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A classification of the modular transformations of infinite dimensional Teichmüller spaces

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Abstract. We classify the modular transformations of infinite dimensional Teichmüller spaces according to the behavior of their orbits. We then consider two classes, stationary and asymptotically elliptic, whose elements have a certain property similar to that of the modular transformations of finite dimensional Teichmüller spaces.

§1. Introduction and preliminaries

We consider the Teichmüller space $T(R)$ of an analytically infinite Riemann surface R and the Teichmüller modular group $\mathrm{Mod}(R)$ acting on $T(R)$, where $T(R)$ is not finite dimensional (moreover not separable) and $\mathrm{Mod}(R)$ is not finitely generated (moreover not countable in most cases). In this paper, we attempt to classify the elements in $\mathrm{Mod}(R)$.

The Teichmüller space $T(R)$ is the set of all Teichmüller equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that $f_1 : R \to R_1$ and $f_2 : R \to R_2$ are Teichmüller equivalent if there exists a conformal homeomorphism $h : R_1 \to R_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to h relative to the ideal boundary at infinity of R_1. We will use the notation o for the basepoint [id] of $T(R)$. It is known that $T(R)$ is a complex Banach manifold. Also it has a metric structure such that the distance between $p_1 = [f_1]$ and $p_2 = [f_2]$ is given by $d(p_1, p_2) = \log K(f)$, where $K(f)$ is the maximal dilatation of an extremal quasiconformal homeomorphism f in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete metric on $T(R)$, which is called the Teichmüller metric. It is known that the Teichmüller metric on $T(R)$ is the same as its Kobayashi metric. See [9], [10], [11] and [18] for fundamental facts on Teichmüller spaces.

A *quasiconformal mapping class* is a homotopy class $[g]$ of quasiconformal automorphisms $g : R \to R$ relative to the ideal boundary at infinity of R. The quasiconformal mapping class group $\mathrm{MCG}(R)$ is the group of all quasiconformal mapping classes. Each $\gamma = [g] \in \mathrm{MCG}(R)$ acts on $T(R)$ from the left such that $\gamma_* : [f] \mapsto [f \circ g^{-1}]$. It is evident from the definition that $\mathrm{MCG}(R)$ acts on $T(R)$

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isometrically with respect to the Teichmüller distance d. Also, it acts biholomorphically on $T(R)$. Let $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ be the homomorphism defined by $\gamma \mapsto \gamma_*$, where $\mathrm{Aut}(T(R))$ denotes the group of all isometric biholomorphic automorphisms of $T(R)$. The image $\mathrm{Im}\,\iota \subset \mathrm{Aut}(T(R))$ is called the *Teichmüller modular group* and is denoted by $\mathrm{Mod}(R)$. Each element $\gamma_* \in \mathrm{Mod}(R)$ is called a *modular transformation*.

For a torus R, it is known that $T(R)$ is biholomorphically and isometrically equivalent to the hyperbolic plane and $\mathrm{MCG}(R)$ is isomorphic to $SL(2,\mathbb{Z})$. Then $SL(2,\mathbb{Z})$ acts on the upper half-plane model $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ as fractional linear transformations and $\mathrm{Mod}(R)$ is identified with $PSL(2,\mathbb{Z})$. This is called the elliptic modular group. Since its elements are Möbius transformations, they can be classified into elliptic, parabolic or hyperbolic.

Due to Bers, this classification of $\mathrm{Mod}(R)$ can be extended to an analytically finite Riemann surface R in general: a modular transformation $\gamma_* \in \mathrm{Mod}(R) - \{\mathrm{id}\}$ is called

- *elliptic* if γ_* has a fixed point in $T(R)$;
- *parabolic* if $\inf_{p \in T(R)} d(\gamma_*(p), p) = 0$ but γ_* has no fixed point in $T(R)$;
- *hyperbolic* if $\inf_{p \in T(R)} d(\gamma_*(p), p) > 0$.

This is closely related to Thurston's classification of the mapping classes such as periodic, reducible and pseudo-Anosov. See [10, §6.5] for this account.

Now we consider our case where R is analytically infinite. Then the variety of the elements in $\mathrm{Mod}(R)$ becomes vast and the behavior of the orbits becomes complicated. Here we summarize major difficulties in classifying these elements in comparison with the analytically finite case: (1) There exists an elliptic element γ_* of infinite order. Actually, a mapping class γ realized as a conformal automorphism of infinite order gives γ_* having a fixed point in $T(R)$. (2) The action γ_* is not necessarily discontinuous and an orbit of γ_* is not necessarily discrete. (3) A boundary of $T(R)$ such as Thurston's where the action of the Teichmüller modular group extends continuously is not yet introduced. Hence, before going to the classification like in the analytically finite case, we need to set more coarse classes just based on the behavior of their orbits.

In Section 2, we define divergent, infinitely discrete and unbounded types according to logical possibility of orbits. Actually, a recent result due to Markovic [12] implies that boundedness is equivalent to ellipticity in the previous sense. An inclusion relation between infinitely discrete and unbounded types is proved in this section (Theorem 3). We also give some examples of elements belonging to the differences between those classes (Theorem 4).

In the analytically finite case (finite dimensional case), elliptic modular transformations are of bounded type, and parabolic and hyperbolic modular transformations are of divergent type. Namely, there are no elements of intermediate types in this case. When a class of modular transformations has this property, we will say that it satisfies the bounded-divergent dichotomy.

Next in Section 3, we define a class of mapping classes generalizing those for compact Riemann surfaces. An element in this class is called stationary and has a property that there is some compact subset in R that does not go to infinity under the iteration of the mapping class. Of course, every mapping class of an analytically finite Riemann surface satisfies this property. Being stationary guarantees compactness of a family of quasiconformal automorphisms of R whose maximal dilatations are uniformly bounded. As a consequence, we will see that the stationary mapping classes satisfy the bounded-divergent dichotomy (Theorem 6).

Finally, in Section 4, we consider a certain analytic condition that provides similarity to the finite dimensional case. It is given by introducing asymptotic Teichmüller spaces.

The asymptotic Teichmüller space $AT(R)$ of a Riemann surface R is a certain quotient space of the Teichmüller space $T(R)$. A quasiconformal homeomorphism f of R is called *asymptotically conformal* if, for every $\varepsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f)$ of f is less than $1 + \varepsilon$ on $R - V$. Asymptotic equivalence of quasiconformal homeomorphisms can be defined similarly to Teichmüller equivalence just by replacing the word "conformal" with "asymptotically conformal". The *asymptotic Teichmüller space* $AT(R)$ is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f of R. Hence there exists a natural projection $\alpha : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $p = [f] \in T(R)$ to the asymptotic equivalence class $\alpha(p) = [[f]] \in AT(R)$. The fiber over any $\alpha(p) \in AT(R)$ is denoted by T_p, which is a closed separable submanifold of $T(R)$. Similar to the case of $T(R)$, $\gamma = [g] \in \mathrm{MCG}(R)$ acts on $AT(R)$ by $\gamma_{**} : [[f]] \mapsto [[f \circ g^{-1}]]$. In other words, $\mathrm{MCG}(R)$ acts on $T(R)$ by preserving the fibers over $AT(R)$. See [1], [2], [3] and [9, Ch.14] concerning asymptotic Teichmüller spaces.

We investigate a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ that fixes some point $\alpha(p) \in AT(R)$, or more precisely, that keeps some fiber $T_p \subset T(R)$ invariant. We call such an element asymptotically elliptic. For an analytically finite Riemann surface R, the asymptotic Teichmüller space $AT(R)$ consists of one point o and hence every modular transformation keeps the fiber $T_o = T(R)$ invariant, namely, they are all asymptotically elliptic. We will prove that the class of the asymptotically elliptic elements also satisfies the bounded-divergent dichotomy (Theorem 10).

In this paper, we only deal with the orbit by a single modular transformation but some of the results are also valid for certain subgroups of $\mathrm{Mod}(R)$ or $\mathrm{Mod}(R)$ itself. Actually, a series of our papers [4], [5], [6], [7], [8], [13], [15] and [16] contain the arguments related to this subject matter. We may say that this is an expository paper of our works restricted to the study of the cyclic dynamics of modular transformations. The whole project will appear in a forthcoming article [17].

§2. Classification of the modular transformations

In this section, we classify the modular transformations of the Teichmüller space into several types according to the behavior of their orbits. Then we investigate the relationship between these types.

Definition. A modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is of *infinitely discrete type* if it is of infinite order and if the orbit $\langle \gamma_* \rangle(p) = \{\gamma_*^n(p)\}_{n \in \mathbb{Z}}$ of each point $p \in T(R)$ has no accumulation points in $T(R)$. Moreover, γ_* is of *divergent type* if the orbit diverges to the point at infinity of $T(R)$ as $n \to \pm\infty$. On the contrary, γ_* is of *bounded type* if the orbit is a bounded set in $T(R)$. Otherwise, γ_* is of *unbounded type*.

It is evident from the definition that, if γ_* is of divergent type, then it is of infinitely discrete type. Note that the orbit $\{\gamma_*^n(p)\}_{n \in \mathbb{Z}}$ is discrete if and only if it is closed. Indeed, if the orbit is closed but not discrete, then it has an accumulation point in it and, by group invariance, it is a perfect set. However, in the complete metric space $T(R)$, it must be a uncountable set, which is impossible for $\{\gamma_*^n(p)\}_{n\in\mathbb{Z}}$.

First we remark that our classification is consistent for any non-trivial element of the cyclic group of a modular transformation.

PROPOSITION 1. *The type of a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is invariant in $\langle\gamma_*\rangle - \{\mathrm{id}\}$. Namely, γ_* is of infinitely discrete (divergent, bounded) type if and only if so is γ_*^k for some and any $k \in \mathbb{Z} - \{0\}$.*

PROOF. If γ_* is of infinitely discrete type, then γ_*^k is clearly of infinitely discrete type for any k. Conversely, if γ_*^k is of infinitely discrete type for some k, then the orbit $\langle\gamma_*^k\rangle(p)$ is discrete and so is its image $\delta_*\{\langle\gamma_*^k\rangle(p)\}$ for every $\delta_* \in \mathrm{Mod}(R)$. By

$$\langle\gamma_*\rangle(p) = \bigcup_{0\le i\le |k|-1} \gamma_*^i\{\langle\gamma_*^k\rangle(p)\},$$

we see that γ_* is of infinitely discrete type. The arguments are the same for divergent type and for bounded type. □

We say that a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is *elliptic* if it has a fixed point p in $T(R)$. This is equivalent to saying that the mapping class γ is realized as a conformal automorphism of the Riemann surface R_p corresponding to p. The Nielsen realization problem for an analytically finite Riemann surface R asserts that a finite subgroup of $\mathrm{MCG}(R)$ is realized as a group of conformal automorphisms, or equivalently, the finite subgroup of $\mathrm{Mod}(R)$ has a common fixed point in $T(R)$. Markovic [12] has extended this theorem to analytically infinite Riemann surfaces. His result on quasisymmetric conjugacy of a uniformly quasisymmetric group implies the following.

THEOREM 2. *A modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is of bounded type if and only if γ_* is elliptic.*

Our first result asserts the inclusion relation between bounded and infinitely discrete types. This result first appeared in [13] modulo the bounded-elliptic equivalence as in Theorem 2.

THEOREM 3. *If a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is of bounded type, then it is not of infinitely discrete type.*

PROOF. We have only to consider the case where γ_* is of infinite order. By Theorem 2, there exists a point $p \in T(R)$ that is fixed by γ_*. Without loss of generality, we may assume that p is the basepoint $o \in T(R)$. Then γ has a conformal representative $g : R \to R$. For an integer $n \in \mathbb{N}$, consider the Teichmüller space $T(R/\langle g^n\rangle)$ in $T(R)$, which is coincident with the fixed point locus of γ_*^n.

In general, if Fuchsian groups have a proper inclusion relation $G_1 \supsetneqq G_2$, then the Teichmüller spaces satisfy $T(\mathbb{H}/G_1) \subsetneqq T(\mathbb{H}/G_2)$ in the universal Teichmüller space $T(\mathbb{H})$ unless G_1 and G_2 are exceptional pairs of Fuchsian groups. See [14]. Hence we have proper inclusion relations

$$T(R/\langle g\rangle) \subsetneqq T(R/\langle g^2\rangle) \subsetneqq T(R/\langle g^4\rangle) \subsetneqq \cdots \subsetneqq T(R/\langle g^{2^k}\rangle) \subsetneqq \cdots$$

in $T(R)$. The properness is also valid locally in $T(R)$ unless the sets in question are empty. The easiest way to see this fact is to take the Bers embedding of these sets in a certain Banach space, where they are realized in Banach subspaces.

Here we see that

$$\mathrm{Cl}\left\{\bigcup_{k=0}^{\infty} T(R/\langle g^{2^k}\rangle)\right\} - \bigcup_{k=0}^{\infty} T(R/\langle g^{2^k}\rangle)$$

is not empty and hence contains a point q. Indeed, if it is empty, then the countable union $\bigcup_{k=0}^{\infty} T(R/\langle g^{2^k}\rangle)$ is complete as a metric subspace of $T(R)$. Hence there exists an integer k_0 such that $T(R/\langle g^{2^{k_0}}\rangle)$ is coincident with the entire union. However, this contradicts the fact that the inclusion relations are proper.

Take a point $q_k \in T(R/\langle g^{2^k}\rangle)$ for each $k \in \mathbb{N}$ so that the Teichmüller distances $d(q, q_k)$ converge to 0 as $k \to \infty$. Then, since $\gamma_*^{2^k}$ fixes q_k, we have

$$d(q, \gamma_*^{2^k}(q)) \le d(q, q_k) + d(q_k, \gamma_*^{2^k}(q_k)) + d(\gamma_*^{2^k}(q_k), \gamma_*^{2^k}(q)) = 2d(q, q_k).$$

Here $\gamma_*^{2^k}(q)$ is distinct from q because q does not belong to $T(R/\langle g^{2^k}\rangle)$. Since $\gamma_*^{2^k}(q)$ converge to q, the orbit $\{\gamma_*^n(q)\}_{n\in\mathbb{Z}}$ is not discrete. □

In the hyperbolic plane $\mathbb{H}$, the orbit of an elliptic Möbius transformation of infinite order is not discrete unless it consists only of the fixed point. The above theorem resembles this fact.

Now we have the following inclusion relations for the classes of modular transformations:

$$\{\text{divergent type}\} \subset \{\text{infinitely discrete type}\} \subset \{\text{unbounded type}\}.$$

We will show that these inclusions are proper by giving examples of Riemann surfaces.

Theorem 4. *There exists a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ for some Riemann surface R that is of unbounded type but not of infinitely discrete type.*

Proof. The Riemann surface R is obtained from the complex plane $\mathbb{C} = \{(x, y)\}$ by removing a countable number of points. These points are defined as follows. Let $h : \mathbb{R} \to [0, 1]$ be a piecewise linear function of period 2 such that $h(x) = x$ for $0 \le x \le 1$ and $h(x) = 2 - x$ for $1 \le x \le 2$. For each positive integer $k \in \mathbb{N}$, set

$$\begin{aligned} P_{2k-1} &= \{(x, y) \mid x \in \mathbb{Z},\ y = \{1 - (k+1)^{-1}\}\, h(2^{-k(k-1)/2}x) + 2k - 1\} \\ P_{2k} &= \{(x, y) \mid x \in \mathbb{Z},\ y = 2k\}. \end{aligned}$$

Then $R = \mathbb{C} - \bigcup_{k\in\mathbb{N}} P_k$.

Next we define a quasiconformal mapping class of R. Consider a Riemann surface $S = \mathbb{C} - \{(x, y) \mid x \in \mathbb{Z},\ y \in \mathbb{N}\}$ and a conformal automorphism $g_S(z) = z + 1$ of S. Let $\eta : S \to R$ be a homeomorphism (not quasiconformal) such that η keeps the x-coordinates invariant. By this requirement, a homotopy class of the homeomorphism η is uniquely determined. Then we define a mapping class γ of R as $\gamma = [\eta \circ g_S \circ \eta^{-1}]$. Although η is not quasiconformal, we can see that γ is a quasiconformal mapping class by choosing a piecewise linear homeomorphism in γ whose maximal dilatation is finite.

We will prove that $\gamma_* \in \mathrm{Mod}(R)$ is of unbounded type but not of infinitely discrete type. Arguments here are also used in [6]. To see unboundedness, consider a sequence $\gamma_*^{2^{m(m-1)/2}}$ and its orbit of the basepoint $o \in T(R)$ for $m \in \mathbb{N}$. Take three punctures

$$(0, 2m-1), \quad (0, 2m), \quad (1, 2m)$$

of R. They are mapped by $\gamma^{2^{m(m-1)/2}}$ to

$$(2^{m(m-1)/2}, 2m-(m+1)^{-1}), \quad (2^{m(m-1)/2}, 2m), \quad (2^{m(m-1)/2}+1, 2m)$$

respectively. The Euclidean distance between the first and the second points changes from 1 to $(m+1)^{-1}$ but the distance between the second and the third points does not change. The maximal dilatation of any quasiconformal homeomorphism of $\mathbb{C}$ that sends those three points to the corresponding three points can be estimated from below and it grows to infinity as $m \to \infty$. This implies that $d(\gamma_*^{2^{m(m-1)/2}}(o), o) \to \infty$ and hence γ_* is of unbounded type.

To see indiscreteness, consider a sequence $\gamma_*^{2\cdot 2^{m(m-1)/2}}$ and its orbit of the basepoint o for $m \in \mathbb{N}$. Note that $P_1, P_2, \ldots, P_{2m}$ are invariant under the $2 \cdot 2^{m(m-1)/2}$-translation. Then P_{2m+1} is the first row that is not invariant under this translation and the Hausdorff distance between P_{2m+1} and its translation is smaller than $1/2^{m-1}$. On the other hand, the shortest distance between P_{2m+1} and $P_{2(m+1)}$ is $(m+2)^{-1}$. Then the ratio of these distances is bounded by $(m+2)/2^{m-1}$. For P_{2k+1} with $k > m$, this ratio is smaller. When we realize the mapping class $\gamma^{2\cdot 2^{m(m-1)/2}}$ as a piecewise linear homeomorphism, the closer this value is to 0, the closer we can make its maximal dilatation to 1. Then, by $(m+2)/2^{m-1} \to 0$ as $m \to \infty$, we see that $d(\gamma_*^{2\cdot 2^{m(m-1)/2}}(o), o) \to 0$ and hence γ_* is not of infinitely discrete type. $\square$

Remark. Similarly, there exists a modular transformation $\gamma_* \in \mathrm{Mod}(R')$ for some Riemann surface R' that is of infinitely discrete type but not of divergent type. For example, R' is obtained by replacing the amplitude $1-(k+1)^{-1}$ in the definition of P_{2k-1} above with $1-2^{-(k-3)}$ for all sufficiently large k. The definition of the quasiconformal mapping class γ is the same. However, since the estimates for showing that this γ_* has the required properties are more complicated, we omit the details here.

We say that a modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is *hyperbolic* if there exists a positive constant $\delta > 0$ such that $d(\gamma_*(p), p) \geq \delta$ for every $p \in T(R)$. There is a problem asking whether or not every hyperbolic modular transformation γ_* is of divergent type even for an analytically infinite Riemann surface R.

§3. Stationary quasiconformal mapping classes

The quasiconformal mapping classes of analytically infinite Riemann surfaces are divided into two categories: stationary and non-stationary. A stationary mapping class is a generalization of a mapping class of a compact Riemann surface.

Definition. A quasiconformal mapping class $\gamma = [g] \in \mathrm{MCG}(R)$ is *stationary* if there exist compact subsurfaces W and W' such that every representative $g_n : R \to R$ of γ^n for every $n \in \mathbb{Z}$ satisfies $g_n(W) \cap W' \neq \emptyset$. Moreover, γ is *pure* if g fixes every topological end of R except a cusp, and γ is *eventually trivial* if there exists a compact subsurface $W \subset R$ such that $g|_U : U \to R$ restricted to each connected component U of $R - W$ except a cusp neighborhood is homotopic to the inclusion map $U \hookrightarrow R$ in R.

It is clear from the definition that an eventually trivial mapping class is pure as well as stationary. Also we have the following result, which is proved in [5].

PROPOSITION 5. *If R has more than two non-cuspidal topological ends, then every pure mapping class is stationary.*

PROOF. By considering a canonical exhaustion of R by a sequence of compact subsurfaces, we have a compact subsurface W whose complement consists of more than two connected components except cusp neighborhoods. If a mapping class γ preserves each non-cuspidal topological end, then every representative g_n of γ^n for every $n \in \mathbb{Z}$ satisfies $g_n(U) \cap U \neq \emptyset$ for at least three connected components U of $R - W$. This forces W to satisfy that $g_n(W) \cap W \neq \emptyset$ and hence γ is stationary.□

It is known that a sequence of normalized quasiconformal mappings in the complex plane whose maximal dilatations are uniformly bounded is sequentially compact in compact open topology. The stationary property of mapping classes corresponds to the normalization in this context and hence such a sequence of mapping classes also has the compactness property if they are uniformly bounded.

In previous works [4] and [8], it has been proved that a modular transformation γ_* corresponding to a stationary mapping class has discrete orbits under a certain boundedness assumption on the hyperbolic geometry of R. We can extend this statement completely to the following form, which is also utilized in [15].

THEOREM 6. *For a stationary quasiconformal mapping class $\gamma \in \mathrm{MCG}(R)$ of infinite order, the modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is of divergent type.*

PROOF. Suppose that γ_* is not of divergent type. Then there exist a constant $K \geq 1$ and an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $d(\gamma_*^{n_k}(o), o) \leq \log K$ for all $k \in \mathbb{N}$. We take a representative $g_k : R \to R$ for each γ^{n_k} whose maximal dilatation is not greater than K, and consider the family $\{g_k\}$ of quasiconformal automorphisms. Since γ is stationary, by passing to a subsequence, we see that g_k converge locally uniformly. In particular, for a compact subsurface W such that the image of the homomorphism $\pi_1(W) \to \pi_1(R)$ induced by the inclusion map $W \hookrightarrow R$ is not cyclic, there exists some k such that $g_k^{-1} \circ g_{k+1}$ restricted to W is homotopic to the inclusion map in R. Consider the mapping class $\delta := \gamma^{n_{k+1}-n_k}$, which maps W trivially. Here, δ is not a trivial element, for otherwise γ would be of finite order. Since we assume that γ_* is not of divergent type, δ_* is not of divergent type either by Proposition 1.

We represent R by a Fuchsian group H acting on the unit disk $\Delta = \{|z| < 1\}$ and take a subgroup H' corresponding to $\pi_1(W)$. The limit set $\Lambda(H')$ is a closed subset of $\partial\Delta$ having more than two points. To the mapping class δ, there corresponds a quasisymmetric automorphism $\bar{\delta} \neq \mathrm{id}$ of $\partial\Delta = \{|z| = 1\}$ that fixes every point of $\Lambda(H')$. Let L be the set of all points that are fixed by $\bar{\delta}$. It is a proper closed subset of $\partial\Delta$ containing $\Lambda(H')$. Take any interval J in $\partial\Delta - L$, which is invariant under $\bar{\delta}$. Since $\bar{\delta}|_J$ is monotonous, for every point $x \in J$, the sequence $\bar{\delta}^n(x)$ converges to j_+ or j_- as $n \to \pm\infty$, where j_+ and j_- are the end points of J. Take some $y \in L$ other than $\{j_+, j_-\}$. Then the cross-ratio $c(\bar{\delta}^n(x), y, j_+, j_-) \in (1, \infty)$ converges to 1 or ∞ as $n \to \pm\infty$. This implies that the maximal dilatation of any representative in δ^n tends to ∞ and hence $\delta_*^n(o) \to \infty$ as $n \to \pm\infty$, that is, δ_* is of divergent type. However, this is a contradiction. □

§4. Asymptotically elliptic modular transformations

In this section, we deal with modular transformations that have fixed points on the asymptotic Teichmüller space.

DEFINITION. A modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is called *asymptotically elliptic* if it has a fixed point on the asymptotic Teichmüller space $AT(R)$. Moreover, $\gamma_* \in \mathrm{Mod}(R)$ is called *asymptotically trivial* if it fixes every point of $AT(R)$.

If a quasiconformal mapping class $\gamma \in \mathrm{MCG}(R)$ is eventually trivial, then the modular transformation $\gamma_* \in \mathrm{Mod}(R)$ is asymptotically trivial. The converse is not true in general, but under the assumption that R satisfies the bounded geometry condition, it is conjectured that the converse should be true.

An elliptic modular transformation is of course asymptotically elliptic, but it is not asymptotically trivial [16]. An example of an asymptotically elliptic modular transformation that is neither elliptic nor asymptotically trivial is constructed by Petrovic [19]. However, this mapping class is stationary.

EXAMPLE 7. There exists a non-stationary mapping class $\gamma \in \mathrm{MCG}(R)$ such that $\gamma_* \in \mathrm{Mod}(R)$ is asymptotically elliptic but is neither elliptic nor asymptotically trivial. Indeed, in [7, §3], we have constructed an example of a non-stationary mapping class γ such that γ_* is of infinitely discrete type and hence it is not elliptic. This is not asymptotically trivial either. By modifying this construction slightly as in Remark 3.4 of that paper, we can make γ_* asymptotically elliptic.

Hereafter, we will show the bounded-divergent dichotomy for asymptotically elliptic modular transformations. We use two lemmas. The first one concerns the change of the cross-ratio under an asymptotically conformal homeomorphism.

LEMMA 8. *Let H be a Fuchsian group acting on the unit disk Δ and $\pi : \Delta \to R = \Delta/H$ the projection to the Riemann surface R. Let $\{(\beta_i, \beta_i')\}_{i \in \mathbb{N}}$ be a sequence of pairs of geodesic lines in Δ with $\beta_i \cap \beta_i' \neq \emptyset$ and let $c(\beta_i, \beta_i')$ be the cross-ratio of the four end points of β_i and β_i' on $\partial\Delta$ so defined as $c(\beta_i, \beta_i') \in (1, \infty)$. Assume that the sequence $\{\pi(\beta_i \cup \beta_i')\}$ diverges to the point at infinity of R as $i \to \infty$. Also assume that $\{c(\beta_i, \beta_i')\}$ are uniformly bounded from above and away from one. Let $g : R \to R$ be an asymptotically conformal automorphism of R and $\tilde{g} : \Delta \to \Delta$ a lift of g to Δ. Then*

$$c(\tilde{g}(\beta_i), \tilde{g}(\beta_i')) - c(\beta_i, \beta_i') \to 0$$

as $i \to \infty$.

PROOF. For each $i \in \mathbb{N}$, we take Möbius transformations φ_i and ψ_i of Δ so that $\varphi_i(\beta_i \cap \beta_i') = \{0\}$ and $\psi_i(\tilde{g}(\beta_i) \cap \tilde{g}(\beta_i')) = \{0\}$. Set $\tilde{g}_i := \psi_i \circ \tilde{g} \circ \varphi_i^{-1}$. Then the complex dilatations $\mu_{\tilde{g}_i}$ of $\tilde{g}_i$ satisfy $\|\mu_{\tilde{g}_i}\|_\infty = \|\mu_{\tilde{g}}\|_\infty < 1$ and converge to 0 almost everywhere as $i \to \infty$. By a property of quasiconformal mappings (cf. [11, p.29]), we see that, for any subsequence of $\{\tilde{g}_i\}$, there exists a further subsequence that converges to a Möbius transformation of Δ uniformly on $\overline{\Delta}$. Since Möbius transformations preserve the cross-ratio, this implies that the difference between $c(\tilde{g}(\beta_i), \tilde{g}(\beta_i'))$ and $c(\beta_i, \beta_i')$ tends to zero. □

Let ℓ be a topological vector space consisting of all the sequences $\xi = (\xi_n)_{n \in \mathbb{Z}}$ of real numbers whose topology is induced from the supremum norm $\|\xi\|_\infty = \sup_{n \in \mathbb{Z}} |\xi_n|$. Let $\sigma : \ell \to \ell$ be the shift operator defined by $\sigma(\xi)_n = \xi_{n+1}$.

LEMMA 9. *Suppose that* $\xi = (\xi_n)_{n\in\mathbb{Z}} \in \ell$ *satisfies the following two conditions for a sequence of integers* $\{n_i\} \subset \mathbb{Z}$*:*

(1) $\|\sigma^{n_i}(\xi) - \xi\|_\infty \le C$ *for some* $C \ge 0$ *and for every* $i \in \mathbb{N}$*;*

(2) *For every* $m \in \mathbb{Z}$*,* $\sigma^m(\xi)_{n_i} - \xi_{n_i} \to 0$ *as* $i \to \infty$*.*

Then $|\xi_m - \xi_0| \le 2C$ *for every* $m \in \mathbb{Z}$ *and in particular* $\|\xi\|_\infty < \infty$*.*

PROOF. Set $s(m) := \sigma^m(\xi) - \xi \in \ell$ and calculate $s(m+n_i) = \sigma^{m+n_i}(\xi) - \xi$ in two ways. In one way, we have

$$s(m+n_i) = \sigma^m(\sigma^{n_i}(\xi)) - \xi = \sigma^m(s(n_i)) + s(m).$$

Since $\|\sigma^m(s(n_i))\|_\infty \le C$ by condition (1), we see that $\|s(m+n_i) - s(m)\|_\infty \le C$. In the other way, we have

$$s(m+n_i) = \sigma^{n_i}(\sigma^m(\xi)) - \xi = \sigma^{n_i}(s(m)) + s(n_i).$$

Since $\|s(n_i)\|_\infty \le C$ by condition (1), we see that $\|s(m+n_i) - \sigma^{n_i}(s(m))\|_\infty \le C$. From those two inequalities, we have

$$\|s(m) - \sigma^{n_i}(s(m))\|_\infty \le 2C.$$

Here we consider the evaluation $\sigma^{n_i}(s(m))_0 = s(m)_{n_i}$ at 0 in the last inequality. By condition (2), this converges to 0 as $i \to \infty$. Therefore we have $|s(m)_0| \le 2C$, in other words, $|\xi_m - \xi_0| \le 2C$ for every $m \in \mathbb{Z}$. □

Now we state our theorem on asymptotically elliptic modular transformations.

THEOREM 10. *An asymptotically elliptic modular transformation* $\gamma_* \in \mathrm{Mod}(R)$ *is either of bounded type (elliptic) or of divergent type.*

PROOF. Let γ_* have a fixed point $\alpha(p) \in AT(R)$. Without loss of generality, we may assume that $\alpha(p) = \alpha(o)$. Suppose that γ_* is not of divergent type. Then there exist a positive constant C and an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ such that $d(\gamma_*^{n_k}(o), o) \le C$ for all $k \in \mathbb{N}$. We represent the Riemann surface R by a Fuchsian group H acting on Δ with the projection $\pi : \Delta \to R$.

We take geodesic lines β and β' in Δ such that $\beta \cap \beta' \neq \emptyset$ and the image $\pi(\beta \cup \beta')$ restricted to the convex core of R is compact. We also take an asymptotically conformal automorphism g of R in the mapping class $\gamma \in \mathrm{MCG}(R)$ and its lift $\tilde g : \Delta \to \Delta$. Consider the cross-ratio $c(\beta, \beta') \in (1,\infty)$ defined by the four end points of β and β'. For each $n \in \mathbb{Z}$, let β_n and β'_n be the geodesic lines in Δ determined by the end points of $\tilde g^n(\beta)$ and $\tilde g^n(\beta')$ respectively, and define also $c(\beta_n, \beta'_n) \in (1,\infty)$. Then set

$$\xi_n = \int_2^{c(\beta_n,\beta'_n)} \rho_{\mathbb{C}-\{0,1\}}(x)dx,$$

where $\rho_{\mathbb{C}-\{0,1\}}(z)|dz|$ is the hyperbolic metric on $\mathbb{C} - \{0,1\}$. This is the signed hyperbolic distance of $c(\beta_n, \beta'_n)$ from 2.

If γ is stationary (meaning that the whole sequence $\{\gamma^n\}_{n\in\mathbb{Z}}$ is stationary by definition), then Theorem 6 implies that γ_* is of finite order, for it is not of divergent type by assumption. By applying a similar argument to the sequence of mapping classes $\{\gamma^{n_k}\}$, we can also show that γ_* is of finite order and in particular of bounded type if the sequence $\{\gamma^{n_k}\}$ is stationary. Hence we have only to consider the case where $\{\gamma^{n_k}\}$ is not stationary. Namely, there exists a subsequence $\{k(i)\}_{i\in\mathbb{N}}$ of

$\{k\}$ such that $g_{k(i)}(\pi(\beta \cup \beta'))$ diverge to the point at infinity as $i \to \infty$ for some representative $g_{k(i)}$ of $\gamma^{n_{k(i)}}$. Hereafter, we replace the indices $k(i)$ by i.

We will show that $\xi := (\xi_n) \in \ell$ together with the sequence $\{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ satisfies the assumptions in the statement of Lemma 9. For condition (1), we consider

$$\xi_{m+n_i} - \xi_m = \int_{c(\beta_m, \beta'_m)}^{c(\tilde{g}^{n_i}(\beta_m), \tilde{g}^{n_i}(\beta'_m))} \rho_{\mathbb{C}-\{0,1\}}(x)dx$$

for every $m \in \mathbb{Z}$ and for every $i \in \mathbb{N}$. It is known that a K-quasiconformal automorphism of Δ changes the cross-ratio by at most $\log K$ with respect to the hyperbolic distance on $\mathbb{C} - \{0,1\}$. Hence the condition that $d(\gamma_*^{n_i}(o), o) \leq C$ for every i implies that $|\xi_{m+n_i} - \xi_m| \leq C$ for every m. Thus we have $\|\sigma^{n_i}(\xi) - \xi\|_\infty \leq C$.

To show that condition (2) is satisfied, we use Lemma 8. From the choice of the sequence $\{n_i\}$, we see that $\pi(\beta_{n_i} \cup \beta'_{n_i})$ diverge to the point at infinity of R as $i \to \infty$. Since $d(\gamma_*^{n_i}(o), o) \leq C$ for every i, we see that $\{c(\beta_{n_i}, \beta'_{n_i})\}$ are uniformly bounded from above and away from one. Note that $\tilde{g}^m$ is a lift of an asymptotically conformal automorphism g^m of R for each $m \in \mathbb{Z}$. Then Lemma 8 can be applied to conclude that

$$\xi_{m+n_i} - \xi_{n_i} = \int_{c(\beta_{n_i}, \beta'_{n_i})}^{c(\tilde{g}^m(\beta_{n_i}), \tilde{g}^m(\beta'_{n_i}))} \rho_{\mathbb{C}-\{0,1\}}(x)dx \to 0.$$

Thus we have $\sigma^m(\xi)_{n_i} - \xi_{n_i} \to 0$ as $i \to \infty$ for every m.

By Lemma 9, we have $|\xi_m - \xi_0| \leq 2C$ for every m. This estimate implies that $\tilde{g}^m$ changes the cross-ratio $c(\beta, \beta')$ by at most $2C$ with respect to the hyperbolic distance on $\mathbb{C} - \{0,1\}$.

Next we take arbitrary four distinct points a_1, a'_1, a_2, a'_2 on $\partial\Delta$ in this order. If the Fuchsian group H is of the first kind, then there exists a sequence of geodesic lines β in Δ whose projections $\pi(\beta)$ are closed geodesics in R and whose end points converge to a_1 and a_2 respectively. If H is of the second kind, we can choose the β so that its end points are not in the limit set and hence $\pi(\beta)$ restricted to the convex core of R is compact. From this fact, we see that the cross-ratio $c(a_1, a_2, a'_1, a'_2)$ can be approximated by the sequence of cross-ratios $\{c(\beta, \beta')\}$ for which our estimate can be applied. Since $\tilde{g}^m$ changes $c(\beta, \beta')$ by at most $2C$, continuity of the cross-ratio shows that $\tilde{g}^m$ changes $c(a_1, a_2, a'_1, a'_2)$ by at most $2C$. It is known that this implies the boundary map $\tilde{g}^m|_{\partial\Delta}$ is M-quasisymmetric, where the constant M depends only on C. Also $d(\gamma_*^m(o), o)$ is bounded by a constant depending only on C. Since this is valid uniformly for every m, we see that the modular transformation γ_* is of bounded type. □

Ideas of the above arguments were inspired by [3] and [20, §4]. Remark that, if γ_* is asymptotically elliptic having a fixed point $\alpha(p)$, then the orbit $\{\gamma_*^n(q)\}_{n \in \mathbb{Z}}$ for any $q \in T_p(R)$ is discrete. This can be seen by using a similar argument as above. In [16], we also discuss this problem in the case where γ_* is elliptic.

References

[1] C. Earle, F. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part I: the complex structure*, In the tradition of Ahlfors and Bers, Contemp. Math., vol. 256, AMS, 2000, pp. 17–38.

[2] C. Earle, F. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part II: the metric structure*, In the tradition of Ahlfors and Bers III, Contemp. Math., vol. 355, AMS, 2004, pp. 187–219.
[3] C. Earle, V. Markovic and D. Saric, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space*, Complex manifolds and hyperbolic geometry, Contemp. Math., vol. 311, AMS, 2002, pp. 87–105.
[4] E. Fujikawa, *Modular groups acting on infinite dimensional Teichmüller spaces*, In the tradition of Ahlfors and Bers III, Contemp. Math., vol. 355, AMS, 2004, pp. 239–253.
[5] E. Fujikawa, *Pure mapping class group acting on Teichmüller space*, preprint.
[6] E. Fujikawa and K. Matsuzaki, *Recurrent and periodic points for isometries of L^∞ spaces*, Indiana Univ. Math. J. **55** (2006), 975–997.
[7] E. Fujikawa and K. Matsuzaki, *Non-stationary and discontinuous quasiconformal mapping class groups*, Osaka J. Math., to appear.
[8] E. Fujikawa, H. Shiga and M. Taniguchi, *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan **56** (2004), 1069–1086.
[9] F. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, SURV 76, American Mathematical Society, 2000.
[10] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer, 1992.
[11] O. Lehto, *Univalent functions and Teichmüller spaces*, GTM 109, Springer, 1986.
[12] V. Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. **19** (2006), 673–715.
[13] K. Matsuzaki, *The action of isotropy subgroups of the modular groups on infinite dimensional Teichmüller spaces*, RIMS Kokyuroku **1270** (2002), 84–87, **MR**1:953 877.
[14] K. Matsuzaki, *Inclusion relations between the Bers embeddings of Teichmüller spaces*, Israel J. Math. **140** (2004), 113–124.
[15] K. Matsuzaki, *Quasiconformal mapping class groups having common fixed points on the asymptotic Teichmüller spaces*, preprint.
[16] K. Matsuzaki, *The action of elliptic modular transformations on the asymptotic Teichmüller spaces*, preprint.
[17] K. Matsuzaki, *Dynamics of Teichmüller modular groups and general topology of moduli spaces*, preprint.
[18] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons, 1988.
[19] I. Petrovic, *A Teichmüller model for period doubling*, In the tradition of Ahlfors and Bers III, Contemp. Math., vol. 355, AMS, 2004, pp. 333–351.
[20] T. Sorvali, *The boundary mapping induced by an isomorphism of covering groups*, Ann. Acad. Sci. Fennicæ, Series A **526** (1972).

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