite.*

PROOF. Passing to a subsequence, we may assume that $\Lambda(\Gamma_n)$ converge to a closed subset $\Lambda \subset S^2$ in the Hausdorff topology and that $\dim \Lambda(\Gamma_n) \le \alpha < 2$ for some α. This implies that Γ_n are geometrically finite by Theorem 3 and $\mathrm{Vol}\,(C_{\Gamma_n})$ are uniformly bounded by Lemma 12. Then by Theorem 13, both $\mathrm{Vol}\,(C_G)$ and $\mathrm{Vol}\,(C_G(\Lambda))$ are finite.

If $\Lambda(G) = S^2$, then $\mathrm{Vol}\,(C_G) < \infty$ implies that G is geometrically finite, which proves the statement. Hence we may assume that $\Lambda(G) \ne S^2$. If the interior of Λ is not empty, then Λ is not coincident with $\Lambda(G)$ and there exists a closed disk D contained in Λ such that $g(D) \cap D = \emptyset$ for any $g \in G - \{1\}$. This implies that the convex hull $H(D)$ of D can be embedded isometrically into $C_G(\Lambda)$, which contradicts the fact that $\mathrm{Vol}\,(C_G(\Lambda))$ is finite. Therefore the interior of Λ is empty, from which it follows that G is analytically finite by Lemma 8. Thus $\mathrm{Vol}\,(C_G) < \infty$ means that G is geometrically finite. □

Main Theorem. *Let $\rho_n : \Gamma_0 \to \Gamma_n$ be an algebraically convergent sequence of homomorphisms of a finitely generated group Γ_0 onto Kleinian groups Γ_n. Suppose that the limit sets $\Lambda(\Gamma_n)$ converge to a closed subset Λ of S^2 in the Hausdorff topology. Further suppose that $\dim \Lambda \geq 1$. Then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda$.*

Proof. Take a subsequence $\Gamma_{n(i)}$ of Γ_n that converges geometrically to a Kleinian group G. First assume that G is geometrically finite. Then, by Lemma 7, $\Lambda(\Gamma_{n(i)}) \to \Lambda(G)$ in the Hausdorff topology. On the other hand, since $\Lambda(\Gamma_n) \to \Lambda$ in the Hausdorff topology, $\Lambda(G) = \Lambda$ and thus $\dim \Lambda(G) \geq 1$. Again by Lemma 7, $\dim \Lambda(\Gamma_{n(i)})$ converge to $\dim \Lambda(G) = \dim \Lambda$. Next assume that G is not geometrically finite. In this case $\dim \Lambda(\Gamma_{n(i)}) \to 2$ by Lemma 14. If the interior of Λ is empty then G is analytically finite by Lemma 8, $\dim \Lambda(G) = 2$ by Theorem 3, and hence $\dim \Lambda = 2$. If the interior of Λ is not empty then clearly $\dim \Lambda = 2$. In both cases we obtain that $\dim \Lambda(\Gamma_{n(i)})$ converge to $\dim \Lambda = 2$. □

§6. The Hausdorff dimension of the geometric limit

In this section, we show supplementary results to the arguments in the proof of our main theorem, which are on the Hausdorff dimension of the geometric limit of a sequence of Kleinian groups. First the following statement is obtained from the main theorem combined with Lemma 10.

Corollary 15. *Let $\rho_n : \Gamma_0 \to \Gamma_n$ be an algebraically convergent sequence of homomorphisms of a finitely generated group Γ_0 onto Kleinian groups Γ_n. Assume that Γ_n converge geometrically to G. If the injectivity radii of N_{Γ_n} in C_{Γ_n} are uniformly bounded and if $\dim \Lambda(G) \geq 1$ then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda(G)$.*

Example 4. In the circumstances of Example 1, we can take a subsequence of totally degenerated groups Γ_n that converges geometrically to a Kleinian group G. Then $\dim \Lambda(\Gamma_n)$ converge to $\dim \Lambda(G)$ by Corollary 15 as well as $\Lambda(\Gamma_n)$ converge to $\Lambda(G)$ in the Hausdorff topology by Lemma 10. Since $\dim \Lambda(\Gamma_n) = 2$ for all n, we have $\dim \Lambda(G) = 2$. Hence G is geometrically infinite by Theorem 3 but analytically finite by Corollary 9. More precisely, $\Omega(G)/G$ is conformally equivalent to $\Omega(\Gamma_n)/\Gamma_n$ for any n.

As a conjecture, the statement of Corollary 15 is still valid even if we do not assume anything about the injectivity radii in the convex cores. If the conjecture that follows Lemma 8 is valid then so is this conjecture.

Next we consider geometrically convergent sequences without assuming them to be algebraically convergent. In this setting, we cannot expect the continuity of the Hausdorff dimension any longer but we can prove lower semi-continuity as in Theorem 6 in a more general case.

Corollary 16. *Suppose that a sequence of Kleinian groups Γ_n with $r(\Gamma_n)$ uniformly bounded converges geometrically to a Kleinian group G. Then*

$$\liminf_{n\to\infty} \dim \Lambda(\Gamma_n) \geq \dim \Lambda(G).$$

PROOF. By Theorem 3, we may assume that all Γ_n are geometrically finite. If G is geometrically finite then the required claim follows from Theorem 6. If G is not geometrically finite then it follows from Lemma 14. □

The statement of Corollary 16 is not always true if we drop the assumption that $r(\Gamma_n)$ are uniformly bounded.

EXAMPLE 5. We can take a sequence of classical Schottky groups Γ_n with $r(\Gamma_n) = n$ that converges geometrically to a Kleinian group G with $\Omega(G) = \emptyset$. Then $\dim \Lambda(\Gamma_n) < 2 - \epsilon$ for some $\epsilon > 0$ by Doyle [8] whereas $\dim \Lambda(G) = 2$.

As an immediate consequence from Corollary 16, lower semi-continuity of the Hausdorff dimension of limit sets under algebraic convergence (Theorem 2) follows because the algebraic limit Γ_∞ is contained in a geometric limit G.

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