CERTAIN INTEGRABILITY OF QUASISYMMETRIC AUTOMORPHISMS OF THE CIRCLE

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In memory of Professor Gehring

ABSTRACT. Using the correspondence between the quasisymmetric quotient and the variation of the cross ratio for a quasisymmetric automorphism g of the unit circle, we establisch a certain integrability of the complex dilatation of a quasiconformal extension of g to the unit disk if the Liouville cocycle for g is integrable. Moreover, under this assumption, we verify regularity properties of g such as being bi-Lipschitz and symmetric.

1. INTRODUCTION

A quasisymmetric automorphism $g : \mathbb{S} \to \mathbb{S}$ of the unit circle $\mathbb{S} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ plays a central role in the quasiconformal theory of Teichmüller spaces. For an orientation-preserving self-homeomorphism g of $\mathbb{S}$, the concept of quasisymmetry can be defined in several ways, but boundedness of the variation of the following quantities under g is being used in some usual definitions, which are known to be equivalent:

(1) the ratio of two intervals given by any three positively ordered points on $\mathbb{S}$;
(2) the cross ratio of four positively ordered points on $\mathbb{S}$.

In both definitions, the variation is taken over all normalized points on $\mathbb{S}$, that is, three consecutive points with equal intervals in the first case and four points of even cross ratio in the second case.

To describe the above definition precisely, it is convenient to use a lift of an orientation-preserving self-homeomorphism g of $\mathbb{S}$ to a self-homeomorphism $\widetilde{g}$ of the real line $\mathbb{R}$ by the universal covering $\mathbb{R} \to \mathbb{S}$ with the correspondence $x \mapsto e^{ix}$.

In general, for an increasing homeomorphic function $h : \mathbb{R} \to \mathbb{R}$, the *quasisymmetric quotient* is defined by

$$m_h(x,t) = \frac{h(x+t) - h(x)}{h(x) - h(x-t)}$$

for every $x \in \mathbb{R}$ and every $t > 0$. If $m_h(x,t)$ is uniformly bounded from above and away from zero, we say that h is *quasisymmetric*. More precisely, h is M-quasisymmetric if there exists a constant $M \geq 1$ such that $1/M \leq m_h(x,t) \leq M$ holds for every $x \in \mathbb{R}$

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and for every $t > 0$. The quasisymmetry of g is defined by that of its lift $\widetilde{g}$. Moreover, a quasisymmetric automorphism g of $\mathbb{S}$ or its lift $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ is called *symmetric* if $m_{\widetilde{g}}(x,t) \to 0$ as $t \to 0$ uniformly, that is, independently of $x \in \mathbb{R}$.

On the other hand, for positively ordered distinct points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{S}$, the cross ratio is defined by

$$[\zeta_1, \zeta_2, \zeta_3, \zeta_4] = \frac{(\zeta_1 - \zeta_3)(\zeta_2 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_2 - \zeta_3)} \in (1, \infty).$$

An orientation-preserving self-homeomorphism g of $\mathbb{S}$ is quasisymmetric if and only if there is some constant $M' \geq 1$ such that

$$\frac{1}{M'} \leq \frac{[g(\zeta_1), g(\zeta_2), g(\zeta_3), g(\zeta_4)]}{[\zeta_1, \zeta_2, \zeta_3, \zeta_4]} \leq M'$$

for any positively ordered distinct points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{S}$ satisfying $[\zeta_1, \zeta_2, \zeta_3, \zeta_4] = 2$.

For $\zeta_j = e^{ix_j}$ $(j = 1, 2, 3, 4)$ with $x_1 < x_2 < x_3 < x_4$ and $x_4 - x_1 < 2\pi$, the cross ratio can be represented by the following integral formula:

$$\log\,[\zeta_1, \zeta_2, \zeta_3, \zeta_4] = \int_{x_3}^{x_4} \int_{x_1}^{x_2} \frac{1}{4\sin^2((x-y)/2)} dxdy.$$

If we assume that an orientation-preserving self-homeomorphism g of $\mathbb{S}$ is absolutely continuous, then the logarithm of the variation of the cross ratio under g is

$$\log \frac{[g(\zeta_1), g(\zeta_2), g(\zeta_3), g(\zeta_4)]}{[\zeta_1, \zeta_2, \zeta_3, \zeta_4]} = \int_{x_3}^{x_4} \int_{x_1}^{x_2} c(g)(x,y)\, dxdy,$$

where

$$c(g)(x,y) = \frac{\widetilde{g}'(x)\widetilde{g}'(y)}{4\sin^2((\widetilde{g}(x) - \widetilde{g}(y))/2)} - \frac{1}{4\sin^2((x-y)/2)}.$$

This is called the *Liouville cocycle* for g. Then its integrable norm

$$\|c(g)\|_1 = \int_{\mathbb{S} \times \mathbb{S} \setminus \Delta} |c(g)(x,y)| dxdy$$

over $\mathbb{S} \times \mathbb{S} \setminus \Delta$ measures the total amount of the variation of the cross ratio under g, where Δ denotes the diagonal set of $\mathbb{S} \times \mathbb{S}$. In particular, $\|c(g)\|_1 < \infty$ implies that g is quasisymmetric.

The corresponding totality of the quasisymmetric quotient $m_{\widetilde{g}}(x,t)$ can be also given by its integral. However, we do not consider this integration directly. Instead, we apply the relationship between $m_{\widetilde{g}}(x,t)$ and the complex dilatation $\widetilde{\mu}(z)$ of a quasiconformal automorphism of the upper half-plane $\mathbb{H} = \{z = x + iy \mid y > 0\}$ obtained by the Beurling-Ahlfors extension of $\widetilde{g}$. Using the holomorphic universal covering $\mathbb{H} \to \mathbb{D} \setminus \{0\}$ defined by $z \mapsto \zeta = e^{iz}$, we have a quasiconformal automorphism of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ extending g whose complex dilatation $\mu(\zeta)$ contains information about the quasisymmetry of g. Then we expect that a certain integral of $\mu(\zeta)$ can be estimated in terms of $\|c(g)\|_1$. This will be a component of the results in this paper, stated as Theorem 4.1.

In addition, the integrability of the Liouville cocycle for g reflects certain regularity of g. Theorem 3.1 shows that g is bi-Lipschitz continuous and Theorem 5.1 shows that g is symmetric. Those three theorems can be unified in the following statement.

Theorem 1.1. *Let $g : \mathbb{S} \to \mathbb{S}$ be an absolutely continuous quasisymmetric automorphism with $\|c(g)\|_1 < \infty$. Then g is bi-Lipschitz and symmetric as well as it extends continuously to a quasiconformal automorphism of $\mathbb{D}$ whose complex dilatation $\mu(\zeta)$ satisfies*

$$\int_{\mathbb{S}} |\mu((1-\tau)e^{ix})|dx \leq a\tau$$

for every $\tau \in (0,1)$, where $a > 0$ is a constant depending only on $\|c(g)\|_1$. In particular,

$$\int_{\mathbb{D}} |\mu(\zeta)|(1-|\zeta|)^{-1} d\xi d\eta \leq \frac{a}{2}$$

follows from this estimate.

In the next section, we will supply necessary facts used for our conclusions mentioned above, especially on (1) the Beurling-Ahlfors extension and (2) the cross ratio and the Liouville cocycle.

2. Preliminaries

2.1. **The Beurling-Ahlfors extension.** An orientation-preserving homeomorphism f of a domain $D \subset \widehat{\mathbb{C}}$ is called *quasiconformal* if it has distributional partial derivatives and satisfies $\|\mu_f\|_\infty < 1$ for the *complex dilatation* $\mu_f(z) = \bar{\partial} f(z)/\partial f(z)$. For a constant $K \geq 1$, f is defined to be K-quasiconformal if

$$\frac{1+\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty} \leq K.$$

A quasiconformal automorphism of $\mathbb{D}$ extends continuously to a quasisymmetric automorphism of $\mathbb{S}$. Conversely, a quasisymmetric automorphism of $\mathbb{S}$ extends continuously to a quasiconformal automorphism of $\mathbb{D}$. In both directions, the quasisymmetric constant M or M' and the quasiconformal constant K appearing in their definitions are related to each other. This fact is also valid for quasiconformal automorphisms of $\mathbb{H}$ and quasisymmetric functions on $\mathbb{R}$.

We review the canonical quasiconformal extension of an arbitrary quasisymmetric function. For a quasisymmetric function $h : \mathbb{R} \to \mathbb{R}$, set

$$\alpha(x,y) = \int_0^1 h(x+ty)dt, \quad \beta(x,y) = \int_0^1 h(x-ty)dt$$

and define

$$F_h^{(r)}(z) = \frac{1}{2}\{\alpha(x,y)+\beta(x,y)\} + \frac{ir}{2}\{\alpha(x,y)-\beta(x,y)\}$$

for $z = x + iy \in \mathbb{H}$. This is called the *Beurling-Ahlfors extension* of h with parameter $r > 0$, and it was proved in [3] that this extension is quasiconformal with certain estimate of its quasiconformal constant. The improvement of the constant as shown in the following theorem is due to Lehtinen [10].

Theorem 2.1. *For an M-quasisymmetric function h of $\mathbb{R}$, its Beurling-Ahlfors extension $F_h^{(r)}$ is a K-quasiconformal automorphism of $\mathbb{H}$ with boundary function h. Here K can be estimated as $K \leq \min\{M^{3/2}, 2M-1\}$ for some suitable choice of r.*

Carleson [4] improveed this theorem in the case where the quasisymmetric quotient $m_h(x,t)$ is uniformly close to one; more precisely, if there is a positive increasing function $\varepsilon(t)$ of $t > 0$ such that

$$(*) \qquad \frac{1}{1+\varepsilon(t)} \leq m_h(x,t) \leq 1 + \varepsilon(t)$$

and $\varepsilon(t)$ gets sufficiently small when $t \to 0$. A typical case occurs when h is a *symmetric* function, which means $\lim_{t\to 0} \varepsilon(t) = 0$ by definition. It was proved in [4] that the complex dilatation $\widetilde{\mu}(z)$ of the Beurling-Ahlfors extension at $z = x + iy$ is dominated by this function $\varepsilon(t)$ for $y = t$. An explicit computation of the constant can be found in [12].

Theorem 2.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a quasisymmetric function that satisfies $(*)$ for a positive increasing function $\varepsilon(t)$. Let $\widetilde{\mu}(z)$ be the complex dilatation of the Beurling-Ahlfors extension $F_h^{(2)}(z)$ of h with parameter $r = 2$. Then $|\widetilde{\mu}(z)| \leq 4\varepsilon(y)$ for every $z = x + iy \in \mathbb{H}$.*

2.2. **Cross ratio and the Liouville cocycle.** For positively ordered distinct points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{S}$, we consider the cross ratio $[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$. We also utilize the following modification which we call the *alternative cross ratio*:

$$\begin{aligned} [\zeta_1, \zeta_2, \zeta_3, \zeta_4]_* &= \frac{[\zeta_2, \zeta_3, \zeta_4, \zeta_1]}{[\zeta_1, \zeta_2, \zeta_3, \zeta_4]} \\ &= \frac{-1}{1 - [\zeta_1, \zeta_2, \zeta_3, \zeta_4]} = -\frac{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}{(\zeta_3 - \zeta_4)(\zeta_1 - \zeta_2)} \in (0, \infty). \end{aligned}$$

Just as in the usual case, the alternative cross ratio is invariant under the action of the Möbius transformations $\mathrm{M\ddot{o}b}(\mathbb{S}) \cong \mathrm{PSL}(2, \mathbb{R})$:

$$[\gamma(\zeta_1), \gamma(\zeta_2), \gamma(\zeta_3), \gamma(\zeta_4)]_* = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]_*$$

for every $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{S})$.

The *Liouville measure* on the space of oriented hyperbolic geodesic lines on the Poincaré model $\mathbb{D}$, which is identified with $\mathbb{S} \times \mathbb{S}$ minus the diagonal set Δ, is given by

$$d\mathcal{L}(x, y) = \frac{1}{4\sin^2((x-y)/2)} dx dy,$$

where x and y are the coordinates of $\mathbb{S}$ under the universal covering $\zeta = e^{ix} : \mathbb{R} \to \mathbb{S}$. The cross ratio can be represented by the integral of this measure. For an absolutely continuous self-homeomorphism g of $\mathbb{S}$, the pull-back of $d\mathcal{L}$ is defined by

$$d(g^*\mathcal{L})(x,y) = \frac{\widetilde{g}'(x)\widetilde{g}'(y)}{4\sin^2((\widetilde{g}(x)-\widetilde{g}(y))/2)}dxdy,$$

where $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ is the lift of g. The *Liouville cocycle* $c(g)(x,y)$ is the density of the signed measure $d(g^*\mathcal{L})(x,y) - d\mathcal{L}(x,y)$ with respect to the Lebesgue measure. For details on the Liouville cocycle, consult the monograph by Navas [13].

By the invariance of the cross ratio under Möb($\mathbb{S}$), we see that the Liouville measure is also invariant under Möb($\mathbb{S}$); $d(\gamma^*\mathcal{L}) = d\mathcal{L}$ for every $\gamma \in$ Möb($\mathbb{S}$). Clearly $\|c(\gamma)\|_1 = 0$ for every $\gamma \in$ Möb($\mathbb{S}$). The norm $\|c(g)\|_1$ measures the difference of g from Möb($\mathbb{S}$).

The alternative cross ratio is also represented by the integral of the Liouville measure:

$$\log\,[\zeta_1,\zeta_2,\zeta_3,\zeta_4]_* = \int_{x_4}^{x_1+2\pi}\int_{x_2}^{x_3}\frac{1}{4\sin^2((x-y)/2)}dxdy - \int_{x_3}^{x_4}\int_{x_1}^{x_2}\frac{1}{4\sin^2((x-y)/2)}dxdy$$

for $\zeta_j = e^{ix_j}$ $(j = 1,2,3,4)$ with $x_1 < x_2 < x_3 < x_4$ and $x_4 - x_1 < 2\pi$. For an absolutely continuous self-homeomorphism g of $\mathbb{S}$, it follows that

$$\log\,\frac{[g(\zeta_1),g(\zeta_2),g(\zeta_3),g(\zeta_4)]_*}{[\zeta_1,\zeta_2,\zeta_3,\zeta_4]_*} = \int_{x_4}^{x_1+2\pi}\int_{x_2}^{x_3} c(g)(x,y)\,dxdy - \int_{x_3}^{x_4}\int_{x_1}^{x_2} c(g)(x,y)\,dxdy.$$

This implies that

$$\begin{aligned}&\left|\log\,\frac{[g(\zeta_1),g(\zeta_2),g(\zeta_3),g(\zeta_4)]_*}{[\zeta_1,\zeta_2,\zeta_3,\zeta_4]_*}\right|\\ \leq\ &\int_{\mathbb{S}}\int_{x_2}^{x_3}|c(g)(x,y)|\,dxdy + \int_{\mathbb{S}}\int_{x_1}^{x_2}|c(g)(x,y)|\,dxdy \leq \|c(g)\|_1.\end{aligned}$$

Let $\mathrm{BL}_+(\mathbb{S})$ denote the group of orientation-preserving bi-Lipschitz automorphisms of $\mathbb{S}$. For every $g \in \mathrm{BL}_+(\mathbb{S})$ and every $\phi \in L^1(\mathbb{S}\times\mathbb{S}\setminus\Delta)$, set

$$(g^*\phi)(x,y) = \phi(g(x),g(y))g'(x)g'(y).$$

This defines a unitary (anti-homomorphic) representation θ of $\mathrm{BL}_+(\mathbb{S})$ on $L^1(\mathbb{S}\times\mathbb{S}\setminus\Delta)$ by $\theta(g) = g^*$. Then the Liouville cocycle $c(g)$ satisfies a cocycle condition associated with the unitary representation θ. Namely,

$$c(g_1g_2) = c(g_2) + \theta(g_2)c(g_1)$$

for any $g_1, g_2 \in \mathrm{BL}_+(\mathbb{S})$.

The use of Liouville cocycles in the study of circle diffeomorphisms has been investigated by Navas. One of his results is the following, see [14].

Theorem 2.3. *Suppose that a non-metabelian subgroup G of the group* $\mathrm{Diff}_+^{3+r}(\mathbb{S})$ $(0 \leq r \leq \infty)$ *of orientation-preserving* C^{3+r}*-diffeomorphisms implies uniform boundedness of*

$\|c(g)\|_1$ *for all* $g \in G$. *Then* G *is conjugate to a subgroup of* $\mathrm{M\ddot{o}b}(\mathbb{S})$ *by an element of* $\mathrm{Diff}_+^{3+r}(\mathbb{S})$.

If a group G of diffeomorphisms of $\mathbb{S}$ is conjugate to a subgroup of $\mathrm{M\ddot{o}b}(\mathbb{S})$, then in particular it satisfies the convergence property introduced by Gehring and Martin [7]. Actually it is known that the convergence property characterizes the homeomorphic conjugation to $\mathrm{M\ddot{o}b}(\mathbb{S})$. Theorem 2.3 asserts that the conjugation can be given by a diffeomorphism of the same regularity as the elements of G.

3. Bi-Lipschitz continuity

In this section, we will show that $g : \mathbb{S} \to \mathbb{S}$ is bi-Lipschitz continuous if $\|c(g)\|_1 < \infty$. Only the Lipschitz continuity of g is used in the proofs of Theorems 4.1 and 5.1 below, but we can prove more, which is of independent interest.

Since Möbius transformations of $\mathbb{S}$ are bi-Lipschitz, to verify this claim, we can normalize g by composing a Möbius transformation so that g fixes three distinct points on $\mathbb{S}$. To make the fixed points symmetric, we prefer using $\omega^0, \omega^1, \omega^2$ for $\omega = \exp(2\pi i/3)$ as a set of such three fixed points on $\mathbb{S}$.

Theorem 3.1. *If an orientation-preserving absolutely continuous self-homeomorphism* g *of* $\mathbb{S}$ *satisfies* $\|c(g)\|_1 < \infty$, *then* g *is bi-Lipschitz continuous. If* g *is normalized by fixing the three points on* $\mathbb{S}$, *then*

$$c\,\exp(-\|c(g)\|_1)\,|x - x'| \le |\widetilde{g}(x) - \widetilde{g}(x')| \le C\exp(\|c(g)\|_1)\,|x - x'|$$

for any $x, x' \in \mathbb{R}$, *where* $C > 0$ *is a universal constant and* $c > 0$ *is a constant depending only on* $\|c(g)\|_1$.

Proof. Dividing the interval between x and x' into several pieces, we have only to show the statement in the case where $|x - x'|$ is sufficiently small, say, $|x - x'| < \pi/18$. We may also assume that $x < x'$. For any two points $e^{ix}, e^{ix'} \in \mathbb{S}$ with this condition, we can choose two points ω and ω' from $\{\omega^0, \omega^1, \omega^2\}$ so that e^{ix} and $e^{ix'}$ lie in the middle part of the circular interval between ω and ω' of length either $2\pi/3$ or $4\pi/3$. Here the middle part means the union of the second and the third quarters of the interval, when divided into four equal parts.

We consider the following variation of the cross ratio of positively ordered points:

$$[\omega, e^{ix}, e^{ix'}, \omega'] = \frac{(\omega - e^{ix'})(e^{ix} - \omega')}{(\omega - \omega')(e^{ix} - e^{ix'})};$$

$$[g(\omega), g(e^{ix}), g(e^{ix'}), g(\omega')] = \frac{(\omega - e^{i\widetilde{g}(x')})(e^{i\widetilde{g}(x)} - \omega')}{(\omega - \omega')(e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x')})}.$$

The logarithm of the ratio of these two values is bounded by $\|c(g)\|_1$, which implies that

$$\exp(-\|c(g)\|_1) \le \left|\frac{e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x')}}{e^{ix} - e^{ix'}}\right| \cdot \left|\frac{(\omega - e^{ix'})(e^{ix} - \omega')}{(\omega - e^{i\widetilde{g}(x')})(e^{i\widetilde{g}(x)} - \omega')}\right| \le \exp(\|c(g)\|_1).$$

Here, the first factor of the middle term is

$$\left|\frac{(e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x')})}{(e^{ix} - e^{ix'})}\right| = \left|\frac{\sin((\widetilde{g}(x) - \widetilde{g}(x'))/2)}{\sin((x - x')/2)}\right|.$$

By $|x-x'| < \pi/18$ and $|\widetilde{g}(x)-\widetilde{g}(x')|/2 < 2\pi/3$, this is comparable to $|\widetilde{g}(x)-\widetilde{g}(x')|/|x-x'|$ having a universal multiplicative error constant.

The second factor of the middle term is bounded away from zero by some universal constant $C' > 0$ since e^{ix} and $e^{ix'}$ sit in the middle part of the interval between ω and ω'. On the other hand, it is bounded from above by some constant $c' > 0$ depending only on $\|c(g)\|_1$. This can be verified as follows.

For each of e^{ix} and $e^{ix'}$, we consider the variation of the cross ratio of positively ordered points given together with $\{\omega^0, \omega^1, \omega^2\}$. By a suitable choice of cyclic permutation, the following values can be taken as their variations:

$$\frac{e^{ix} - \omega'}{e^{i\widetilde{g}(x)} - \omega'} \cdot \frac{e^{i\widetilde{g}(x)} - \omega^*}{e^{ix} - \omega^*}; \qquad \frac{e^{ix'} - \omega}{e^{i\widetilde{g}(x')} - \omega} \cdot \frac{e^{i\widetilde{g}(x')} - \omega^*}{e^{ix'} - \omega^*}.$$

These values are bounded from above by $\exp(\|c(g)\|_1)$. Here $\omega^* \in \{\omega^0, \omega^1, \omega^2\}$ is the farthest point from e^{ix} and $e^{ix'}$ respectively; $e^{i\widetilde{g}(x)}$ and $e^{i\widetilde{g}(x')}$ cannot be close to ω^*. Hence, also using the fact that e^{ix} and $e^{ix'}$ are in the middle part of the interval between ω and ω', we see that $e^{i\widetilde{g}(x)}$ cannot get close to ω' and $e^{i\widetilde{g}(x')}$ cannot get close to ω. Thus the denominator $|(\omega - e^{i\widetilde{g}(x')})(e^{i\widetilde{g}(x)} - \omega')|$ of the fraction in question is bounded away from zero by a constant depending only on $\|c(g)\|_1$.

Plugging these estimates in the previous inequality, we obtain the required estimate of $|\widetilde{g}(x) - \widetilde{g}(x')|$ in terms of $|x - x'|$. □

Corollary 3.2. *The set of all orientation-preserving absolutely continuous self-homeomorphisms g of $\mathbb{S}$ with $\|c(g)\|_1 < \infty$ is a subgroup of the group $\mathrm{BL}_+(\mathbb{S})$ of bi-Lipschitz automorphisms of $\mathbb{S}$.*

Proof. By Theorem 3.1, the condition $\|c(g)\|_1 < \infty$ implies that g is bi-Lipschitz continuous. Consider the unitary representation θ of $\mathrm{BL}_+(\mathbb{S})$ on $L^1(\mathbb{S}\times\mathbb{S}\setminus\Delta)$. The Liouville cycle satisfies

$$c(g_1 g_2) = c(g_2) + \theta(g_2)c(g_1)$$

for any $g_1, g_2 \in \mathrm{BL}_+(\mathbb{S})$. This in particular implies that $\|c(g_1g_2)\|_1 \le \|c(g_1)\|_1 + \|c(g_1)\|_1$ and $\|c(g^{-1})\|_1 = \|c(g)\|_1$. Thus the statement follows. □

4. Integrability

In this section, we prove that the integrability of the Liouville cocycle for an absolutely continuous quasisymmetric automorphism g implies a certain integrability of the complex dilatation of some quasiconformal extension of g. By virtue of Theorem 3.1, we may assume that g is bi-Lipschitz hereafter. Also, we always assume that self-homeomorphisms of $\mathbb{S}$ are orientation-preserving.

Theorem 4.1. *Let $g : \mathbb{S} \to \mathbb{S}$ be a bi-Lipschitz automorphism with $\|c(g)\|_1 < \infty$. Then g extends continuously to a quasiconformal automorphism of $\mathbb{D}$ whose complex dilatation $\mu(\zeta)$ satisfies*

$$\int_{\mathbb{S}} |\mu((1-\tau)e^{ix})|dx \leq a\tau$$

for every $\tau \in (0,1)$, where $a > 0$ is a constant depending only on $\|c(g)\|_1$.

We set up the proof of Theorem 4.1. By the invariance of the norm of the Liouville cocycle and the complex dilatation of the quasiconformal extension under post-composition of a Möbius transformation, we may assume that g fixes three distinct points on $\mathbb{S}$, say, $\omega^0, \omega^1, \omega^2$ for $\omega = \exp(2\pi i/3)$ as before. These points divide the circle $\mathbb{S}$ into three circular intervals $I_0 = [\omega^1, \omega^2)$, $I_1 = [\omega^2, \omega^0)$ and $I_2 = [\omega^0, \omega^1)$. We define a map $\omega : \mathbb{S} \to \{\omega^0, \omega^1, \omega^2\}$ by $\omega(\zeta) = \omega^k$ if $\zeta \in I_k$ for $k = 0, 1, 2$.

For three consecutive points $e^{i(x-t)}$, e^{ix} and $e^{i(x+t)}$ on $\mathbb{S}$, we pick up $\omega(e^{i(x+t)}) \in \mathbb{S}$ as the fourth point. Representing $\omega(e^{i(x+t)}) = e^{i\widetilde{\omega}}$ by an appropriate $\widetilde{\omega} \in \mathbb{R}$, we first compare the quasisymmetric quotient defined by the three points with the variation of the alternative cross ratio defined by the four points. As before, we pass to the lift $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ of g via the universal covering $\zeta = e^{ix} : \mathbb{R} \to \mathbb{S}$.

Lemma 4.2. *Assume that $\|c(g)\|_1 < \infty$ and g fixes $\omega^0, \omega^1, \omega^2$. Then there exist a universal constant $B > 0$ and a constant $t_0 > 0$ depending only on $\|c(g)\|_1$ such that*

$$\begin{aligned} &\left| \log \frac{[e^{i\widetilde{g}(x-t)}, e^{i\widetilde{g}(x)}, e^{i\widetilde{g}(x+t)}, e^{i\widetilde{\omega}}]_*}{[e^{i(x-t)}, e^{ix}, e^{i(x+t)}, e^{i\widetilde{\omega}}]_*} - \log m_{\widetilde{g}}(x,t) \right| \\ \leq\ & B\{t + \widetilde{g}(x+t) - \widetilde{g}(x-t)\} \end{aligned}$$

for every $x \in \mathbb{R}$ and for every $t \in \mathbb{R}$ with $0 < t \leq t_0$.

Proof. We may assume that $0 < t \leq \pi/6$ by choosing $t_0 \leq \pi/6$. First we estimate

$$[e^{i(x-t)}, e^{ix}, e^{i(x+t)}, e^{i\widetilde{\omega}}]_* = \frac{|e^{i\widetilde{\omega}} - e^{i(x-t)}|}{|e^{i\widetilde{\omega}} - e^{i(x+t)}|}.$$

We may further assume that $e^{i(x+t)} \in I_0$ and $e^{i\widetilde{\omega}} = \omega(e^{i(x+t)}) = 1$. Then

$$\frac{|e^{i\widetilde{\omega}} - e^{i(x-t)}|}{|e^{i\widetilde{\omega}} - e^{i(x+t)}|} = \frac{\sin((x-t)/2)}{\sin((x+t)/2)} = 1 - \frac{2\cos(x/2)\sin(t/2)}{\sin((x+t)/2)}.$$

Since $\pi/3 \leq (x+t)/2 < 2\pi/3$ and $\pi/4 \leq x/2 < 2\pi/3$ by the above assumptions, we see that

$$1 - \frac{\sqrt{2}}{\sqrt{3}}\,t \leq [e^{i(x-t)}, e^{ix}, e^{i(x+t)}, e^{i\widetilde{\omega}}]_* \leq 1 + \frac{1}{\sqrt{3}}\,t.$$

Next, we will apply a similar estimate to the first factor of

$$[e^{i\widetilde{g}(x-t)}, e^{i\widetilde{g}(x)}, e^{i\widetilde{g}(x+t)}, e^{i\widetilde{\omega}}]_* = \frac{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x-t)}|}{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x+t)}|} \cdot \frac{|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x+t)}|}{|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x-t)}|}.$$

As before, we have

$$\begin{aligned} \frac{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x-t)}|}{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x+t)}|} &= \frac{\sin(\widetilde{g}(x-t)/2)}{\sin(\widetilde{g}(x+t)/2)} \\ &= 1 - \frac{2\cos\{(\widetilde{g}(x+t)+\widetilde{g}(x-t))/4\}\sin\{(\widetilde{g}(x+t)-\widetilde{g}(x-t))/4\}}{\sin(\widetilde{g}(x+t)/2)}. \end{aligned}$$

Here $\pi/3 \le \widetilde{g}(x+t)/2 < 2\pi/3$ still holds, but in order to assume

$$\frac{\pi}{4} \le \frac{\widetilde{g}(x+t)+\widetilde{g}(x-t)}{4} < \frac{2\pi}{3},$$

we have to choose $t_0 > 0$ so that $\widetilde{g}(x+t) - \widetilde{g}(x-t) \le \pi/3$ whenever $t \le t_0$. By Theorem 3.1,

$$t_0 = \frac{\pi}{6} \cdot \frac{\exp(-\|c(g)\|_1)}{C}$$

with $C \ge 1$ is appropriate for this purpose. Hence, if $0 < t \le t_0$, then

$$1 - \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\widetilde{g}(x+t)-\widetilde{g}(x-t)}{2} \le \frac{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x-t)}|}{|e^{i\widetilde{\omega}} - e^{i\widetilde{g}(x+t)}|} \le 1 + \frac{1}{\sqrt{3}} \cdot \frac{\widetilde{g}(x+t)-\widetilde{g}(x-t)}{2}.$$

For the estimate of the second factor, we use a fact that $|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x+t)}|$ is comparable to $\widetilde{g}(x+t) - \widetilde{g}(x)$ with an multiplicative error factor dominated by this difference. This is also true for $|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x-t)}|$. More precisely,

$$\begin{aligned} 1 &> \frac{|e^{i\widetilde{g}(x+t)} - e^{i\widetilde{g}(x)}|}{|\widetilde{g}(x+t)-\widetilde{g}(x)|} = \frac{\sin\{(\widetilde{g}(x+t)-\widetilde{g}(x))/2\}}{(\widetilde{g}(x+t)-\widetilde{g}(x))/2} > \left(1 + \frac{\widetilde{g}(x+t)-\widetilde{g}(x)}{2}\right)^{-1}; \\ 1 &> \frac{|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x-t)}|}{|\widetilde{g}(x)-\widetilde{g}(x-t)|} = \frac{\sin\{(\widetilde{g}(x)-\widetilde{g}(x-t))/2\}}{(\widetilde{g}(x)-\widetilde{g}(x-t))/2} > \left(1 + \frac{\widetilde{g}(x)-\widetilde{g}(x-t)}{2}\right)^{-1}. \end{aligned}$$

Here we use

$$0 < \frac{\widetilde{g}(x+t)-\widetilde{g}(x)}{2} < \frac{2\pi}{3}; \quad 0 < \frac{\widetilde{g}(x)-\widetilde{g}(x-t)}{2} < \frac{2\pi}{3}$$

for these estimates. Thus we have

$$m_{\widetilde{g}}(x,t)\left(1 + \frac{\widetilde{g}(x+t)-\widetilde{g}(x)}{2}\right)^{-1} \le \frac{|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x+t)}|}{|e^{i\widetilde{g}(x)} - e^{i\widetilde{g}(x-t)}|} \le m_{\widetilde{g}}(x,t)\left(1 + \frac{\widetilde{g}(x)-\widetilde{g}(x-t)}{2}\right).$$

All the above estimates allow us to conclude that if $0 < t \leq t_0$ then

$$\begin{aligned}
&\left|\log \frac{[e^{i\widetilde{g}(x-t)}, e^{i\widetilde{g}(x)}, e^{i\widetilde{g}(x+t)}, e^{i\widetilde{\omega}}]_*}{[e^{i(x-t)}, e^{ix}, e^{i(x+t)}, e^{i\widetilde{\omega}}]_*} - \log m_{\widetilde{g}}(x,t)\right| \\
\leq\ & -\log\left(1 - \frac{\sqrt{2}}{\sqrt{3}}t\right) - \log\left(1 - \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\widetilde{g}(x+t) - \widetilde{g}(x-t)}{2}\right) \\
& + \log\left(1 + \frac{\widetilde{g}(x+t) - \widetilde{g}(x-t)}{2}\right) \\
\leq\ & B\{t + \widetilde{g}(x+t) - \widetilde{g}(x-t)\},
\end{aligned}$$

where $B > 0$ is some universal constant involved with an estimate of $-\log(1-x)$ in terms of x. This estimate is possible because t and $(\widetilde{g}(x+t) - \widetilde{g}(x-t))/2$ are bounded from above suitably. □

Next, we take the supremum of the quasisymmetric quotient $m_{\widetilde{g}}(x,t)$ over the interval $[x-t, x+t]$ to estimate the complex dilatation $\widetilde{\mu}$ of the Beurling-Ahlfors extension $F^{(2)}_{\widetilde{g}}$ of $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ with parameter $r = 2$. We define the optimal quasisymmetric constant for $\widetilde{g}$ by

$$M(\widetilde{g}) = \sup_{x\in\mathbb{R},\ t>0} \max\{m_{\widetilde{g}}(x,t), m_{\widetilde{g}}(x,t)^{-1}\}.$$

Proposition 4.3. *Assume that $M(\widetilde{g}) < \infty$. Then there is a constant $b > 0$ depending only on $M(\widetilde{g})$ such that*

$$|\widetilde{\mu}(x+iy)| \leq b \sup_{x',y'} |\log m_{\widetilde{g}}(x',y')|$$

holds for every $x + iy \in \mathbb{H}$, where the supremum is taken over all $x' \in \mathbb{R}$ and $y' > 0$ with $[x'-y', x'+y'] \subset [x-y, x+y]$.

Proof. For a quasisymmetric function $\widetilde{g} : \mathbb{R} \to \mathbb{R}$, the Beurling-Ahlfors extension $F^{(2)}_{\widetilde{g}} : \mathbb{H} \to \mathbb{H}$ is defined by

$$F^{(2)}_{\widetilde{g}}(z) = \frac{1}{2}\{\alpha(x,y) + \beta(x,y)\} + i\{\alpha(x,y) - \beta(x,y)\},$$

where

$$\alpha(x,y) = \int_0^1 \widetilde{g}(x+ty)dt; \qquad \beta(x,y) = \int_0^1 \widetilde{g}(x-ty)dt.$$

The partial derivatives α_x, α_y, β_x and β_y at $z = x + iy$ can be represented by the values of $\widetilde{g}$ only in the interval $[x-y, x+y]$. Especially, the complex dilatation $\widetilde{\mu}(z)$ of $F^{(2)}_{\widetilde{g}}(z)$ at $z = x + iy$ is estimated in terms of the quasisymmetric quotients $m_{\widetilde{g}}(x',y')$ for all x' and y' with $[x'-y', x'+y'] \subset [x-y, x+y]$. This can be found in Carleson [4]. See also

[12] for an exposition of this result. Then the local version of Theorem 2.2 is valid. More precisely, defining $\varepsilon(x,y)$ by

$$1+\varepsilon(x,y)=\sup\{\max\{m_{\widetilde{g}}(x',y'),m_{\widetilde{g}}(x',y')^{-1}\}:[x'-y',x'+y']\subset[x-y,x+y]\},$$

we have $|\widetilde{\mu}(x+iy)|\leq 4\varepsilon(x,y)$. Taking the logarithm in the above equation, we can find a suitable constant $b>0$ depending only on $M(\widetilde{g})$. □

Remark. By a similar but simpler argument as in the proof of Lemma 4.2, we see that $M(\widetilde{g})$ can be estimated in terms of $\|c(g)\|_1$.

Now we are ready to prove the theorem.

Proof of Theorem 4.1. By post-composition of a Möbius transformation, we may assume that g fixes $\omega^0,\omega^1,\omega^2$. Then take the lift $\widetilde{g}:\mathbb{R}\to\mathbb{R}$ of g and the Beurling-Ahlfors extension $F^{(2)}_{\widetilde{g}}:\mathbb{H}\to\mathbb{H}$ of $\widetilde{g}$. Its complex dilatation $\widetilde{\mu}$ satisfies

$$|\widetilde{\mu}(x+iy)|\leq b\sup_{x',y'}|\log m_{\widetilde{g}}(x',y')|$$

by Proposition 4.3. Note that $F^{(2)}_{\widetilde{g}}$ satisfies $F^{(2)}_{\widetilde{g}}(z+2\pi)=F^{(2)}_{\widetilde{g}}(z)+2\pi$ since $\widetilde{g}$ satisfies it too. By projecting $F^{(2)}_{\widetilde{g}}$ down by the holomorphic universal covering $\zeta=e^{iz}:\mathbb{H}\to\mathbb{D}\setminus\{0\}$ and filling 0 as its fixed point, we have a quasiconformal automorphism of $\mathbb{D}$ continuously extending g whose complex dilatation $\mu(\zeta)$ ($\zeta\in\mathbb{D}$) satisfies $|\mu(\zeta)|=|\widetilde{\mu}(z)|$ (a.e.). If we write $\zeta=(1-\tau)e^{ix}$, then $\tau=1-e^{-y}$ or $y=-\log(1-\tau)$. For a fixed $\tau\in(0,1)$, the integration of $\mu((1-\tau)e^{ix})$ over $x\in[0,2\pi)$ yields

$$\int_{\mathbb{S}}|\mu((1-\tau)e^{ix})|dx\leq b\int_{\mathbb{S}}\sup_{x',t'}|\log m_{\widetilde{g}}(x',t')|dx$$

for $t=-\log(1-\tau)$, where the supremum is taken over all x' and t' with $[x'-t',x'+t']\subset[x-t,x+t]$.

Next, applying Lemma 4.2, we see that if $0<t\leq t_0$ then

$$\begin{aligned}&\int_{\mathbb{S}}\sup_{x',t'}|\log m_{\widetilde{g}}(x',t')|dx\\ \leq\ &\int_{\mathbb{S}}\sup_{x',t'}\left|\log\frac{[e^{i\widetilde{g}(x'-t')},e^{i\widetilde{g}(x')},e^{i\widetilde{g}(x'+t')},e^{i\widetilde{\omega}}]_*}{[e^{i(x'-t')},e^{ix'},e^{i(x'+t')},e^{i\widetilde{\omega}}]_*}\right|dx\\ &+\int_{\mathbb{S}}\sup_{x',t'}B\{t'+\widetilde{g}(x'+t')-\widetilde{g}(x'-t')\}dx.\end{aligned}$$

Here the first term after the inequality sign is bounded by

$$\begin{aligned} &\int_{\mathbb{S}} \sup_{x',t'} \left(\int_{\mathbb{S}} \int_{x'-t'}^{x'+t'} |c(g)(u,v)| du dv \right) dx \\ \leq \; &\int_{\mathbb{S}} \left(\int_{\mathbb{S}} \int_{x-t}^{x+t} |c(g)(u,v)| du dv \right) dx = 2\|c(g)\|_1 t. \end{aligned}$$

The second term is bounded by

$$\int_{\mathbb{S}} B\{t + \widetilde{g}(x+t) - \widetilde{g}(x-t)\} dx = 6\pi B t.$$

Gathering all the above estimates together, we can conclude that

$$\int_{\mathbb{S}} |\mu((1-\tau)e^{ix})| dx \leq b(2\|c(g)\|_1 + 6\pi B) t$$

if $t = -\log(1-\tau) \leq t_0$. On the other hand, since $|\mu(\zeta)| < 1$, the integral is bounded by 2π for any $\tau \in (0,1)$. Hence we can find a constant $a > 0$ such that the integral is bounded by $a\tau$ for any $\tau \in (0,1)$. Since t_0 and b depend only on $\|c(g)\|_1$ and B is universal, the constant a can be taken depending only on $\|c(g)\|_1$. □

5. Asymptotic conformality

In this section, applying the arguments from previous sections, we investigate the relationship between the vanishing order of the complex dilatation of the quasiconformal extension of g and its integrability with respect to the hyperbolic metric on $\mathbb{D}$ under the assumption $\|c(g)\|_1 < \infty$.

We say that a quasiconformal automorphism of $\mathbb{D}$ is *asymptotically conformal* if its complex dilatation $\mu(\zeta)$ vanishes at the boundary $\partial\mathbb{D} = \mathbb{S}$, that is, $\operatorname{ess.sup}_{|\zeta| \geq 1-\tau} |\mu(\zeta)| \to 0$ as $\tau \to 0$. On the other hand, a quasisymmetric automorphism g of $\mathbb{S}$ is symmetric if the quasisymmetric quotient $m_{\widetilde{g}}(x,t)$ of the lift $\widetilde{g}$ satisfies $m_{\widetilde{g}}(x,t) \to 1$ as $t \to 0$ uniformly on $x \in \mathbb{R}$. Then the boundary extension of an asymptotically conformal automorphism of $\mathbb{D}$ to $\mathbb{S}$ is a symmetric automorphism g of $\mathbb{S}$ and conversely a symmetric automorphism g of $\mathbb{S}$ extends to an asymptotically conformal automorphism of $\mathbb{D}$. This was essentially proved by Carleson [4]. See Becker and Pommerenke [2] and Gardiner and Sullivan [6] for properties of symmetric automorphisms of $\mathbb{S}$ and asymptotically conformal homeomorphisms of $\mathbb{D}$.

Our first application is as follows.

Theorem 5.1. *Let $g : \mathbb{S} \to \mathbb{S}$ be a bi-Lipschitz automorphism of $\mathbb{S}$. If $\|c(g)\|_1 < \infty$, then g is a symmetric automorphism of $\mathbb{S}$.*

Proof. We will show that $m_{\widetilde{g}}(x,t) \to 1$ uniformly as $t \to 0$. By Theorem 3.1 and Lemma 4.2, we have only to prove that

$$\begin{aligned} &\log \frac{[e^{i\widetilde{g}(x-t)}, e^{i\widetilde{g}(x)}, e^{i\widetilde{g}(x+t)}, e^{i\widetilde{\omega}}]_*}{[e^{i(x-t)}, e^{ix}, e^{i(x+t)}, e^{i\widetilde{\omega}}]_*} \\ = &\int_{\widetilde{\omega}}^{x-t+2\pi} \int_x^{x+t} c(g)(u,v)\, dudv - \int_{x+t}^{\widetilde{\omega}} \int_{x-t}^{x} c(g)(u,v)\, dudv \end{aligned}$$

tends to 0 uniformly as $t \to 0$. Its absolute value is bounded by

$$\int_{\mathbb{S}} \int_{x-t}^{x+t} |c(g)(u,v)| dudv = \int_{x-t}^{x+t} F(u) du,$$

where $F(u) = \int_{\mathbb{S}} |c(g)(u,v)| dv$. Since $F(u)$ is integrable, its indefinite integral is a uniformly continuous function on $\mathbb{S}$. Hence the integral from $x-t$ to $x+t$ tends to 0 uniformly as $t \to 0$. □

This theorem implies that the quasiconformal extension of g to $\mathbb{D}$ is asymptotically conformal. To measure the vanishing order of the complex dilatation of an asymptotically conformal automorphism quantitatively, we define

$$\kappa_\mu(\tau) = \operatorname{ess.sup}_{|\zeta| \geq 1-\tau} |\mu(\zeta)|$$

for every Beltrami coefficient μ on $\mathbb{D}$ and for every $\tau \in (0,1)$. The following result was first proved by Carleson [4] for an asymptotically conformal automorphism of the upper half-plane $\mathbb{H}$ where the boundary is restricted to $\mathbb{R}$ and not including ∞. The case of the unit disk $\mathbb{D}$ was treated by Anderson, Becker and Lesley [1].

Theorem 5.2. *If the complex dilatation $\mu(\zeta)$ of a quasiconformal automorphism of $\mathbb{D}$ satisfies*

$$\int_0^1 \frac{\kappa_\mu(\tau)}{\tau} d\tau < \infty,$$

then it extends to a quasisymmetric automorphism g of $\mathbb{S}$ that is continuously differentiable. On the other hand, if

$$\int_0^1 \frac{\kappa_\mu(\tau)^2}{\tau} d\tau < \infty,$$

then g is absolutely continuous with local L^q-derivatives ($q < \infty$).

Our second application is concerning the integrability of the complex dilatation with respect to the hyperbolic metric $\rho_{\mathbb{D}}(\zeta)|d\zeta|$ on $\mathbb{D}$ when we assume that an asymptotically conformal automorphism satisfies the condition as in Theorem 5.2 in addition to the integrability of the Liouville cocycle for its boundary extension.

Theorem 5.3. *Suppose that a bi-Lipschitz automorphism g of $\mathbb{S}$ satisfies $\|c(g)\|_1 < \infty$ and has a quasiconformal extension to $\mathbb{D}$ whose complex dilatation μ satisfies*

$$\int_0^1 \frac{\kappa_\mu(\tau)^\beta}{\tau} d\tau < \infty$$

for some $\beta > 0$. Then μ satisfies

$$\int_{\mathbb{D}} |\mu(\zeta)|^p \rho_{\mathbb{D}}^2(\zeta) d\xi d\eta < \infty$$

for every $p \geq 1 + \beta$.

Proof. By Theorem 4.1, there is a constant $a > 0$ such that

$$\int_0^{2\pi} |\mu((1-\tau)e^{ix})| dx \leq a\tau.$$

Replacing $|\mu((1-\tau)e^{ix})|$ to the power $p-1 \geq \beta$ with its supremum, we have

$$\int_0^{2\pi} |\mu((1-\tau)e^{ix})|^p dx \leq \sup_{0 \leq x < 2\pi} |\mu((1-\tau)e^{ix})|^{p-1} \int_0^{2\pi} |\mu((1-\tau)e^{ix})| dx \leq a\tau\kappa_\mu(\tau)^\beta.$$

Therefore

$$\begin{aligned} \int_{\mathbb{D}} |\mu(\zeta)|^p \rho_{\mathbb{D}}^2(\zeta) d\xi d\eta &= \int_0^1 \int_0^{2\pi} |\mu((1-\tau)e^{ix})|^p \frac{4(1-\tau)}{\tau^2(2-\tau)^2} dx d\tau \\ &\leq \int_0^1 \int_0^{2\pi} |\mu((1-\tau)e^{ix})|^p \tau^{-2} dx d\tau \\ &\leq a \int_0^1 \kappa_\mu(\tau)^\beta \tau^{-1} d\tau < \infty. \end{aligned}$$

This proves the statement. □

It is easy to check that if g be a diffeomorphism of $\mathbb{S}$ with an α-Hölder continuous derivative for some $\alpha \in (0,1)$, then the quasisymmetric quotient $m_{\widetilde{g}}(x,t)$ for the lift $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ satisfies $m_{\widetilde{g}}(x,t) = 1 + O(t^\alpha)$ uniformly as $t \to 0$ (see [12]). Then by Theorem 2.2 we have a quasiconformal automorphism of $\mathbb{H}$ extending $\widetilde{g}$ whose complex dilatation $\widetilde{\mu}(z)$ satisfies $|\widetilde{\mu}(z)| = O(y^\alpha)$ uniformly as $y = \operatorname{Im} z \to 0$. By projecting down this by the holomorphic universal covering $\zeta = e^{iz} : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ and filling 0 as its fixed point, we have a quasiconformal automorphism of $\mathbb{D}$ extending g whose complex dilatation $\mu(\zeta)$ satisfies $|\mu(\zeta)| = O(\tau^\alpha)$ uniformly as $\tau = 1 - |\zeta| \to 0$. In particular, it satisfies

$$\int_0^1 \frac{\kappa_\mu(\tau)^\beta}{\tau} d\tau < \infty$$

for any $\beta > 0$. Hence the next corollary immediately follows from Theorem 5.3.

Corollary 5.4. *Let g be a diffeomorphism of $\mathbb{S}$ with an α-Hölder continuous derivative for some $\alpha \in (0,1)$ and with $\|c(g)\|_1 < \infty$. Then there is a quasiconformal automorphism of $\mathbb{D}$ that is the extension of g whose complex dilatation μ satisfies*

$$\int_{\mathbb{D}} |\mu(\zeta)|^p \rho_{\mathbb{D}}^2(\zeta) d\xi d\eta < \infty$$

for every $p > 1$.

Integrable complex dilatations with respect to the hyperbolic metric and subspaces of the universal Teichmüller space defined by such complex dilatations were first studied by Cui [5] for the case $p = 2$ and then extended by Takhtajan and Teo [15]. Generalization to the case $p > 2$ was done by Guo [8] and Tang [16]. The paper, Hu and Shen [9] and Wu [17] also contain certain characterizations of quasisymmetric automorphisms whose quasiconformal extensions have integrable complex dilatations.

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