CONFORMAL CONJUGATION OF FUCHSIAN GROUPS FROM THE FIRST KIND TO THE SECOND KIND

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Dedicated to Professor Yukio Kusunoki on the occasion of his seventieth birthday

0. Introduction

We say a Riemann surface R has extension or R is extendable if it is conformally embedded in a larger Riemann surface $\hat{R}$ as a proper subregion. In this paper, we shall investigate the relationship between the extendability of R and its Fuchsian model. Let G be a (possibly infinitely generated) Fuchsian group acting on the unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. The limit set $\Lambda(G)$ of G is the set of all accumulation points of the orbit $G(0) = \cup_{g \in G} g(0)$ on $\partial\Delta$. It is known that G acts properly discontinuously on $\Omega(G) = \hat{\mathbb{C}} - \Lambda(G)$. We call G of *the first kind* if $\Lambda(G) = \partial\Delta$ and of *the second kind* otherwise. If G is of the second kind, then $R = \Delta/G$ has "visible" ideal boundary $(\partial\Delta - \Lambda(G))/G$ called border. However, for Fuchsian groups of the first kind, we can obtain little information, from the limit set itself, about the ideal boundary of R (cf. [6]), and about the extendability of R.

On the other hand, there is the following statement about the conformal deformation of Fuchsian group of the first kind:

Folklore. *Let G be a Fuchsian group of the first kind, and f a conformal map of Δ such that $G_f = fGf^{-1}$ is a Kleinian group. Then $f(\Delta)$ is an invariant component of the region of discontinuity of G_f. In particular, $\partial f(\Delta)$ is the limit set of G_f.*

At glance, the folklore seems to be true because the image of the orbit via f also accumulates to the limit set of G_f which seems to be $\partial f(\Delta)$. For instance, in a well-known paper [3] concerning Kleinian groups, the above folklore seems to be taken for granted (see [3] Theorem 6 and the proof), but it did not cause any problem because it was applied for finitely generated Fuchsian groups of the first kind, whose Riemann surfaces are closed except for at most a finite number of punctures. Under the assumption of the folklore, it is obvious that $\Lambda(G_f) \subset \partial f(\Delta)$ since G_f acts on $f(\Delta)$. If $\Lambda(G_f)$ is a proper subset of $\partial f(\Delta)$, the Riemann surface Δ/G is extendable beyond the ideal boundary $(\partial f(\Delta) - \Lambda(G_f))/G_f$. Therefore the extendability of Riemann surfaces is related to the folklore.

Extendability of Riemann surfaces has been well investigated (cf. Sario-Oikawa [11] X.5). Recently, Sakai [10] has completely classified all the types of extensions. Among their works, what we are especially interested in is the extension whose inclusion map gives homotopy equivalence. Surprising is that there are Riemann surfaces not admitting border (namely, whose Fuchsian representations are of the first kind), but admitting such extensions.

In this paper, we *disprove* the folklore. Moreover, we characterize Fuchsian groups for which the folklore is true. This is our main theorem and the statement is in §1. We shall give a proof in §2. Next in §3, a concrete example of a Fuchsian group of the first kind for which the folklore does not hold is constructed.

Recently, Hamilton [4] considered simultaneous uniformization of a pair of Riemann surfaces by a Kleinian group in a more general situation than the original one due to Bers [2]. But it seems to the authors that he tried to show the folklore in a part of the proof. In §4, we give some remarks about this generalized simultaneous uniformization.

Finally certain related results are given in §5. In particular, we see that a Fuchsian group whose horocyclic limit set is of full measure satisfies the folklore.

The authors are grateful to the referee for his/her helpful comments.

1. Statement of Main Theorem.

We obtain an answer to the question what Fuchsian groups hold the folklore in the introduction.

Main Theorem. *Let R be a complete hyperbolic Riemann surface and G the Fuchsian group acting on the unit disk Δ which uniformizes R. Then the following are equivalent.*

- (S-1) *Suppose that f is a conformal map of Δ such that fGf^{-1} is a Kleinian group. Then $f(\Delta)$ is an invariant component of the region of discontinuity of fGf^{-1}.*
- (S-2) *Suppose that f is a conformal map of Δ into Δ such that fGf^{-1} is a Fuchsian group acting on Δ. Then f is a Möbius transformation.*
- (S-3) *R has no homotopic extensions.*
- (S-4) *Suppose that f is a conformal map of Δ into Δ such that fGf^{-1} is a Fuchsian group acting on Δ. Then fGf^{-1} is of the first kind.*
- (S-5) *R has no disks with D-ideal boundary.*

Here we should define "homotopic extension" and "disk with D-ideal boundary" which appear in the statement of the main theorem.

We say an open Riemann surface R has *homotopic extension* if there exists a conformal map ι of R into another Riemann surface $\hat{R}$ such that $\iota(R)$ is a proper subregion of $\hat{R}$ and ι gives homotopy equivalence.

We say a compact subset E of $\mathbb{C}$ belongs to the class N_D if the unbounded component V of $\mathbb{C}-E$ belongs to the class O_{AD} of Riemann surfaces, in other words, V admits no non-constant holomorphic functions with finite Dirichlet integrals.

Definition 1 (Disk with D-ideal boundary). Let R be a Riemann surface and let U be a simply connected subregion of R whose relative boundary $\partial_R U$ in R consists of Jordan arcs. We call U a disk with D-ideal boundary if for the Riemann map h of U onto the unit disk Δ, $\partial\Delta - h(\partial_R U)$ does not belong to N_D.

The reason why a disk with D-ideal boundary is related to our problem will be seen by Proposition 1 in the next section.

Remark. Sakai [10] further classified the disks with D-ideal boundary into two cases; they are either simply connected skirts or disks with crowded ideal boundaries in his terms.

Definition 2 (Stably of the first kind, S-type). We say that a Fuchsian group is stably of the first kind or simply of S-type if it satisfies the conditions in the main theorem. We denote by $\mathcal{S}$ the class of all the Fuchsian groups stably of the first kind. We also use the term S-type for Riemann surfaces if they are represented by Fuchsian groups of S-type.

If a Fuchsian group G is finitely generated and of the first kind, then $R = \Delta/G$ is a closed Riemann surface with at most a finite number of punctures. Thus it satisfies (S-3) and G is a Fuchsian group stably of the first kind. A Riemann surface without planar ends is of S-type if and only if it admits no extensions.

Remark. In this paper, we are interested in the conditions under which a Fuchsian group of the first kind is conjugated to a Fuchsian group of the second kind. On the other hand, it is known that any non-elementary Fuchsian group of the second kind is conjugated to a Fuchsian group of the first kind by some conformal map. Sugawa [13] has referred to this fact. See Theorem 3 in [4] as literature.

2. Proof of Main Theorem

Before the proof of our main theorem, we prepare a result which is used at an essential step in our argument (cf. Sario-Oikawa [11] X.5B).

Proposition 1. *Let E be a closed set in $\partial\Delta$. Then the following are equivalent.*

1. *There exists a conformal map φ of Δ into Δ such that $\varphi(\Delta) \neq \Delta$, φ has a continuous extension to $\partial\Delta - E$ denoted by the same letter φ, and $\varphi(\partial\Delta - E) \subset \partial\Delta$.*
2. *E does not belong to N_D.*

Proof. First suppose that E does not belong to N_D. By conjugation, we may consider the proposition on the upper half plane H and assume E to be a bounded closed set on the real axis $\mathbb{R}$. Let $P_0(z)$ and $P_1(z)$ be the extremal horizontal and vertical slit maps of $\mathbb{C} - E$ respectively with the Laurent expansion $z + a_1 z^{-1} + a_2 z^{-2} + \cdots$ at ∞. Then we see $P_0(z) \equiv z$ because $\mathbb{C} - E$ is already an extremal horizontal slit plane ([11] IX.4B), and $P_1(\bar{z}) = \overline{P_1(z)}$ because $\mathbb{C} - E$ is symmetric with respect to $\mathbb{R}$. Since $E \notin N_D$, we know $P_0(z) \neq P_1(z)$, in other words, the span does not vanish ([11] VI.2D). This implies that P_1 is not a Möbius transformation though it maps H into H and $\mathbb{R} - E$ on $\mathbb{R}$. Thus the image of H must be properly contained in H.

Conversely suppose that E belongs to N_D. Let φ be a conformal map of Δ into Δ continuously extending to $\partial\Delta - E$ so that $\varphi(\partial\Delta - E) \subset \partial\Delta$. By the reflection principle, φ further extends to a conformal map of $\mathbb{C} - E$. Then again by [11] VI.2D, it must be the restriction of a Möbius transformation. Thus we have $\varphi(\Delta) = \Delta$. □

Proof of Main Theorem. (S-1) ⇒ (S-2): Suppose that there is a conformal map f of Δ into Δ such that fGf^{-1} is a Fuchsian group but f is not a Möbius transformation. Then $f(\Delta)$ is a proper subset of Δ and Δ is a subset of the invariant component of the region of discontinuity of the Fuchsian group fGf^{-1}. This contradicts (S-1).

(S-2) ⇔ (S-3): Suppose that $R = \Delta/G$ admits homotopic extension. Then there exists a Riemann surface $\hat{R}$ such that R is a proper subregion of $\hat{R}$ and the inclusion $\iota : R \to \hat{R}$ is a homotopy equivalent conformal map. Let $\hat{G}$ be a Fuchsian group acting on Δ which represents $\hat{R}$ and let $f : \Delta \to \Delta$ be a lift of ι. This is conformal and satisfying $fGf^{-1} = \hat{G}$ because ι gives the homotopy equivalence. Since $\iota(R)$ is a proper subregion of $\hat{R}$, we conclude that $f(\Delta)$ is a proper subset of Δ. In particular, f is not Möbius. Thus (S-2) implies (S-3), and the converse is now easy.

(S-3) ⇒ (S-4): If G is of the second kind, R has homotopic extension. Hence any Fuchsian group satisfying (S-3) is of the first kind. We have seen that (S-3) is equivalent to (S-2). Therefore (S-4) follows from (S-2) and (S-3).

(S-4) ⇒ (S-5): Suppose that $R = \Delta/G$ has a disk with D-ideal boundary U. From its definition, there exists a conformal map h of U onto Δ such that $E = \partial\Delta - h(\partial_R U)$ does not belong to N_D. Considering φ as in Proposition 1, we verify that there exist a Riemann surface $\hat{R}$ with $R \subset \hat{R}$ and a simply connected subset $\hat{U}$ of $\hat{R}$ such that U is a proper subset of $\hat{U}$. Take a connected component α (continuum) of $\hat{U} - U$ and consider a Riemann surface $R' = \hat{R} - \alpha$. We denote by G' a Fuchsian group representing R'. Lifting the homotopy equivalent inclusion map $\iota : R \to R'$ to Δ, we have a conformal map $f : \Delta \to \Delta$ such that $fGf^{-1} = G'$. But, from the construction of R', the Fuchsian group G' is of the second kind. Thus we have a contradiction.

(S-5) ⇒ (S-1): Let G be a Fuchsian group such that $R = \Delta/G$ satisfies (S-5). Suppose that there exists a conformal map f for which (S-1) does not hold. Namely, $G_f = fGf^{-1}$ is a Kleinian group but $f(\Delta)$ is not an invariant component of the region of discontinuity of G_f. Then $f(\Delta)$ is a proper subregion of $\Omega(G_f)$, the region of discontinuity of G_f. Take a point $p \in \partial f(\Delta) \cap \Omega(G_f)$ and a small disk U_p centered at p such that $g(U_p) \cap U_p = \emptyset$ for every $g \in G_f - \{id\}$. Let V be a connected component of $f(\Delta) \cap U_p$. Obviously it is simply connected. Let ψ be a conformal map of V onto the unit disk Δ. Then the set $\partial V \cap f(\Delta)$ is mapped to a relatively open subset O of $\partial\Delta$ via ψ. We denote by E the complement of O with respect to $\partial\Delta$. We regard V as a simply connected domain in Δ/G. Then, from (S-5), V is not a disk with D-ideal boundary, and thus $E \in N_D$. On the other hand, V is a proper subdomain of U_p and O is mapped on ∂U_p via $\varphi = \psi^{-1}$. Therefore, from Proposition 1, we see that E does not belong to N_D. This is a contradiction. □

3. Example

Now we construct a Fuchsian group of the first kind but not stably of the first kind. Actually, Example 1 in Sakai [10] is a Riemann surface represented by such a Fuchsian group. We will explain it with a little modification.

We take a closed set E in $\partial\Delta$ which is totally disconnected and satisfying the conditions of Proposition 1 (cf. [11] IX. 4G-4I). Then we have a conformal map φ of Δ into Δ such that $\varphi(\partial\Delta - E) \subset \partial\Delta$ and $\varphi(\Delta) \subsetneqq \Delta$. Let $P' = \{p'_n\}_{n=1}^{\infty}$ be a discrete subset in $\varphi(\Delta)$ such that the derived set is equal to $\partial\varphi(\Delta) \cap \partial\Delta = \varphi(\partial\Delta - E)$. Then $R' = \varphi(\Delta) - P'$ is conformally equivalent to $R = \Delta - P$ via φ^{-1}, where $P = \varphi^{-1}(P')$. Note that the derived set of P is $\partial\Delta$ because E is totally disconnected.

From the construction, R admits a homotopic extension. Indeed, it is a proper subregion of a Riemann surface which is conformally equivalent to $\Delta - P'$. Namely, R is not of S-type.

On the other hand, if we know R is of the first kind, we obtain the desired example.

Lemma 1. *Let $P = \{p_n\}_{n=1}^{\infty}$ be a discrete subset of Δ such that the derived set of P is $\partial\Delta$. Then the Fuchsian group G which represents $R = \Delta - P$ is of the first kind.*

Actually, we show a more general result which induces Lemma 1.

Theorem 1. *Let R be a hyperbolic planar domain. We consider a connected component K of ∂R which is a continuum. For a point p in K and for a neighborhood U of p, we denote by $\mathcal{B}(U,p)$ the set of components of $U - K$ whose boundary contains p. If every $V \in \mathcal{B}(U,p)$ for every U, p and K satisfies $V \cap \partial R \neq \emptyset$, then the Fuchsian group G which represents R is of the first kind.*

Remark. The converse of Theorem 1 is not true. Moreover, the assumption on a planar domain R is not conformally invariant; namely, even if R and R' are conformally equivalent and R satisfies the condition about ∂R as in Theorem 1, R' may not. A counterexample is given just by R and R' in the above example.

Proof of Theorem 1. Suppose that G is of the second kind, and acting on the upper half plane $H = \{z = (x,y) \mid y > 0\}$. Then there exists a rectangle

$$Q = \{z = (x,y) \mid a < x < b,\ 0 < y < c\}$$

such that $g(Q) \cap Q = \emptyset$ for every $g \in G - \{id\}$. The universal cover $\pi : H \to R$ restricted to Q is a conformal map onto a simply connected domain $\pi(Q) \subset R$. We can take three vertical segments

$$l_i = \{z = (x,y) \mid x = x_i,\ 0 < y < c\}\ (i = 1,2,3;\ a < x_1 < x_2 < x_3 < b)$$

in Q such that the limits

$$\lim_{l_i \ni z \to (x_i,0)} \pi(z) = \zeta_i$$

exist in $\partial R \cap \partial\pi(Q)$ for $i = 1,2,3$ and $\{\zeta_i\}_{i=1,2,3}$ are distinct. This is possible because of the Fatou-Riesz theorem about non-tangential limits of bounded analytic functions. It is clear that $\{\zeta_i\}_{i=1,2,3}$ are in the same component K of ∂R. We consider the smaller rectangle

$$Q' = \{z = (x,y) \mid x_1 < x < x_3,\ 0 < y < c\}$$

and its image $\pi(Q')$. Since the both ends of the arc $\partial_R\pi(Q')$ are bounded away from ζ_2, we can choose a neighborhood U of ζ_2 so that $U \cap \partial_R\pi(Q') = \emptyset$. Then there is a component of $U - K$ in which $\pi(l_2)$ is accessible to ζ_2. Since it is contained in $\pi(Q')$, it has no points of ∂R. Therefore the assumption of the theorem does not hold for $p = \zeta_2$. $\square$

Remark. Sakai [10] actually took a two-sheeted unlimited branched covering of Δ so that the projection of the set of the branch points is P. While our example is of genus 0 and has planar ends, his example is of infinite genus and has no planar ends and it also gives a Fuchsian group of the first kind but not stably of the first kind.

4. Simultaneous uniformization

Let R and R^* be hyperbolic Riemann surfaces for which there is an orientation reversing homeomorphism $\Psi : R \to R^*$. Bers proved that if R and R^* are of finite area, then there is a Kleinian group Γ simultaneously uniformizing R and R^* (cf. [2]). This means that Γ has two invariant components W and W^* of the region of discontinuity $\Omega(\Gamma)$ so that the Riemann surfaces W/Γ and W^*/Γ are conformally equivalent to R and R^* respectively. In his paper [4], Hamilton considered the simultaneous uniformization for the general case where the Fuchsian groups corresponding to R and R^* are of the first kind. However his proof relies on the folklore in the introduction, which contradicts our result in the previous section. See Theorem 4 or Corollary 1 in [4]. Thus we think that this generalized simultaneous uniformization (Theorem 1 in [4]) is still open.

In this section, we show the simultaneous uniformization theorem for Riemann surfaces R and R^* of S-type with an orientation reversing homeomorphism $\Psi : R \to R^*$. Of course, Hamilton's proof is applicable at least to this case. But several complicated arguments are not necessary any longer. We will prove the theorem much simply.

Theorem 2. *Let R and R^* be Riemann surfaces of S-type. If there is an orientation reversing homeomorphism $\Psi : R \to R^*$, then there is a Kleinian group Γ which simultaneously uniformizes R and R^*.*

Proof. Let G be a Fuchsian group acting on the unit disk Δ such that $R = \Delta/G$, and G^* acting on the exterior of the unit disk $\Delta^* = \{z \in \hat{\mathbb{C}} \mid |z| > 1\}$ such that $R^* = \Delta^*/G^*$. Lifting Ψ to Δ, we have an orientation reversing homeomorphism $\Phi : \Delta \to \Delta^*$ which induces the isomorphism $\theta : G \to G^*$. The G^* acts not only on Δ^* but also on Δ. Then $\Psi^{-1} \circ J : \Delta \to \Delta$ induces θ^{-1}, where J is the reflection with respect to the unit circle. Let R_n be the exhaustion of R by topologically finite Riemann surfaces and G_n the corresponding exhaustion of G by finitely generated subgroups of the second kind. Further, set $G_n^* = \theta(G_n)$. Of course, $\Psi^{-1} \circ J$ induces $\theta^{-1} : G_n^* \to G_n$ for each n. Since they are finitely generated, we may replace $\Psi^{-1} \circ J$ with a quasiconformal automorphism ω_n of Δ such that $\omega_n g^* \omega_n^{-1} = \theta^{-1}(g^*)$ for every $g^* \in G_n^*$.

Let μ_n be the complex dilatation of ω_n. We define a Beltrami coefficient $\hat{\mu_n}$ on $\hat{\mathbb{C}}$ such that $\hat{\mu_n} = \mu_n$ on Δ and $\hat{\mu_n} = 0$ elsewhere. Then there is a quasiconformal automorphism F_n of $\hat{\mathbb{C}}$ with the complex dilatation $\hat{\mu}_n$ and with the normalization that the Laurent expansion at ∞ is

$$F_n(z) = z + \frac{b_{n1}}{z} + \frac{b_{n2}}{z^2} + \cdots .$$

This conjugates the Fuchsian group G_n^* to a Kleinian group Γ_n. We define

$$f_n^* = F_n : \Delta^* \to W_n^* \ ; \ f_n = F_n \circ \omega_n^{-1} : \Delta \to W_n.$$

They are conformal and satisfying $f_n^* \theta(g) f_n^{*-1} = f_n g f_n^{-1} \in \Gamma_n$ for every $g \in G_n$.

The sequences $\{f_n\}$ and $\{f_n^*\}$ constitute normal families. Passing to subsequences, we may assume that $f_n \to f$ and $f_n^* \to f^*$ uniformly on each compact subset in Δ and Δ^* respectively. In virtue of the above normalization, we see that the limit f^* is conformal. Therefore $f_n^*\theta(g)f_n^{*-1}$ converges to a Möbius transformation $\gamma_g = f^*\theta(g)f^{*-1}$ for each $g \in G$, and f^* conjugates G^* to a Kleinian group Γ.

On the other hand, we can prove that f is not a constant map, from the fact that $f_n g f_n^{-1}$ also converge to γ_g. For the proof, we prepare the following lemma.

Lemma 2. *If conformal maps f_n of Δ converge to a constant map $f \equiv a$ and Möbius transformations $\gamma_n = f_n g f_n^{-1}$ converge to a Möbius transformation γ_g, then the point a is fixed by γ_g.*

Proof. γ_n converge to γ_g uniformly and satisfy that $\gamma_n \circ f_n(z) = f_n \circ g(z)$ for $z \in \Delta$. Taking the limit as $n \to \infty$, we have $\gamma_g(a) = a$. □

Choosing non-commutative elements g_1 and g_2 in G, we see that γ_{g_1} and γ_{g_2} do not commute. If f is a constant map a, the above lemma shows that they both have the same fixed point a, and thus they must commute. This contradiction proves that f is non-constant, and hence conformal.

Let W be the image of Δ by the conformal map f, and W^* the image of Δ^* by f^*. The Kleinian group $\Gamma = fGf^{-1} = f^*G^*f^{*-1}$ acts on both W and W^* properly discontinuously and keeps each of them invariant. The quotient W/Γ is conformally equivalent to R and W^*/Γ is to R^*. Further, by the assumption that R and R^* are of S-type, we know that W and W^* are the components of the region of discontinuity of Γ (Main Theorem). Thus Γ simultaneously uniformizes R and R^*. □

Remark. At this stage, we cannot eliminate the possibility that Γ may have other components of the region of discontinuity than W and W^* yet. An example of a Kleinian group with two invariant components and other components was constructed in [1].

Conversely, if Γ is a Kleinian group which has two invariant components W and W^* of the region of discontinuity, the Riemann surfaces W/Γ and W^*/Γ should be topologically equivalent by an orientation reversing homeomorphism. In Section 6 of [4], there is a proof about this fact, but there the Riemann surfaces W/Γ and W^*/Γ are assumed to be of the first kind. But even if Γ is a Kleinian group whose region of discontinuity consists of two invariant components W and W^*, the Riemann surfaces W/Γ and W^*/Γ can be of the second kind. In other words, the Riemann mappings of W and W^* may conjugate Γ to Fuchsian groups of the second kind. This was remarked in [1]. An example of such a Kleinian group was explained in [5] Note 3.1.3. Thus the proof in [4] should be slightly changed.

5. Supplemental results

(A) *Quasiconformal invariance.* It is an interesting problem to consider whether a property of a Riemann surface is preserved under quasiconformal deformations. Here we prove that the class $\mathcal{S}$ has quasiconformal invariance. In other words;

Theorem 3. *Let R be a Riemann surface of S-type and R' a quasiconformal deformation of R. Then R' is also of S-type.*

This is known from quasiconformal invariance of N_D (cf. [12] II.14-15). A weaker claim but enough for our application is easily derived from Proposition 1; we include it here.

Lemma 3. *Let E be a closed set in $\partial\Delta$ and σ a quasiconformal automorphism of $\mathbb{C}$ which maps Δ onto Δ. If E belongs to N_D, then so does $E' = \sigma(E)$.*

Proof. If not, by Proposition 1, there is a conformal map φ' of Δ into Δ such that $\varphi'(\Delta) \subsetneqq \Delta$ and $\varphi'(\partial\Delta - E') \subset \partial\Delta$. By the measurable Riemann mapping theorem, there is a quasiconformal automorphism τ of Δ such that $\varphi = \tau \circ \varphi' \circ \sigma$ is conformal. This maps Δ into Δ such that $\varphi(\Delta) \subsetneqq \Delta$ and $\varphi(\partial\Delta - E) \subset \partial\Delta$. This means that E does not belong to N_D by Proposition 1. □

Proof of Theorem 3. Suppose that R' is not of S-type. Then there exists a disk U' with D-ideal boundary in R'. Under a conformal map $h' : U' \to \Delta$, let us denote $E' = \partial\Delta - h'(\partial_{R'} U')$. It does not belong to N_D. Let $q : R \to R'$ be a quasiconformal homeomorphism and set $U = q^{-1}(U')$. Again, by using a conformal map $h : U \to \Delta$, we define $E = \partial\Delta - h(\partial_R U)$. Then the composition $\sigma = h' \circ q \circ h^{-1} : \Delta \to \Delta$ is a quasiconformal automorphism, extends to $\mathbb{C}$ and maps E onto E'. By Lemma 3, we see that E does not belong to N_D. Hence R has a disk with D-ideal boundary, and it is not of S-type. □

(B) *Conservative Fuchsian groups.* In [9], Pommerenke introduced a certain class of Fuchsian groups (accessible type), and gave several characterizations of these groups. One of them is a condition about the horocyclic limit sets. For a Fuchsian group G, we say in this paper that $x \in \partial\Delta$ is a horocyclic limit point of G if in some horodisk tangent to $\partial\Delta$ at x, there are points of $G(0)$ which accumulate to x. The set of horocyclic limit point is denoted by $\Lambda_h(G)$, which is of course a subset of the limit set $\Lambda(G)$. Generally, the action of G on $\partial\Delta$ divides it into the two parts up to null sets: the dissipative part and the conservative part. The former is the part where G has a measurable fundamental set. According to Theorem 1 in [9], we know that the conservative part of G coincides with $\Lambda_h(G)$. In this sense, we say G is *conservative* if $\Lambda_h(G)$ has full Lebesgue measure on $\partial\Delta$, and denote the class of conservative Fuchsian groups by $\mathcal{C}$. Cofinite area Fuchsian groups are conservative and conservative Fuchsian groups are of the first kind.

Here we see a relation between the classes $\mathcal{S}$ and $\mathcal{C}$. The key fact is the following Proposition 3 derived from Theorem 2 in [9], characterizing Riemann surfaces whose Fuchsian groups are conservative (abusing notation, these Riemann surfaces are also called conservative).

Definition 3 (Disk with B-ideal boundary). Let R be a Riemann surface and let U be a simply connected subregion of R whose relative boundary $\partial_R U$ in R consists of Jordan arcs. We call U a disk with B-ideal boundary if for the Riemann map h of U onto the unit disk Δ, $\partial\Delta - h(\partial_R U)$ has positive linear measure.

Proposition 3. *A Riemann surface is conservative if and only if it does not have a disk with B-ideal boundary.*

Let E be a compact subset of $\mathbb{C}$. We say that the set E belongs to the class N_B if the unbounded component V of $\mathbb{C} - E$ belongs to the class O_{AB}, in other words, the planar domain V admits no non-constant bounded analytic functions. For a compact subset E in $\partial\Delta$, it is known that E belongs to N_B if and only if the linear measure of E is zero (cf. [12] II.10A). It is known that $O_{AB} \subset O_{AD}$ (cf. [12] I.9A), and we have $N_B \subset N_D$ from the definition. Accordingly, a disk with D-ideal boundary is with B-ideal boundary. Therefore we have the following theorem.

Theorem 4. *If a Fuchsian group G is conservative, in other words, if $\Lambda_h(G)$ has full measure on $\partial\Delta$, then it is stably of the first kind.*

Furthermore, the inclusion $\mathcal{C} \subset \mathcal{S}$ is strict. Indeed, the class $\mathcal{C}$ is not quasiconformally invariant (cf. [7] Theorem 7.1) but the class $\mathcal{S}$ is quasiconformally invariant by Theorem 3, and hence they are not coincident.

Another proof of Theorem 4, which is pointed out by the referee, is obtained from a result by Nagel, Rudin and Shapiro [8]. They proved in particular that a conformal map of the unit disk Δ into Δ has horocyclic limits almost everywhere on $\partial\Delta$. Let G be a conservative Fuchsian group and $f : \Delta \to \Delta$ a conformal map such that fGf^{-1} is a Fuchsian group acting on Δ. If $f(\Delta)$ is a proper subset of Δ, then there is a subset of positive measure in $\Lambda_h(G)$ where f has horocyclic limits inside Δ. However as the orbits $(fGf^{-1})(f(0))$ are conjugate to the orbits $G(0)$, they must only cluster on $\partial\Delta$. This implies that $f(\Delta) = \Delta$, in other words, f must be a Möbius transformation. Thus the condition (S-2) in Main Theorem is satisfied and G is stably of the first kind.

Finally, we remark about a sufficient condition for a planar Riemann surface $R = \Delta - P$ to be in $\mathcal{S}$, where $P = \{p_n\}_{n=1}^{\infty}$ is a discrete subset of Δ. We define the conical derived set $L_c(P)$ of P as follows: $x \in \partial\Delta$ belongs to $L_c(P)$ if there is a Stolz angular region with the vertex x where P accumulates to x. Pommerenke proved that $R = \Delta - P$ belongs to $\mathcal{C}$ if and only if $L_c(P)$ has full Lebesgue measure on $\partial\Delta$ (cf. [7] Lemma 2.1). Therefore we see from Theorem 4 that R belongs to $\mathcal{S}$ if $L_c(P)$ has full measure. We can also see that $L_c(P)$ does not have full measure for the example $R = \Delta - P$ in §3.

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