AN ESTIMATE OF THE MAXIMAL DILATATIONS OF QUASICONFORMAL AUTOMORPHISMS OF ANNULI

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ABSTRACT. We introduce a certain extremal problem for quasiconformal automorphisms of annuli and give upper and lower estimates for the minimal value of their maximal dilatations.

1. INTRODUCTION

In the theory of quasiconformal mapping, various types of extremal problems have been studied following the pioneering works on rectangles and doubly connected domains by Grötzsch and Teichmüller (see e.g. [1], [6]). An extremal problem asks about the minimal value of the maximal dilatations under a certain condition on quasiconformal maps as well as the map attaining the minimum. In this note, we introduce the following extremal problem on quasiconformal automorphisms of annuli.

Let $A = \{z \in \mathbb{C} \mid 1 \leq |z| \leq R\}$ denote an annulus and f a quasiconformal automorphism of A satisfying a normalization condition $f(1) = 1$. The modulus of A is defined by $\mathrm{mod}(A) = (2\pi)^{-1} \log R$. For such a map $f : A \to A$, we define a constant
$$\tau(f) := \max_{0 \leq \theta < 2\pi} |\arg f(Re^{i\theta}) - \theta|,$$
where the branch of the argument is determined continuously from $\arg(f(1)) = 0$.

Problem. *Estimate the maximal dilatation $K(f)$ of f from below in terms of $\tau = \tau(f)$, namely, give a constant $c_R(\tau)$ satisfying*
$$K(f) \geq c_R(\tau)$$
for all quasiconformal automorphisms $f : A \to A$ with $f(1) = 1$. Furthermore, find the best possible constant and the extremal map if there is such one.

A motivation for this problem lies in an estimate of the Teichmüller distance when we deform a Riemann surface S by twisting along a simple closed geodesic c on S ([7], [8]). In other words, we consider how much dilatation is necessary for a quasiconformal homeomorphism of S to yield a given amount of twist along c. The intention of setting the above problem is to supply several results which can be applied to this kind of estimate more generally.

It is relatively easy to see that, for each fixed $R > 1$, there is some constant $c_R(\tau)$ such that $c_R(\tau) > 1$ for $\tau > 0$ and $c_R(\tau) \to \infty$ as $\tau \to \infty$. In this note, we will find better and concrete estimates as well as determine the order of convergence and divergence as $\tau \to 0$ and $\tau \to \infty$. Especially, when $\tau \to \infty$, we will have the asymptotically best estimate.

2. Candidates for the extremal map

For the annulus $A = \{1 \leq |z| \leq R\}$, we consider a holomorphic universal cover

$$\tilde{A} = \{\zeta = \xi + i\eta \in \mathbb{C} \mid 0 \leq \xi \leq \log R\}$$

with the covering projection $z = \exp \zeta$ and the covering transformation group J generated by $j(\zeta) = \zeta + 2\pi i$. For a quasiconformal automorphism $f : A \to A$ satisfying $f(1) = 1$, we consider its lift $\tilde{f}$ to the universal cover $\tilde{A}$ such that $\tilde{f}(2\pi in) = 2\pi in$ for every integer $n \in \mathbb{Z}$.

For a given constant $\tau \geq 0$, we define a canonical affine automorphism $\tilde{f}_1(\zeta)$ of $\tilde{A}$ compatible with J by

$$\tilde{f}_1(\xi + i\eta) = \xi + i\left(\eta + \frac{\tau\xi}{\log R}\right)$$

for $\zeta = \xi + i\eta$. Its projection to A is given by

$$f_1(z) = z \exp\left(i\frac{\log|z|}{\log R}\tau\right).$$

Then the maximal dilatation of f_1 (or $\tilde{f}_1$) is

$$K(f_1) = \left[\left\{1 + \left(\frac{\tau}{2\log R}\right)^2\right\}^{\frac{1}{2}} + \frac{\tau}{2\log R}\right]^2.$$

In particular, we have

$$K(f_1) = \frac{\tau^2}{(\log R)^2} + O(1) \quad (\tau \to \infty);$$
$$K(f_1) = 1 + \frac{\tau}{\log R} + O(\tau^2) \quad (\tau \to 0).$$

One may think that f_1 should be the extremal quasiconformal map and the best possible constant for $c_R(\tau)$ is $K(f_1)$. However, this is not true, as we will see below.

Next, we restrict τ to $[0, 2\pi)$ and define a canonical stretching automorphism $\tilde{f}_2(\zeta)$ of $\tilde{A}$ compatible with J by

$$\tilde{f}_2(\xi + i\eta) = \begin{cases} \xi + i\eta \dfrac{\pi + \tau/2}{\pi - \tau/2} & (0 \leq \eta \leq \pi - \frac{\tau}{2} \mod 2\pi) \\[2mm] \xi + i(\eta - 2\pi)\dfrac{\pi - \tau/2}{\pi + \tau/2} + 2\pi i & (\pi - \frac{\tau}{2} \leq \eta \leq 2\pi \mod 2\pi). \end{cases}$$

The maximal dilatation of its projection $f_2 : A \to A$ is

$$K(f_2) = \frac{\pi + \tau/2}{\pi - \tau/2} = 1 + \frac{\tau}{\pi} + O(\tau^2) \quad (\tau \to 0).$$

Hence, if $\log R < \pi$, then $K(f_1) > K(f_2)$ for all sufficiently small $\tau > 0$. This means that the quasiconformal automorphism f_1 of A is not an extremal one for our problem in this case. Moreover, by considering the following *hybrid map* f_0 between f_1 and f_2, we always have a better estimate for the constant $c_R(\tau)$.

We still restrict τ to $[0, 2\pi)$ and take positive numbers α and β so that $\alpha + \beta = 1$. We define an automorphism $\tilde{f}_0(\zeta)$ of $\tilde{A}$ compatible with J by

$$
\tilde{f}_0(\xi + i\eta)
$$
$$
= \begin{cases}
\xi + i\left(\eta \dfrac{\pi + \alpha\tau/2}{\pi - \alpha\tau/2} + \dfrac{\beta\tau\xi}{\log R} \right) & (0 \le \eta \le \pi - \frac{\tau}{2} \mod 2\pi) \\[2mm]
\xi + i\left((\eta - 2\pi) \dfrac{\pi - \alpha\tau/2}{\pi + \alpha\tau/2} + 2\pi + \dfrac{\beta\tau\xi}{\log R} \right) & (\pi - \frac{\tau}{2} \le \eta \le 2\pi \mod 2\pi).
\end{cases}
$$

Its projection $f_0 : A \to A$ is our hybrid quasiconformal automorphism with the ratio $\alpha : \beta$ for f_1 and f_2. We will specify the values of α and β later.

We calculate the maximal dilatation $K(f_0)$. The partial derivatives of $\tilde{f}_0$ are

$$
\partial_\zeta \tilde{f}_0 = \frac{1}{2}\left(1 + \frac{\pi \pm \alpha\tau/2}{\pi \mp \alpha\tau/2} + i\frac{\beta\tau}{\log R} \right); \quad
\partial_{\bar{\zeta}} \tilde{f}_0 = \frac{1}{2}\left(1 - \frac{\pi \pm \alpha\tau/2}{\pi \mp \alpha\tau/2} + i\frac{\beta\tau}{\log R} \right),
$$

where the upper signs of $\pm$ and $\mp$ are applied to $0 \le \eta \le \pi - \frac{\tau}{2}$ and the lower signs to $\pi - \frac{\tau}{2} \le \eta \le 2\pi$. Then, by setting

$$
a = \frac{\pi \pm \alpha\tau/2}{\pi \mp \alpha\tau/2}; \quad b = \frac{\beta\tau}{\log R},
$$

we have the complex dilatation

$$
\mu_{\tilde{f}_0} = \frac{\partial_{\bar{\zeta}} \tilde{f}_0}{\partial_\zeta \tilde{f}_0} = \frac{1 - a^2 + b^2 + 2abi}{(1 + a)^2 + b^2}.
$$

Therefore

$$
\begin{aligned}
\|\mu_{f_0}\| = \|\mu_{\tilde{f}_0}\| &= \left\{ \frac{(1 - a)^2 + b^2}{(1 + a)^2 + b^2} \right\}^{\frac{1}{2}} \\[2mm]
&= \left\{ \left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\beta}{\log R}\right)^2 \right\}^{\frac{1}{2}} \frac{\tau}{2} + O(\tau^2) \qquad (\tau \to 0);
\end{aligned}
$$

$$
K(f_0) = \frac{1 + \|\mu_{f_0}\|}{1 - \|\mu_{f_0}\|} = 1 + \left\{ \left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\beta}{\log R}\right)^2 \right\}^{\frac{1}{2}} \tau + O(\tau^2) \qquad (\tau \to 0).
$$

If we choose

$$
\alpha = \frac{\pi^2}{(\log R)^2 + \pi^2}; \quad \beta = \frac{(\log R)^2}{(\log R)^2 + \pi^2},
$$

then we have

$$
K(f_0) = 1 + \frac{\tau}{\sqrt{(\log R)^2 + \pi^2}} + O(\tau^2) \qquad (\tau \to 0).
$$

Therefore, $K(f_0) < K(f_1)$ and $K(f_0) < K(f_2)$ for all sufficiently small $\tau > 0$.

Conjecture. *The constant $c_R(\tau)$ in the extremal problem can be taken so that*

$$
c_R(\tau) = 1 + \frac{\tau}{\sqrt{(\log R)^2 + \pi^2}} + o(\tau) \quad (\tau \to 0).
$$

3. An estimate using angular variation

In this section, we will give an estimate of the constant $c_R(\tau)$ under a relatively less strict condition by introducing the angular variation. However, our result obtained by this method is far from sharp and does not meet the desired answer to our extremal problem. Nevertheless, we present it here because its arguments themselves might be useful.

For a quasiconformal automorphism f of the annulus A with $f(1) = 1$, we define the *maximal angular variation* by

$$\omega(f) := \max_{0 \leq \theta < 2\pi} \max_{1 \leq r, r' \leq R} |\arg f(re^{i\theta}) - \arg f(r'e^{i\theta})|,$$

where $\arg f(z)$ takes the branch of $\arg f(1) = 0$.

For example, $\omega(f_1) = \tau$ and $\omega(f_2) = 0$ for the canonical quasiconformal automorphisms defined in the previous section. Note that, by considering the image of a radial segment $[1, R]$ in A, we always have

$$\tau(f) - 2\pi \leq \omega(f).$$

We divide the estimate into two cases according to the comparison of $\tau = \tau(f)$ with $\omega = \omega(f)$:

    A (small angular variation) $\omega \leq \tau/3$;
    B (large angular variation) $\omega \geq \tau/3$.

Note that, in Case A, an additional condition $\tau \leq 3\pi$ is forced to be required. Indeed, by $\tau - 2\pi \leq \omega$, the assumption $\omega \leq \tau/3$ gives such restriction to τ.

First, we deal with Case A as follows.

Lemma 3.1. *Let f be a quasiconformal automorphism of the annulus A with $f(1) = 1$. If $\omega(f) \leq \tau(f)/3$, then the maximal dilatation $K(f)$ of f satisfies*

$$K(f) \geq 1 + \frac{\tau}{6\pi}.$$

Proof. Let $\theta_0 \in [0, 2\pi)$ be the angle for which $\tau = \tau(f)$ is attained, namely,

$$\tau = |\arg f(Re^{i\theta_0}) - \theta_0|.$$

Take the radial segments $\gamma_1 = [1, R]$ and $\gamma_2 = [e^{i\theta_0}, Re^{i\theta_0}]$. If $\arg f(Re^{i\theta_0}) - \theta_0 > 0$, we consider a circular quadrilateral D in the annulus A bounded by γ_1 and γ_2 in the positive direction from γ_1, and if $\arg f(Re^{i\theta_0}) - \theta_0 < 0$, we take such D in the negative direction from γ_1. Without loss of generality, we have only to deal with the positive case. Consider the image $f(D)$ of D under f and compare the modulus $\mathrm{mod}(f(D))$ with $\mathrm{mod}(D) = \theta_0 / \log R$.

Since $f(1) = 1$, we see that $\arg f(r) \leq \omega$ for every $r \in [1, R]$. Hence $f(D)$ contains a radial segment $\gamma_1' = [e^{i\omega}, Re^{i\omega}]$. On the other hand, since $f(Re^{i\theta_0}) = Re^{i(\theta_0 + \tau)}$, we see that $\arg f(re^{i\theta_0}) \geq \theta_0 + \tau - \omega$ for every $r \in [1, R]$. Hence $f(D)$ contains a radial segment $\gamma_2' = [e^{i(\theta_0 + \tau - \omega)}, Re^{i(\theta_0 + \tau - \omega)}]$. Therefore $f(D)$ contains a smaller circular quadrilateral D' bounded by γ_1' and γ_2'.

By the monotone principle of the modulus, we have

$$\mathrm{mod}(f(D)) \geq \mathrm{mod}(D') = \frac{\theta_0 + \tau - 2\omega}{\log R} \geq \frac{\theta_0 + \tau/3}{\log R}.$$

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Since the maximal dilatation of f satisfies $K(f) \geq \mathrm{mod}(f(D))/\mathrm{mod}(D)$, we obtain that

$$K(f) \geq \frac{\theta_0 + \tau/3}{\theta_0} \geq 1 + \frac{\tau}{6\pi}.$$

This completes the proof. □

Next, we consider Case B. In this case, an estimate will be given in terms of an increasing function of ω. Then the condition $\omega \geq \tau/3$ yields the corresponding estimate by τ.

Lemma 3.2. *Let f be a quasiconformal automorphism of the annulus A with $f(1) = 1$. If $6\pi \geq \omega(f) \geq \tau(f)/3$, then the maximal dilatation $K(f)$ of f satisfies*

$$K(f) \geq 1 + \frac{\tau^3}{1458\pi(\log R)^2 + 18\pi\tau^2 - \tau^3}.$$

Proof. We see from [2, Lemma 2.2] (see also [9, Lemma 5.1]) that, if $\omega(f) = \omega \leq 6\pi$, then

$$K(f) \geq \frac{2\pi/\log R}{2\pi/\log R - g(u)} = 1 + \frac{g(u)}{2\pi/\log R - g(u)},$$

where $u = \omega/(3\log R)$ and $g(u) = u^3/(1 + u^2)$. (Actually, this result is obtained by a clever choice of a conformal density which can estimate the extremal length of a certain curve family.) Since $g(u)$ is an increasing function of u, by replacing $u = \omega/(3\log R)$ with $u = \tau/(9\log R)$, we obtain the required inequality. □

It is easy to see that

$$\frac{\tau^3}{1458\pi(\log R)^2 + 18\pi\tau^2 - \tau^3} < \frac{\tau}{18\pi - \tau} < \frac{\tau}{6\pi}$$

holds for $\tau \leq 3\pi$. Then Lemmata 3.1 and 3.2 conclude the following.

Proposition 3.3. *Let f be a quasiconformal automorphism of the annulus A with $f(1) = 1$. If $\omega(f) \leq 6\pi$, then the maximal dilatation $K(f)$ of f satisfies*

$$K(f) \geq 1 + \frac{\tau^3}{1458\pi(\log R)^2 + 18\pi\tau^2 - \tau^3}.$$

4. AN ESTIMATE USING THE FAMILY OF RADIAL SEGMENTS

In this section, we deal with the case where $\tau(f)$ is large and obtain an inequality for $K(f)$ from below, which will be an asymptotically sharp estimate. When $\tau(f)$ is greater than 2π, all the radial segments of A must be moved by f. Then we will have such an estimate by considering the extremal length of the curve family of these radial segments. A similar argument has been given in [7] and [8].

Theorem 4.1. *Let f be a quasiconformal automorphism of the annulus A with $f(1) = 1$. If $\tau(f) > 2\pi$, then the maximal dilatation $K(f)$ of f satisfies*

$$K(f) \geq 1 + \left(\frac{\tau - 2\pi}{\log R}\right)^2.$$

Proof. We denote the two boundary components of A by $\partial_1 A = \{|z| = 1\}$ and $\partial_2 A = \{|z| = R\}$. Let $F = \{\beta_\theta\}_{\theta \in [0, 2\pi)}$ be a curve family in A consisting of all the radial segments $\beta = \beta_\theta$ connecting $e^{i\theta} \in \partial_1 A$ and $Re^{i\theta} \in \partial_2 A$. We consider the extremal length

$$\lambda(F) = \sup_\rho \frac{\{\inf_{\beta \in F} \int_\beta \rho(z)|dz|\}^2}{\iint_A \rho(z)^2 dxdy}$$

of the curve family F, where the supremum is taken over all Borel measurable non-negative functions $\rho(z)$ on A (see e.g. [1], [6], [11]). Then, similar to the case of a curve family consisting of all curves connecting $\partial_1 A$ and $\partial_2 A$,

$$\rho_0(z)|dz| = \frac{|dz|}{|z|}$$

is the extremal metric, which is just the push-forward of the euclidean metric on the universal cover $\tilde{A} = \{\zeta \in \mathbb{C} \mid 0 \leq \mathrm{Re}\,\zeta \leq \log R\}$ by $z = \exp\zeta$. By this metric, the length of any radial segment β is $\log R$ and the area of A is $2\pi \log R$. In particular $\lambda(F) = (2\pi)^{-1} \log R$.

Consider the length $\int_{f(\beta)} \rho_0(z)|dz|$ for any $\beta \in F$. This coincides with the euclidean length of the lift $\tilde{f}(\tilde{\beta})$ of $f(\beta)$ to $\tilde{A}$. If $\tau = \tau(f) > 2\pi$, then the difference of the imaginary part of the endpoints of $\tilde{f}(\tilde{\beta})$ is not less than $\tau - 2\pi$. Then the euclidean length of $\tilde{f}(\tilde{\beta})$ is greater than or equal to

$$\left\{(\log R)^2 + (\tau - 2\pi)^2\right\}^{\frac{1}{2}}.$$

Therefore the extremal length $\lambda(f(F))$ of the curve family $f(F)$ can be estimated by

$$\lambda(f(F)) \geq \frac{(\log R)^2 + (\tau - 2\pi)^2}{2\pi \log R}.$$

The extremal lengths of the curve families and the maximal dilatation of f satisfy $\lambda(f(F)) \leq K(f)\lambda(F)$. Hence

$$\frac{1 + \{(\tau - 2\pi)/\log R\}^2}{2\pi/\log R} \leq K(f)\frac{\log R}{2\pi},$$

from which we have $K(f) \geq 1 + \{(\tau - 2\pi)/\log R\}^2$. $\qquad\square$

Remark. The same argument as above also gives a similar estimate

$$K(f) \geq 1 + \left(\frac{\omega}{\log R}\right)^2$$

by using the angular variation $\omega = \omega(f)$. Since $\omega \geq \tau - 2\pi$, Theorem 4.1 can be shown through this result.

This theorem implies that we can take the constant $c_R(\tau)$ so that

$$c_R(\tau) = \frac{\tau^2}{(\log R)^2} + O(\tau) \quad (\tau \to \infty).$$

On the other hand, the maximal dilatation of the canonical quasiconformal automorphism f_1 defined in Section 2 has this asymptotic expansion regarding the top term. Hence we can say that f_1 asymptotically attains the minimal value or f_1 is an asymptotically extremal map when τ tends to the infinity.

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Corollary 4.2. *The constant $c_R(\tau)$ in the extremal problem can be taken so that*

$$c_R(\tau) = \frac{\tau^2}{(\log R)^2} + O(\tau) \quad (\tau \to \infty).$$

5. AN ESTIMATE USING HYPERBOLIC METRIC

We will try to find a better estimate when τ is small. The estimate obtained in Section 3 is $K(f) \geq 1 + O(\tau^3)$ $(\tau \to 0)$. We will improve the small order of τ into $O(\tau)$. In an oral communication, Toshiyuki Sugawa noticed the author of the following method of extending a normalized quasiconformal automorphism f of the annulus A to a quasiconformal automorphism of the three-punctured sphere $\mathbb{C} - \{0, 1\}$ and estimating $K(f)$ by using Teichmüller's theorem (see [5]).

Theorem 5.1. *Let f be a quasiconformal automorphism of the annulus A with $f(1) = 1$. If $\tau = \tau(f) \leq \pi$, then the maximal dilatation $K(f)$ of f satisfies*

$$K(f) \geq \exp\left(\frac{\tau}{C_R}\right) \geq 1 + \frac{\tau}{C_R}$$

for some constant $C_R > 0$ depending only on R. Moreover,

$$K(f) \geq 1 + \frac{\tau}{\log R + C} + o(\tau) \quad (\tau \to 0),$$

where $C = \Gamma(1/4)^4/(2\pi)^2$.

Proof. By the iterated application of the reflection with respect to the boundary ∂A, the quasiconformal automorphism $f : A \to A$ extends to a quasiconformal automorphism $\hat{f}$ of $\mathbb{C} - \{0\}$ that fixes R^{2n} for every $n \in \mathbb{Z}$. In particular, $\hat{f}$ can be regarded as a quasiconformal automorphism of $\mathbb{C} - \{0, 1\}$. Note that the maximal dilatation $K(f)$ is the same as $K(\hat{f})$.

Let $\lambda(z)|dz|$ be the hyperbolic metric (with constant curvature -1) on $\mathbb{C} - \{0, 1\}$ and d its hyperbolic distance. It is known that the density function $\lambda(z)$ can be estimated as

$$\lambda(z) \geq \frac{1}{|z| \, (|\log|z|\,| + C)},$$

where $C = \lambda(-1)^{-1} = \Gamma(1/4)^4/(2\pi)^2$ ([3], [4]; see also [10] for improved estimates). On the other hand, Teichmüller's theorem ([5]) implies

$$K(\hat{f}) \geq \sup_{z \in \mathbb{C} - \{0,1\}} \exp d(\hat{f}(z), z).$$

Let $\theta_0 \in [0, 2\pi)$ be the angle for which $\tau = \tau(f)$ is attained. For the point $z = Re^{i\theta_0}$ and $\hat{f}(z) = Re^{i(\theta_0 \pm \tau)}$, we have

$$d(Re^{i(\theta_0 \pm \tau)}, Re^{i\theta_0}) = \inf_\gamma \int_\gamma \lambda(z)|dz| \geq \inf_\gamma \int_\gamma \frac{|dz|}{|z| \, (|\log|z|\,| + C)},$$

where the infimum is taken over all paths γ connecting $Re^{i(\theta_0 \pm \tau)}$ and $Re^{i\theta_0}$.

If we choose a circular path γ_0 of radius R, then

$$\int_{\gamma_0} \frac{|dz|}{|z| \, (|\log|z|\,| + C)} = \frac{\tau}{\log R + C} \leq \frac{\pi}{\log R + C}.$$

Hence, in order to estimate the distance, we have only to consider the paths γ whose lengths do no exceed this value. In particular, we may assume that each γ

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stays within an annulus $\{1/R' \leq |z| \leq R'\}$ defined by a constant R' depending only on R. This gives an estimate

$$\inf_{\gamma} \int_{\gamma} \frac{|dz|}{|z| \left(|\log|z|\,| + C\right)} \geq \inf_{\gamma} \int_{\gamma} \frac{|dz|}{|z|(\log R' + C)} = \frac{\tau}{C_R},$$

where we set $C_R = \log R' + C$. Hence $K(f)$ can be estimated from below by $\exp(\tau/C_R)$.

When $\tau \to 0$, the constant R' can be taken arbitrarily close to R. This implies that

$$\exp\left(\frac{\tau}{C_R}\right) = 1 + \frac{\tau}{\log R + C} + o(\tau) \quad (\tau \to 0).$$

Thus we have the required estimate of $K(f)$. $\qquad\square$

Remark. The argument using the hyperbolic metric also yields an estimate for $K(f)$ when $\tau \to \infty$. To do this, we use the modular function $p : \mathbb{H} \to \mathbb{C} - \{0,1\}$ on the upper half-plane $\mathbb{H} = \{\mathrm{Im}\,\zeta > 0\}$, which is the universal covering map of $\mathbb{C} - \{0,1\}$. Here we assume that $\{0,1,\infty\}$ corresponds to $\{0,1,\infty\}$ respectively by p. Then p maps the positive imaginary axis $i(0,\infty)$ in $\mathbb{H}$ onto the negative axis $(0,-\infty)$ in $\mathbb{C} - \{0,1\}$ and satisfies $p(i) = -1$. We lift the quasiconformal automorphism $\hat{f}$ to $\mathbb{H}$ against p and apply Teichmüller's theorem. For instance, by considering the case $\tau = 2\pi n$ for an integer n, we can obtain

$$K(\hat{f}) \geq \exp d_{\mathbb{H}}(iY_R, iY_R + 2n) \geq \left(\frac{2n}{Y_R}\right)^2,$$

where $d_{\mathbb{H}}$ is the hyperbolic distance on $\mathbb{H}$ and Y_R is a positive number satisfying $p(iY_R) = -R$. Then we may use an estimate $\pi Y_R \leq \log R + \pi$ ([10, Lemma 5.4]) to obtain

$$K(f) \geq \frac{\tau^2}{(\log R + \pi)^2} + O(\tau) \quad (\tau \to \infty).$$

However, this is not better than the estimate obtained in Theorem 4.1 or Corollary 4.2.

The hybrid map f_0 introduced in Section 2 satisfies

$$K(f_0) = 1 + \frac{\tau}{\sqrt{(\log R)^2 + \pi^2}} + O(\tau^2) \qquad (\tau \to 0),$$

which means that $c_R(\tau)$ must satisfy

$$c_R(\tau) \leq 1 + \frac{\tau}{\sqrt{(\log R)^2 + \pi^2}} + O(\tau^2) \quad (\tau \to 0).$$

On the other hand, Theorem 5.1 implies that $c_R(\tau)$ can be taken so that

$$c_R(\tau) \geq 1 + \frac{\tau}{\log R + C} + o(\tau) \quad (\tau \to 0),$$

where $C = \lambda(-1)^{-1} \approx 4.4 > \pi$. In sum:

Corollary 5.2. *The constant $c_R(\tau)$ in the extremal problem can be taken so that*

$$1 + \frac{\tau}{\log R + C} + o(\tau) \leq c_R(\tau) \leq 1 + \frac{\tau}{\sqrt{(\log R)^2 + \pi^2}} + O(\tau^2) \quad (\tau \to 0).$$

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