TWISTS AND GROMOV HYPERBOLICITY OF RIEMANN SURFACES

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Abstract. The main aim of this paper is to study whether the Gromov hyperbolicity is preserved under some transformations on Riemann surfaces (with their Poincaré metrics). In fact, although quasiconformal maps between Riemann surfaces preserve hyperbolicity, we show that arbitrary twists along simple closed geodesics do not preserve it, in general.

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1. Introduction

In the 1980s, M. Gromov (cf. [1]) introduced a notion of abstract hyperbolic spaces, and it has thereafter been studied and developed by many authors, e.g. [2]-[5]. Initially, the research was mainly centered on hyperbolic group theory; lately researchers have been increasing their interest in more direct studies of certain spaces endowed with metrics which are used in geometric function theory, e.g. [6]-[28]. A primary question we naturally ask is whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. A classical example of a Gromov hyperbolic space is a simply connected Riemannian manifold with sectional curvature $K \leq -k^2 < 0$.

One of the important problems when studying a geometric property is to consider its stability under appropriate deformations, in other words, to determine what type of perturbations preserve this property. Aiming this, we study the stability of Gromov hyperbolicity in this paper. In [12], the authors proved that a certain strong change in the metric preserves hyperbolicity for Denjoy domains.

It is well known that Gromov hyperbolicity is invariant under quasi-isometries between geodesic metric spaces; however, usual deformations in the context of Riemann surfaces are given by quasiconformal maps, and it is natural to ask whether they preserve hyperbolicity or not.

In this paper we first notice that quasiconformal maps between Riemann surfaces preserve hyperbolicity (see Theorem 3.2). In the context of Riemann surfaces of topologically finite type, twists along disjoint simple closed geodesics are always realized by quasiconformal maps; however, for Riemann surfaces of topologically infinite type, we prove that the twists do not preserve hyperbolicity in general (see Theorem 6.2), although they preserve hyperbolicity in some cases (see Theorems 5.3 and 3.4). In order to achieve this goal and propose further problems, we need to prove certain results about hyperbolicity of Denjoy domains (see Theorems 4.2 and 4.4).

2. Background on Gromov spaces and previous results

For a geodesic metric space (X, d) and $x_1, x_2, \dots, x_n \in X$, a *geodesic polygon* $P = \{x_1, x_2, \dots, x_n\}$ is the union of n geodesics $J_1 := [x_1, x_2], J_2 := [x_2, x_3], \dots, J_n := [x_n, x_1]$. We say that P is δ-*thin* for a constant $\delta \geq 0$ if for every $x \in J_i$ we have that $d(x, \bigcup_{j \neq i} J_j) \leq \delta$. The space (X, d) is *Gromov δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. In order to simplify the notation, we say that X is δ-hyperbolic or just hyperbolic instead of saying that (X, d) is Gromov δ-hyperbolic.

It is easy to check that in a δ-hyperbolic space, every geodesic polygon with n sides is $(n-2)\delta$-thin. If we have a triangle with two identical vertices, we call it a *bigon*; obviously, every bigon in a δ-hyperbolic space is δ-thin.

A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geq 1$, $b \geq 0$ such that

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

A such function is called an (a, b)-*quasi-isometry*. We say that the image of f is ε-*full* (for some $\varepsilon \geq 0$) if $d_Y(y, f(X)) \leq \varepsilon$ for every $y \in Y$.

We say that a property holds *quantitatively*, if it holds with a constant depending only on the constants in the assumptions.

Theorem 2.1. ([29, p. 88]) *Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is hyperbolic, then X is hyperbolic, quantitatively. Besides, if the image of f is ε-full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic, quantitatively.*

Recall that the universal cover of any domain $\Omega \subset \mathbb{C}$, with at least two finite boundary points, is the unit disk $\mathbb{D}$. In Ω we can define the Poincaré metric, i.e. the metric obtained by projecting the metric $ds = 2|dz|/(1 - |z|^2)$ of the unit disk by any universal covering map $\pi : \mathbb{D} \longrightarrow \Omega$. Alternatively, we may use the upper half plane $\mathbb{H}$ with the metric $ds = |dz|/y$ as the universal cover. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1 and, in particular, Ω is a geodesic metric space. We denote the distance on Ω by d_Ω and the length of a curve in Ω by L_Ω.

The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1 (that is, non-expanding), when we consider the respective Poincaré metrics.

A *Denjoy domain* Ω is a domain in the complex plane whose boundary is contained in the real axis. Since $\Omega \cap \mathbb{R}$ is an open set in $\mathbb{R}$, it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega\cap\mathbb{R} = \bigcup_{n\in\Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n\in\Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where Λ is countably infinite, since if Λ is finite then Ω is hyperbolic by [25, Proposition 3.2] or [11, Proposition 3.5].

Definition 2.2. Let Ω be a Denjoy domain. Then we have $\Omega\cap\mathbb{R} = \bigcup_{n\ge0}(a_n, b_n)$ for some pairwise disjoint intervals. We say that a curve in Ω is a *fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_0, b_0) and (a_n, b_n) for some $n > 0$; we denote by γ_n the fundamental geodesic corresponding to n and define its length by $2l_n := L_\Omega(\gamma_n)$.

Theorem 2.3. ([18, Theorem 5.1]) *Let Ω be a Denjoy domain with its Poincaré metric. Then the following conditions are quantitatively equivalent:*

1. *Ω is δ-hyperbolic.*
2. *There exists a constant c_1 such that $d_\Omega(z, \Omega\cap\mathbb{R}) \le c_1$ for every $z \in \bigcup_{n\ge1}\gamma_n$.*
3. *There exists a constant c_2 such that every geodesic bigon in Ω with vertices in $\mathbb{R}$ is c_2-thin.*

We need a stronger version of this result.

Theorem 2.4. *Let Ω be a Denjoy with $\Omega\cap\mathbb{R} = \bigcup_{n\ge0}(a_n, b_n)$. Then the following conditions are quantitatively equivalent:*

1. *Ω is δ-hyperbolic.*
2. *There exists a constant c_1 such that $d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) \le c_1$ for every $z \in \bigcup_{n\ge1}\gamma_n$.*

Proof. Assume first (1): Ω is δ-hyperbolic. Theorem 2.3 says that there exists a constant c_1 such that $d_\Omega\big(z, \bigcup_{n\ge0}(a_n, b_n)\big) \le c_1$ for every $z \in \bigcup_{n\ge1}\gamma_n$. Then every $z \in \bigcup_{n\ge1}\gamma_n$ with $d_\Omega\big(z, (a_0, b_0)\big) > c_1$ verifies

$$d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) = d_\Omega\big(z, \bigcup_{n\ge0}(a_n, b_n)\big) \le c_1\,.$$

By continuity, we have $d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) \le c_1$ for every $z \in \bigcup_{n\ge1}\gamma_n$ with $d_\Omega\big(z, (a_0, b_0)\big) \ge c_1$. Now, if $z \in \gamma_n$ for some n with $d_\Omega\big(z, (a_0, b_0)\big) < c_1$ and $d_\Omega\big(z, (a_n, b_n)\big) > c_1$, then there exists $z' \in \gamma_n$ with $d_\Omega(z, z') \le c_1$ and $d_\Omega\big(z', (a_0, b_0)\big) = c_1$, and therefore $d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) \le 2c_1$. Consequently, $d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) \le 2c_1$ for every $z \in \bigcup_{n\ge1}\gamma_n$.

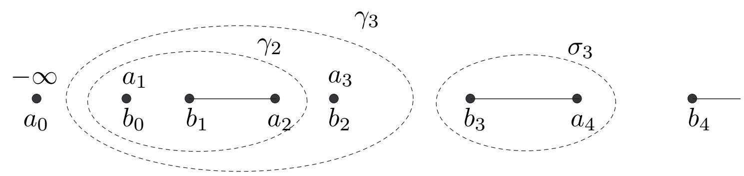
Assume now (2). Then

$$d_\Omega\big(z, \bigcup_{n\ge0}(a_n, b_n)\big) \le d_\Omega\big(z, \bigcup_{n\ge1}(a_n, b_n)\big) \le c_1$$

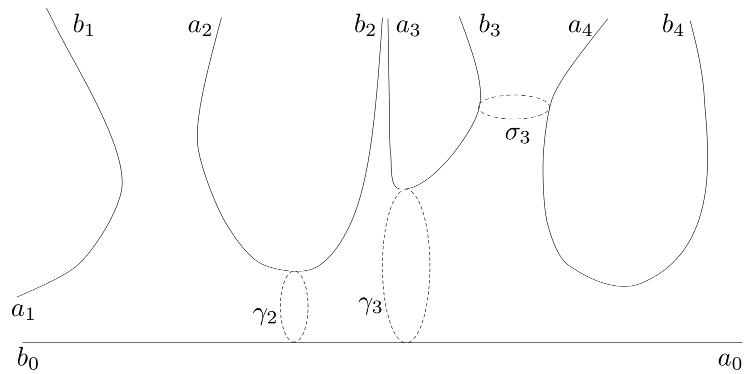
for every $z \in \bigcup_{n\ge1}\gamma_n$, and Theorem 2.3 implies that Ω is δ-hyperbolic, quantitatively. □

Definition 2.5. A *train* is a Denjoy domain Ω with $\Omega\cap\mathbb{R} = \bigcup_{n=0}^{\infty}(a_n, b_n)$, such that $-\infty \le a_0$ and $b_n \le a_{n+1}$ for every n and $\lim_{n\to\infty} a_n = \infty$. We say that a train is *tight* if $b_n = a_{n+1}$ for every n.

A curve in a train Ω is a *second fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{n+1}, b_{n+1}) for some $n \ge 0$; we denote by σ_n the second fundamental geodesic corresponding to n and define its length by $2r_n := L_\Omega(\sigma_n)$ (see figure below). If $b_n = a_{n+1}$, we define σ_n as the puncture at this point and $r_n = 0$.



(a) Train seen as a subset of the complex plane.



(b) The same train seen with "Euclidean eyes".

Remark 2.6. (1) We can also define a train to be a Denjoy domain Ω such that $\mathbb{R}\setminus\Omega$ consists of an "increasing" sequence of closed intervals or punctures.

(2) Recall that in every free homotopy class there exists a single simple closed geodesic, assuming that punctures are simple closed geodesics with length equal to zero. That is why both the fundamental geodesic and the second fundamental geodesic are unique for every n.

A train is tight if and only if every second fundamental geodesic is a puncture. Tight trains are important since they are the simplest examples of infinite ends; furthermore, in a tight train it is possible to give a fairly precise description of the ending geometry. See, e.g. [30], [31], [32], where they call a similar but more general surface (allowing twists) a flute space.

Definition 2.7. A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere from which three open disks are removed and whose boundary curves are simple closed geodesics. A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere from which n open disks and m points are removed ($n + m = 3$) such that the n boundary curves are simple closed geodesics and the m deleted points are punctures.

Given three non-negative numbers a, b, c, there is a unique (up to conformal mapping) generalized Y-piece such that their boundary curves have lengths a, b, c (see e.g. [33, p.109]). They are standard pieces for constructing Riemann surfaces. A clear description of these Y-pieces and their use are given in [34, Chapter X.3] and [33, Chapter 3].

If $a_0 = -\infty$, then the fundamental geodesics $\{\gamma_n\}$ and $\{\sigma_n\}$ of a train Ω divide it into generalized Y-pieces and annuli beyond the second fundamental geodesics called funnels. However, when $a_0 \neq -\infty$, we need one more piece for this decomposition, which is a simply connected domain around the interval $(-\infty, a_0]$ called a half-disk. See [35]. A train is determined by the values of their lengths $\{l_n\}_{n>0}$ and $\{r_n\}_{n\geq 0}$.

In [23] the authors prove that it is possible to obtain a similar decomposition (as a union generalized Y-pieces, annuli and half-disks) of any complete surface with arbitrary curvature.

Here we introduce a powerful and simple characterization for a train to be hyperbolic. In particular, this can be applied to the case where l_n is a non-decreasing sequence.

Theorem 2.8. ([22, Theorem 3.18]) *Let us consider a train Ω with $l_m \le l_n + c_1$ for some $c_1 \ge 0$ and for every positive integer numbers $m \le n$.*

1. *If $\{l_n\}$ is a bounded sequence, then Ω is hyperbolic.*
2. *If $\lim_{n\to\infty} l_n = \infty$, then Ω is hyperbolic if and only if $\{r_n\}$ is a bounded sequence and*

$$\sum_{k=n}^{\infty} e^{-l_k} \le c_2\, e^{-l_n}, \qquad \text{for every } n > 1\,. \tag{2.9}$$

holds for some constant c_2.

Remark 2.10. Note that the hypothesis "$l_m \le l_n + c_1$ for $m \le n$" in Theorem 2.8 implies that $\{l_n\}$ is either a bounded sequence or a sequence with limit ∞.

Finally in this section, we give a criterion of hyperbolicity for geodesic metric spaces in general which can be easily applied to Riemann surfaces.

Definition 2.11. Let (X, d) be a geodesic metric space, and let $\{X_n\}_n \subseteq X$ be a family of geodesic metric subspaces such that $X = \bigcup_n X_n$ and that $\eta_{n,m} := X_n \cap X_m$ are compact sets. Further, assume that the set $X \setminus \eta_{n,m}$ for any n and m is not connected, and that any $a \in X_n \setminus \eta_{n,m}$ and $b \in X_m \setminus \eta_{n,m}$ are in different components of $X \setminus \eta_{n,m}$ for $m \ne n$. If there exists positive constants c_1 and c_2 such that $\operatorname{diam}_{X_n}(\eta_{n,m}) \le c_1$ for every n, m, and $d_{X_n}(\eta_{n,m}, \eta_{n,k}) \ge c_2$ for every n and $m \ne k$, we say that $\{X_n\}_n$ is a (c_1, c_2)-*tree decomposition* of X.

Theorem 2.12. ([25, Theorem 2.4]) *Let us consider a geodesic metric space X and a family of geodesic metric subspaces $\{X_n\}_n \subseteq X$ which is a (c_1, c_2)-tree decomposition of X. Then X is δ-hyperbolic if and only if there exists a constant δ' such that X_n is δ'-hyperbolic for every n.*

See Theorems 2.9 and 2.19 in [19] for further results.

3. Quasiconformal maps preserve Gromov hyperbolicity

In this section, we will show that any quasiconformal homeomorphism between Riemann surfaces preserves their Gromov hyperbolicity. Actually, it is well-known that any bi-Lipschitz map in the following sense preserves the hyperbolicity.

Definition 3.1. Let $f : R \longrightarrow R'$ be a surjective homeomorphism between Riemann surfaces with the Poincaré metric and d_R and $d_{R'}$ are the distances on R and R', respectively. If there exists a constant $a \ge 1$ such that

$$\frac{1}{a} d_R(x, y) \le d_{R'}(f(x), f(y)) \le a d_R(x, y)$$

for any x and y in R, then f is called a *bi-Lipschitz homeomorphism.* Assume further that f is a diffeomorphism and let ds and ds' be the line elements of the Poincaré metric on R and R', respectively. If there exists a constant $a \ge 1$ such that

$$\frac{1}{a} ds(x) \le ds'(f(x)) \le a ds(x)$$

for any x in R, then f is called a *bi-Lipschitz diffeomorphism.*

Note that if $f : R \longrightarrow R'$ is an isometry, then it is a bi-Lipschitz diffeomorphism. Also a bi-Lipschitz homeomorphism is a quasi-isometry. Any quasi-isometry preserves hyperbolicity.

We consider the universal covering maps $\pi : \mathbb{D} \longrightarrow R$ and $\pi' : \mathbb{D} \longrightarrow R$ with the covering transformation groups (Fuchsian groups) G and G', respectively. Any homeomorphism $f : R \longrightarrow R'$ lifts to an automorphism $\tilde{f}$ of $\mathbb{D}$ satisfying $f \circ \pi = \pi' \circ \tilde{f}$, or equivalently $\tilde{f} G \tilde{f}^{-1} = G'$. Then f is bi-Lipschitz if and only if $\tilde{f}$ is bi-Lipschitz.

Any quasiconformal automorphism $\tilde{f}$ of $\mathbb{D}$ extends to the boundary $\partial\mathbb{D}$ as a quasisymmetric automorphism $\bar{f}$. If $\tilde{f}$ satisfies the compatibility condition $\tilde{f} G \tilde{f}^{-1} = G'$ for the Fuchsian groups, then $\bar{f} G \bar{f}^{-1} = G'$ on $\partial\mathbb{D}$. Conversely, there is a way of extending a quasisymmetric automorphism $\bar{f}$ of $\partial\mathbb{D}$ to $\mathbb{D}$. The conformal barycentric extension due to Douady and Earle [36] gives such an extension $E(\bar{f})$ which is a bi-Lipschitz diffeomorphism of $\mathbb{D}$ onto itself. Moreover, if $\bar{f}$ is compatible with the Fuchsian groups G and G', then so is $E(\bar{f})$ for G and G'. The bi-Lipschitz constant a for $E(\bar{f})$ can be taken depending only on the quasisymmetric constant $m(\bar{f})$ of $\bar{f}$. Here $m(\bar{f})$ can be defined as the infimum of the maximal dilatations k of quasiconformal automorphisms $\mathbb{D}$ that have the boundary value $\bar{f}$ (see [36]).

In virtue of the conformal barycentric extension, we can prove the required result.

Theorem 3.2. *Assume that there is a k-quasiconformal homeomorphism f of a Riemann surface R onto another Riemann surface R'. Then R is hyperbolic if and only if R' is hyperbolic, quantitatively.*

Proof. We take a lift $\tilde{f} : \mathbb{D} \longrightarrow \mathbb{D}$ of $f : R \longrightarrow R'$ and then the quasisymmetric extension $\bar{f} : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$. This satisfies the compatibility condition $\bar{f}G\bar{f}^{-1} = G'$ for the Fuchsian groups of R and R'. The barycentric extension $E(\bar{f})$ of $\bar{f}$ is a bi-Lipschitz diffeomorphism of $\mathbb{D}$ satisfying $E(\bar{f})GE(\bar{f})^{-1} = G'$. As is mentioned above, the bi-Lipschitz constant a for $E(\bar{f})$ depends only on the maximal dilatation of $\tilde{f}$, which is the constant k for the quasiconformal homeomorphism f. Hence it projects to a bi-Lipschitz diffeomorphism of R onto R' with the same constant a. This in particular implies that R and R' are quasi-isometric and hence they are hyperbolic at the same time. Moreover, by Theorem 2.1, the constants δ for the hyperbolicity of R and R' are depending on a, and hence on k. □

Here we illustrate a concrete example of twists on Riemann surfaces which preserve the hyperbolicity.

Definition 3.3. We say that a Riemann surface S has an *l-decomposition* for a constant $l > 0$ if there exist generalized Y-pieces $\{Y_n\}_{n\in N}$ and funnels $\{F_m\}_{m\in M}$ having pairwise disjoint interiors such that:

- $S = \left(\bigcup_{n\in N} Y_n\right) \cup \left(\bigcup_{m\in M} F_m\right)$,
- $L_S(\gamma) \le l$ for every simple closed geodesic $\gamma \subset \bigcup_{n\in N} \partial Y_n \setminus \bigcup_{m\in M} \partial F_m$.

Theorem 3.4. *Let us consider a Riemann surface S with an l-decomposition for $l > 0$. Denote by S' a Riemann surface obtained from S by any amount of twist around the boundary geodesics of $\{Y_n\}_{n\in N}$, which are the generalized Y-pieces corresponding to the l-decomposition of S. Then S is hyperbolic if and only if S' is hyperbolic, quantitatively.*

Proof. By [27, Theorem 3.7] there exists a graph G related to S (an l-skeleton of S, with the notation in [27]), such that S is hyperbolic if and only if G is hyperbolic (with control on the hyperbolicity constants). We obtain the same result for S' with a graph G'. This finishes the proof since, by construction, $G' = G$ (the graph depends on the lengths of $\{\partial Y_n\}_{n\in N}$, but not on the twist). □

Actually, we can prove that there exists a quasiconformal homeomorphism between the Riemann surfaces S and S' in Theorem 3.4 whose quasiconformal constant is estimated by l. See, e.g. [37, Theorem 3.1]. Basically, the reason of this fact is as follows. If the lengths of simple closed geodesics are uniformly bounded, the collar lemma implies that there are annular neighborhoods of the simple closed geodesics whose conformal moduli are uniformly large. Then the twists can be realized on these annuli by quasiconformal maps with uniformly bounded dilatations.

On the other hand, we will construct later an example of a Gromov hyperbolic Riemann surface S such that another Riemann surface S^0 obtained from S by giving certain twists along mutually disjoint simple closed geodesics $\{\gamma_n\}$ is not hyperbolic. If the lengths of $\{\gamma_n\}$ were uniformly bounded, then this deformation of hyperbolic structure would be given quasiconformally, namely, there would be a quasiconformal homeomorphism of S onto S^0. However, Theorem 3.2 implies that such a deformation cannot break Gromov hyperbolicity and hence we must seek the example of the above S and S^0 beyond this uniform twist deformation.

4. On Gromov hyperbolicity of bitrains

Definition 4.1. A *bitrain* Ω is a Denjoy domain with $\Omega \cap \mathbb{R} = \bigcup_{n\in\mathbb{Z}^*}(a_n, b_n)$ for some pairwise disjoint intervals, such that $b_m \le a_n$ for every $m < n$ (here $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$).

We denote by ω_n the simple closed geodesic which just intersects $\mathbb{R}$ in (a_{-n}, b_{-n}) and (a_n, b_n) for some $n > 0$; we define $2h_n := L_\Omega(\omega_n)$.

We denote by η_n the simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and $(a_{n+\operatorname{sgn} n}, b_{n+\operatorname{sgn} n})$ for some $n \in \mathbb{Z}^*$; we define $2s_n := L_\Omega(\eta_n)$. If $[a_n, b_n] \cap [a_{n+\operatorname{sgn} n}, b_{n+\operatorname{sgn} n}] \neq \varnothing$, we define η_n as the puncture at this intersection point.

The purpose of this section is to develop criteria which allow to decide or discard the hyperbolicity of a bitrain.

Theorem 4.2. *Let Ω be a bitrain with its Poincaré metric, such that $s_n \le c_1$ for every $n > 0$. Then the following conditions are quantitatively equivalent:*

1. *Ω is δ-hyperbolic.*
2. *There exists a constant c_2 such that $d_\Omega(z, \Omega \cap \mathbb{R}) \le c_2$ for every $z \in \cup_{n\ge1}\omega_n$.*

Proof. Assume first (1): Ω is δ-hyperbolic. Fix $n > 0$ and $z \in \omega_n$. Denote by z_1, z_2 the points in $\omega_n \cap \mathbb{R}$; then ω_n is a geodesic bigon with vertices z_1, z_2. By symmetry, we have only to consider the case $z \in \Omega \cap \overline{\mathbb{H}}$. Since ω_n is δ-thin, we have $d_\Omega(z, \Omega \cap \mathbb{R}) \le d_\Omega(z, \omega_n \cap \{\mathbb{C} \setminus \mathbb{H}\}) \le \delta$.

Assume now (2). Let us define $(\alpha_0, \beta_0) = (a_{-1}, b_{-1})$, $(\alpha_{2k}, \beta_{2k}) = (a_k, b_k)$ for $k > 0$, $(\alpha_{2k+1}, \beta_{2k+1}) = (a_{-k-2}, b_{-k-2})$ for $k \ge 0$. Then $\Omega \cap \mathbb{R} = \bigcup_{n\ge0}(\alpha_n, \beta_n)$. Let γ_n be the simple closed geodesic in Ω which intersects $\mathbb{R}$ just in (α_0, β_0) and (α_n, β_n). By Theorem 2.3, we just need to show that there exists a constant c_3 such that $d_\Omega(z, \Omega \cap \mathbb{R}) \le c_3$ for every $z \in \bigcup_{n\ge1}\gamma_n$.

We first consider the case that $z \in \bigcup_{k\ge1}\gamma_{2k}$. If $z \in \gamma_2 = \omega_1$, we have directly $d_\Omega(z, \Omega \cap \mathbb{R}) \le c_2$. Hence we assume that $z \in \gamma_{2k}$ for $k > 1$ and fix this integer k. Without loss of generality, we may assume that $z \in \overline{\mathbb{H}}$. Hereafter, for a Jordan curve g in Ω, we denote by $\operatorname{int} g$ the bounded connected component of $\Omega \setminus g$ (i.e., the set of points surrounded by g) in the relative topology of $\mathbb{C}$.

If $z \in \operatorname{int}\omega_2$, let us consider the geodesic heptagon P_1 contained in the closure of $\operatorname{int}\omega_2 \cap \mathbb{H}$, with sides contained in γ_{2k}, ω_2, (a_2, b_2), η_1, (a_1, b_1), ω_1 and (a_{-1}, b_{-1}). (When $k = 2$, γ_4 does not intersect ω_2. In this case, we assume P_1 to be a hexagon and modify the argument below.) The heptagon P_1 bounds a simply connected domain in Ω, and therefore it can be lifted to the unit disk, which is δ_0-hyperbolic for $\delta_0 := \log(1 + \sqrt{2}\,)$ (see [38, p. 130]). Hence, P_1 is $5\delta_0$-thin in Ω and there exists $w \in P_1 \setminus \gamma_{2k}$ with $d_\Omega(z, w) \le 5\delta_0$. Then the argument is divided into the following cases:

- If $w \in (a_2, b_2) \cup (a_1, b_1) \cup (a_{-1}, b_{-1})$, then $d_\Omega(z, \Omega \cap \mathbb{R}) \le 5\delta_0$;
- If $w \in \omega_1 \cup \omega_2$, then $d_\Omega(z, \Omega \cap \mathbb{R}) \le d_\Omega(z, w) + d_\Omega(w, \Omega \cap \mathbb{R}) \le 5\delta_0 + c_2$;
- If $w \in \eta_1$, then $d_\Omega(w, \Omega \cap \mathbb{R}) \le c_1/2$ (since $L_\Omega(\eta_1 \cap \mathbb{H}) = s_1 \le c_1$) and $d_\Omega(z, \Omega \cap \mathbb{R}) \le 5\delta_0 + c_1/2$.

Then we conclude $d_\Omega(z, \Omega \cap \mathbb{R}) \le 5\delta_0 + c_2 + c_1/2$ for every $z \in \operatorname{int}\omega_2$.

If $z \in \operatorname{int}\omega_k \setminus \operatorname{int}\omega_{k-1}$ with $k > 2$, let us consider the geodesic pentagon P_2 contained in the closure of $\{\operatorname{int}\omega_k \setminus \operatorname{int}\omega_{k-1}\} \cap \mathbb{H}$, with sides contained in γ_{2k}, (a_k, b_k), η_{k-1}, (a_{k-1}, b_{k-1}) and ω_{k-1}. Since P_2 is $3\delta_0$-thin in Ω, a similar argument to the previous one gives that $d_\Omega(z, \Omega \cap \mathbb{R}) \le 3\delta_0 + c_2 + c_1/2$. If $z \in \operatorname{int}\omega_{j+1} \setminus \operatorname{int}\omega_j$ with $1 < j < k-1$, let us consider the geodesic hexagon P_3 contained in the closure of $\{\operatorname{int}\omega_{j+1} \setminus \operatorname{int}\omega_j\} \cap \mathbb{H}$, with sides contained in γ_{2k}, ω_{j+1}, (a_{j+1}, b_{j+1}), η_j, (a_j, b_j) and ω_j. Since P_3 is $4\delta_0$-thin in Ω, a similar argument to the previous one gives that $d_\Omega(z, \Omega \cap \mathbb{R}) \le 4\delta_0 + c_2 + c_1/2$.

If $z \in \bigcup_{k\ge0}\gamma_{2k+1}$, then we also obtain from a similar argument that $d_\Omega(z, \Omega \cap \mathbb{R}) \le 5\delta_0 + c_2 + c_1/2$.

Therefore, Theorem 2.3 implies that Ω is hyperbolic, quantitatively. □

Definition 4.3. We say that a bitrain is *symmetric* if $(a_{-n}, b_{-n}) = (-b_n, -a_n)$ for every $n > 0$.

Note that, by symmetry, the imaginary axis minus $\{0\}$ consists of two geodesic lines in every symmetric bitrain.

A symmetric bitrain is determined by the values of the sequences $\{h_n\}_{n>0}$ and $\{s_n\}_{n>0}$.

Theorem 4.4. *A symmetric bitrain with $s_n \le c$ for every n is hyperbolic if and only if the train with parameters $l_n := h_n/2$ and $r_n := s_n$ is hyperbolic, quantitatively.*

Proof. Denote by Ω a symmetric bitrain and define $\Omega_+ := \Omega \cap \overline{\mathbb{H}}$. A Riemann surface Ω_1 obtained from Ω_+ by identifying the point iy with $-iy$ for every $y > 0$ can be conformally (and isometrically) mapped onto a train Ω_2 by $f : \Omega_1 \longrightarrow \Omega_2$. If $\Omega \cap \mathbb{R} = \bigcup_{n\in\mathbb{Z}^*}(\alpha_n, \beta_n)$, we can write $\Omega_2 \cap \mathbb{R} = \bigcup_{n\ge0}(a_n, b_n)$, where $(a_n, b_n) = f\big((\alpha_n, \beta_n)\big)$ for $n > 0$, and (a_0, b_0) is the image of the imaginary axis minus $\{0\}$ by f. Since Ω

is symmetric, the geodesics ω_n meet orthogonally the imaginary axis. Therefore, for each $n > 0$, the image by f of $\omega_n \cap \{z \in \mathbb{C} : \Re z \ge 0\}$ is a (smooth) simple closed geodesic in Ω_2 joining (a_0, b_0) with (a_n, b_n), and therefore it is the geodesic γ_n in Ω_2. Furthermore, the parameters of Ω and Ω_2 correspond as $l_n = h_n/2$ and $r_n = s_n$.

By Theorem 4.2, Ω is hyperbolic if and only if $d_\Omega\big(z, \bigcup_{\mathbb{Z}^*}(\alpha_n, \beta_n)\big) \le c_1$ for every $z \in \bigcup_{n\ge 1}\omega_n$; by symmetry, this is equivalent to that $d_\Omega\big(z, \bigcup_{n>0}(\alpha_n, \beta_n)\big) \le c_1$ for every $z \in \bigcup_{n\ge1}\omega_n$ with $\Re z \ge 0$. This is also equivalent to that $d_{\Omega_2}\big(w, \bigcup_{n>0}(a_n, b_n)\big) \le c_1$ for every $w \in \bigcup_{n\ge1}\gamma_n$, and by Theorem 2.4, this is equivalent to the hyperbolicity of Ω_2, quantitatively. □

5. Twists and hyperbolicity of trains

We will show later that infinite twists, in general, do not preserve hyperbolicity (see Theorem 6.2). However, in this section, we prove that they do preserve hyperbolicity for a large class of examples constructed by trains (Theorems 5.3).

First of all, we need a technical lemma.

Lemma 5.1. *If $\{x_n\}$ is a sequence with $x_n \ge 0$ and*

$$\sum_{k=n}^{\infty} x_k \le c\, x_n$$

for every $n \ge 1$, then

$$\sum_{k=n}^{\infty} x_k^p \le c_p x_n^p,$$

for every $n \ge 1$ and $p > 0$, where $c_p := 1$ if $c \le 1$; $c_p := c^p/(1 - c_0^p)$ and $c_0 := (c-1)/c$ if $c > 1$.

Proof. Assume first that $c \le 1$; then $\sum_{k=1}^{\infty} x_k \le x_1$ and, consequently, $x_n = 0$ for every $n \ge 2$; therefore, the result is direct.

Assume now that $c > 1$. If we define $z_n := \sum_{k=n}^{\infty} x_k$, then $x_n = z_n - z_{n+1}$ and $z_n \le c(z_n - z_{n+1})$, for every $n \ge 1$, and we just need to prove that

$$\sum_{k=n}^{\infty} (z_k - z_{k+1})^p \le c_p (z_n - z_{n+1})^p,$$

for every $n \ge 1$ and $p > 0$. We have

$$c\, z_{n+1} \le c\, z_n - z_n\,, \qquad z_{n+1} \le c_0 z_n\,.$$

Then

$$\sum_{k=n}^{\infty} (z_k - z_{k+1})^p \le \sum_{k=n}^{\infty} z_k^p \le \sum_{k=n}^{\infty} (c_0^{k-n} z_n)^p = \frac{1}{1 - c_0^p}\, z_n^p \le c_p (z_n - z_{n+1})^p,$$

for every $n \ge 1$ and $p > 0$, and the lemma is proved. □

Now we define the twists which are interesting for us.

Definition 5.2. Given a bitrain Ω and $n > 0$, we denote by μ_n the simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{-n-1}, b_{-n-1}). Let us define $2j_n := L_\Omega(\mu_n)$. Denote by Y_n the generalized Y-piece in Ω bounded by ω_n, μ_n and η_{-n}, and by Y_n' the generalized Y-piece in Ω bounded by μ_n, ω_{n+1}, and η_n We denote by Ω^* the *canonical twist* of Ω, defined as the train obtained from Ω by twisting angle π the generalized Y-pieces Y_n for every $n \ge 1$.

If Ω is a symmetric bitrain, then Ω^* is the train with parameters $l_{2k-1} := h_k$, $l_{2k} := j_k$, $r_{2k-1} := s_k$ and $r_{2k} := s_k$ for every $k > 0$.

Theorem 5.3. *Let us consider any symmetric bitrain Ω such that $s_n \le c_0$ for some $c_0 \ge 0$ and for every n, and $h_m \le h_n + c_1$ for some $c_1 \ge 0$ and for every $m \le n$. Then Ω is hyperbolic if and only if Ω^* is hyperbolic.*

Proof. First of all, let us note that the hypothesis $h_m \le h_n + c_1$ for every $m \le n$ implies that we have either $\sup_n h_n < \infty$ or $\lim_{n\to\infty} h_n = \infty$.

Assume first that $h_n \le c$ for every n and for some constant c. By Theorem 4.4, Ω is hyperbolic if and only if the train Ω_1 with parameters $l_n := h_n/2 \le c/2$ and $r_n := s_n$ is hyperbolic. Theorem 2.8 gives that Ω_1 is hyperbolic, and hence Ω is hyperbolic.

If we denote by $\{l_n^*\}_{n>0}$ and $\{r_n^*\}_{n\ge0}$ the parameters of the train Ω^*, then $l_{2k-1}^* = h_k \le c$, $l_{2k}^* = j_k$, $r_{2k-1}^* = s_k \le c_0$ and $r_{2k}^* = s_k \le c_0$ for every $k > 0$. Let us consider $X_n = Y_n \cup Y_n'$ in Ω, called a generalized X-piece, which is bounded by ω_n, ω_{n+1}, η_n and η_{-n}. Since these 4 outer loops of X_n have lengths less than or equal to $\max\{2c, 2c_0\}$, Bers' Theorem (see [39]) implies that there exists a simple closed geodesic i_n in X_n that is not an outer loop with $L_\Omega(i_n) \le c_*$, where c_* is a constant depending only on c and c_0. By the symmetry of X_n, we see that i_n and μ_n have the same length, and hence $2j_n = L_\Omega(\mu_n) \le c_*$. Consequently, $l_n^* \le \max\{c, c_*/2\}$ for every $n > 0$, and Theorem 2.8 gives that Ω^* is hyperbolic.

Assume now that $\lim_{n\to\infty} h_n = \infty$. By Theorem 4.4, Ω is hyperbolic if and only if the train Ω_1 with parameters $l_n := h_n/2$ and $r_n := s_n \le c_0$ is hyperbolic. Since $\lim_{n\to\infty} l_n = \infty$ and $l_m \le l_n + c_1/2$ for every $m \le n$, Theorem 2.8 gives that Ω_1 is hyperbolic if and only if

$$\sum_{k=n}^{\infty} e^{-h_k/2} \le c_2\, e^{-h_n/2}, \qquad \text{for every } n > 1 .$$

By Lemma 5.1 (with $p = 2$ and $p = 1/2$), this is equivalent to

$$\sum_{k=n}^{\infty} e^{-h_k} \le c_3\, e^{-h_n}, \qquad \text{for every } n > 1 . \tag{5.4}$$

If we denote by $\{l_n^*\}_{n>0}$ and $\{r_n^*\}_{n\ge0}$ the parameters of the train Ω^*, then $l_{2k-1}^* = h_k$, $l_{2k}^* = j_k$, $r_{2k-1}^* = s_k \le c_0$ and $r_{2k}^* = s_k \le c_0$ for every $k > 0$.

We claim now that $l_m^* \le l_n^* + c_4$ for every $m \le n$, and that (5.4) is equivalent to

$$\sum_{k=n}^{\infty} e^{-l_k^*} \le c_5\, e^{-l_n^*}, \qquad \text{for every } n > 1 . \tag{5.5}$$

Since $l_m^* \le l_n^* + c_4$ for every $m \le n$ and $\lim_{k\to\infty} l_{2k-1}^* = \infty$, we deduce that $\lim_{n\to\infty} l_n^* = \infty$. Then Theorem 2.8 gives that (5.5) is equivalent to the hyperbolicity of Ω^*.

We first prove the equivalence of (5.4) and (5.5). It is clear that (5.5) implies (5.4) with $c_3 = c_5$. Conversely, assume that (5.4) holds.

For each $n > 0$, let us consider the geodesic octagon O_n in $\Omega \cap \overline{\mathbb{H}}$ with sides $\omega_n \cap \overline{\mathbb{H}}$, $\omega_{n+1} \cap \overline{\mathbb{H}}$, $\eta_n \cap \overline{\mathbb{H}}$, $\eta_{-n} \cap \overline{\mathbb{H}}$, and the four segments joining their endpoints contained in (a_n, b_n), (a_{n+1}, b_{n+1}), (a_{-n}, b_{-n}) and (a_{-n-1}, b_{-n-1}). Since Ω is a symmetric bitrain, the shortest geodesic g_n in O_n joining ω_n and ω_{n+1} splits O_n into two isometric hexagons H_n and H_n'.

Since $\mu_n \cap \overline{\mathbb{H}}$ is the shortest geodesic in O_n joining (a_n, b_n) and (a_{-n-1}, b_{-n-1}), we have for every $n > 0$

$$\frac{h_n + h_{n+1}}{2} \le j_n \le \frac{h_n + h_{n+1}}{2} + L_\Omega(g_n) .$$

Hyperbolic trigonometry for the hexagon H_n (see e.g. [40, p. 161]) gives

$$\cosh L_\Omega(g_n) = \frac{\cosh s_n + \cosh(h_n/2)\cosh(h_{n+1}/2)}{\sinh(h_n/2)\sinh(h_{n+1}/2)} .$$

Since $\inf_n h_n > 0$ and $s_n \le c_2$ for every n, there exist a constant c_6 such that $L_\Omega(g_n) \le c_6$. Consequently, we have for every $n > 0$

$$\frac{h_n + h_{n+1}}{2} \le j_n \le \frac{h_n + h_{n+1}}{2} + c_6 .$$

Then, we also have $j_n \le h_{n+1} + c_6 + c_1/2$.

Therefore, for every $n > 1$,

$$\sum_{k=n}^{\infty} e^{-j_k} \le \sum_{k=n}^{\infty} e^{-(h_k+h_{k+1})/2} \le e^{c_1/2} \sum_{k=n}^{\infty} e^{-h_k} \le c_3\, e^{c_1/2} e^{-h_n}, \tag{5.6}$$

and hence

$$\sum_{k=n}^{\infty}\left(e^{-h_k}+e^{-j_k}\right)\le c_3\left(1+e^{c_1/2}\right)e^{-h_n}. \tag{5.7}$$

We also have

$$\sum_{k=n+1}^{\infty} e^{-h_k}\le c_3\, e^{-h_{n+1}}\le c_3\, e^{c_1/2}e^{-(h_n+h_{n+1})/2}\le c_3\, e^{c_6+c_1/2}e^{-j_n},$$

and using (5.6)

$$\sum_{k=n+1}^{\infty} e^{-j_k}\le c_3\, e^{c_1/2}e^{-h_{n+1}}\le c_3\, e^{c_1+c_6}e^{-j_n}.$$

Therefore,

$$e^{-j_n}+\sum_{k=n+1}^{\infty}\left(e^{-j_k}+e^{-h_k}\right)\le\left(1+2\,c_3\, e^{c_1+c_6}\right)e^{-j_n}. \tag{5.8}$$

Finally, using (5.7) and (5.8), we deduce (5.5) with $c_5=1+2\,c_3\, e^{c_1+c_6}$.

We then prove $l_m^*\le l_n^*+c_4$ for every $m\le n$. Fix $m\le n$. We have

$$\begin{aligned}
h_m &\le h_n+c_1\,,\\
h_m &= \frac{h_m-c_1+h_m-c_1}{2}+c_1\le\frac{h_n+h_{n+1}}{2}+c_1\le j_n+c_1\,,\\
j_m &\le \frac{h_m+h_{m+1}}{2}+c_6\le\frac{h_n+h_{n+1}}{2}+c_1+c_6\le j_n+c_1+c_6\,,\\
j_m &\le \frac{h_m+h_{m+1}}{2}+c_6\le\frac{h_{n+1}+h_{n+1}}{2}+c_1+c_6=h_{n+1}+c_1+c_6\,,
\end{aligned}$$

and thus conclude $l_m^*\le l_n^*+c_1+c_6$ for every $m\le n$. □

6. An example of infinite twists not preserving hyperbolicity

In this section, we give an example of infinite twists of a Riemann surface which do not preserve the Gromov hyperbolicity. First, we need a criterion which tells that a Riemann surface is not hyperbolic. The following theorem is a useful one.

Theorem 6.1. ([21, Theorem 5.2]) *Let us consider a non-exceptional Riemann surface S, and $X_n^1, X_n^2\subset S$ bordered surfaces such that $X_n^1\cap X_n^2=\partial X_n^1\cap\partial X_n^2=\eta_n^1\cup\eta_n^2$, and $d_{X_n^2}(\eta_n^1,\eta_n^2)\ge k_n$ for every n. If $\lim_{n\to\infty}k_n=\infty$, then S is not hyperbolic.*

Then we can construct our example from Theorems 2.12 and 6.1.

Theorem 6.2. *There exist a hyperbolic Riemann surface S and a non-hyperbolic Riemann surface S^0, such that S^0 can be obtained by twisting some Y-pieces of S.*

Proof. For each $n\ge 1$ let us consider two (isometric) Y-pieces Y_n and Y_n' such that the lengths of the three simple closed geodesics $\gamma_{n,1},\gamma_{n,2},\gamma_{n,3}\subset\partial Y_n$ (respectively, $\gamma_{n,1}',\gamma_{n,2}',\gamma_{n,3}'\subset\partial Y_n'$) are $L_{Y_n}(\gamma_{n,1})=L_{Y_n}(\gamma_{n,2})=2$ and $L_{Y_n}(\gamma_{n,3})=2n$ (respectively, $L_{Y_n'}(\gamma_{n,1}')=L_{Y_n'}(\gamma_{n,2}')=2$ and $L_{Y_n'}(\gamma_{n,3}')=2n$). Define the bordered Riemann surface Z_n as the surface obtained by pasting $\gamma_{n,3}$ and $\gamma_{n,3}'$ "in a symmetric way", which means that the shortest geodesic in Z_n joining $\gamma_{n,1}$ and $\gamma_{n,1}'$ meets orthogonally $\gamma_{n,3}=\gamma_{n,3}'$. Now, X_n is the bordered Riemann surface obtained by attaching to Z_n two funnels with boundary length 2, identifying the boundaries of the funnels with $\gamma_{n,2}$ and $\gamma_{n,2}'$.

Let us consider a sequence of (isometric) Y-pieces $\{Y_{0,n}\}_{n\ge1}$ such that the lengths of the three simple closed geodesics in $\partial Y_{0,n}$ are all 2. Define the bordered Riemann surface Z_0 as the surface obtained by pasting $\{Y_{0,n}\}_{n\ge1}$ in the following way: we identify a simple closed geodesic in $\partial Y_{0,n}$ with a simple closed geodesic in $\partial Y_{0,n+1}$ for every $n\ge 1$ in the symmetric way as above. Now, X_0 is the bordered Riemann surface obtained by attaching to Z_0 a funnel with boundary length 2, identifying the boundary of the funnel with a simple closed geodesic $\partial Z_0 \cap Y_{0,1}$.

We construct our Riemann surface S by identifying, for every $n \ge 1$, $\gamma_{n,1}$ with the simple closed geodesic $\partial X_0 \cap Y_{0,2n-1}$ and $\gamma'_{n,1}$ with the simple closed geodesic $\partial X_0 \cap Y_{0,2n}$.

We are going to prove that $\{X_n\}_{n\ge0}$ is a (c_1,c_2)-tree decomposition of S. Note first that $\eta_{n,m} = \varnothing$ for every $n,m \ge 1$, and $\eta_{n,0} = \gamma_{n,1} \cup \gamma'_{n,1}$ for every $n \ge 1$.

We have $d_{X_n}(\gamma_{n,1},\gamma'_{n,1}) = 2\, d_{Y_n}(\gamma_{n,1},\gamma_{n,3})$. Hyperbolic trigonometry (see e.g. [40, p. 161]) gives

$$\cosh d_{Y_n}(\gamma_{n,1},\gamma_{n,3}) = \frac{\cosh 1 + \cosh 1 \cosh n}{\sinh 1 \sinh n}\,.$$

Hence, there exists a constant c_0 such that $d_{X_n}(\gamma_{n,1},\gamma'_{n,1}) \le c_0$ for every $n \ge 1$. We also have $d_{X_0}(\gamma_{n,1},\gamma'_{n,1}) \le c_0$ for every $n \ge 1$. Consequently,

$$\operatorname{diam}_{X_n}(\eta_{n,0}) \le \operatorname{diam}_{X_n}(\gamma_{n,1}) + d_{X_n}(\gamma_{n,1},\gamma'_{n,1}) + \operatorname{diam}_{X_n}(\gamma_{n,1}) \le 2 + c_0 =: c_1$$

for every $n \ge 1$ and, in a similar way, $\operatorname{diam}_{X_0}(\eta_{n,0}) \le c_1$ for every $n \ge 1$. Furthermore,

$$d_{X_0}(\eta_{n,0},\eta_{m,0}) \ge d_{X_0}(\eta_{n,0},\eta_{n+1,0}) = 2\operatorname{Arccosh}\frac{\cosh 1 + \cosh^2 1}{\sinh^2 1} =: c_2$$

for every $m,n \ge 1$ with $m \ne n$. Then, we have proved that $\{X_n\}_{n\ge0}$ is a (c_1,c_2)-tree decomposition of S.

Note that, for every $n \ge 1$, the four outer loops of X_n have length 2. Therefore, by [27, Theorem 3.4] (see [21, Theorem 5.6] for further results), there exists a constant δ' such that X_n is δ'-hyperbolic for every $n \ge 1$. On the other hand, since X_0 is a train such that the lengths of the fundamental geodesics are bounded, it is hyperbolic by Theorem 2.8. Hence, Theorem 2.12 gives that S is hyperbolic.

Let us define S^0 as the Riemann surface obtained from S by twisting angle $\pi/2$ along the geodesic $\gamma'_{n,3}$ (with respect to $\gamma_{n,3}$) for every $n \ge 1$. Denote by p_n, q_n the points in $\gamma_{n,1}, \gamma'_{n,1}$ (respectively) satisfying

$$d_{X_n}(\gamma_{n,1},\gamma'_{n,1}) = d_{X_n}(p_n,q_n) = 2\operatorname{Arccosh}\frac{\cosh 1 + \cosh 1 \cosh n}{\sinh 1 \sinh n}\,.$$

For each $n \ge 1$, denote by X_n^0 the X-piece obtained from X_n by twisting angle $\pi/2$ along the geodesic $\gamma'_{n,3}$.

Triangle inequality gives that

$$d_{X_n^0}(p_n,q_n) \ge \frac{n}{2} - 2\operatorname{Arccosh}\frac{\cosh 1 + \cosh 1 \cosh n}{\sinh 1 \sinh n} \ge \frac{n}{2} - c_0$$

for every $n \ge 1$, and consequently,

$$d_{X_n^0}(\gamma_{n,1},\gamma'_{n,1}) \ge d_{X_n^0}(p_n,q_n) - 2 \ge \frac{n}{2} - c_0 - 2 =: k_n$$

for every $n \ge 1$.

For each $n \ge 1$, let us consider $X_n^2 := X_n^0$, and let X_n^1 be the closure of $S^0 \setminus X_n^2$. Define $\eta_n^1 := \gamma_{n,1}$ and $\eta_n^2 := \gamma'_{n,1}$. Since $\lim_{n\to\infty} k_n = \infty$, Theorem 6.1 gives that S^0 is not hyperbolic. $\square$

7. Problems on trains

In this section, we raise a problem asking whether certain twists on a train preserve its hyperbolicity or not. The train in question is constructed as follows.

Let Ω be a train such that the lengths of the fundamental geodesics are l_n for $n \ge 1$ and the lengths of the second fundamental geodesics satisfy $r_n = 0$ for $n \ge 1$. We assume that $\{l_n\}$ is an increasing sequence and diverges to ∞ as n tends to ∞. This is a tight train which is a union of generalized Y-pieces $\{Y_n\}_{n\ge1}$ and possibly a half-disk, where each Y_n has one puncture p_n and two geodesic boundary components γ_n and γ_{n+1}. By Theorem 2.8 (see also [18, Theorem 5.12]), we see that if $l_n = an$ for a constant $a > 0$, then Ω is hyperbolic.

We define a pentagon $Y_n^+ = Y_n \cap \Omega^+$ for each $n \ge 1$, where $\Omega^+ = \Omega \cap \overline{\mathbb{H}}$. Set $\gamma_n^+ = \gamma_n \cap Y_n^+$ and denote the two end points of γ_n^+ by q_n (the closer one to p_n) and o_n. We have two sides γ_n^+ and γ_{n+1}^+ of Y_n^+. The side of Y_n^+ in $\mathbb{R}$ opposite to the vertex p_n is denoted by λ_n. The vertices of Y_n^+ are p_n, q_n, o_n, o_{n+1} and q_{n+1}.

For the hyperbolic length $L_\Omega(\lambda_n)$ of the arc λ_n, we know from a formula on pentagon (cf. [40, Theorem 7.18.1]) that

$$\cosh L_\Omega(\lambda_n) = \frac{\cosh l_n \cosh l_{n+1} + 1}{\sinh l_n \sinh l_{n+1}} \,,$$

which gives that

$$1 + \frac{2}{\sinh^2 l_{n+1}} < \cosh L_\Omega(\lambda_n) < 1 + \frac{2}{\sinh^2 l_n}$$

for all n large enough. From this, we have an estimate $1/\sinh l_{n+1} < L_\Omega(\lambda_n) < 2/\sinh l_n$ for all sufficiently large n. Hence, if $l_n = an$, then the sum $\sum_{n=1}^\infty L_\Omega(\lambda_n)$ converges. This implies that a half-disk is necessary for the above decomposition of Ω. This condition is also equivalent to that the Denjoy domain Ω can be represented by the unit disk $\mathbb{D}$ from which a countably many points on $\mathbb{R}$ are removed.

The shortest geodesic arc in Y_n^+ from the vertex q_n to the side γ_{n+1}^+ is denoted by α_n and its other endpoint in γ_{n+1}^+ is denoted by q'_{n+1}. Also the subarc of γ_{n+1}^+ between q'_{n+1} and o_{n+1} is denoted by γ'_{n+1}. To estimate the length $L_\Omega(\alpha_n)$, we consider a quadrilateral with the sides α_n, γ_n^+, λ_n and γ'_{n+1}. Let ϕ_n be the angle at the vertex q_n in this quadrilateral. From hyperbolic trigonometry on quadrilateral (cf. [40, Theorem 7.17.1]), we obtain that

$$\cosh L_\Omega(\alpha_n) = \frac{\cosh L_\Omega(\lambda_n)}{\sin \phi_n} = \sqrt{1 + \cosh^2 l_n \sinh^2 L_\Omega(\lambda_n)} \,.$$

Here we have

$$\begin{aligned} \sinh^2 L_\Omega(\lambda_n) &= \cosh^2 L_\Omega(\lambda_n) - 1 \\ &= \frac{(\cosh l_n \cosh l_{n+1} + 1)^2 - (\cosh^2 l_n - 1)(\cosh^2 l_{n+1} - 1)}{\sinh^2 l_n \sinh^2 l_{n+1}} \\ &= \left(\frac{\cosh l_n + \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}}\right)^2 , \end{aligned}$$

and by using this equality we also have

$$1 + \left(\frac{\cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}}\right)^2 < 1 + \cosh^2 l_n \sinh^2 L_\Omega(\lambda_n) < 1 + 4\left(\frac{\cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}}\right)^2 .$$

Then $\cosh L_\Omega(\alpha_n) > \sqrt{2}$ for all $n \ge 1$ and $\lim_{n\to\infty} \cosh L_\Omega(\alpha_n) \le \sqrt{5}$. This implies that there are constants $c_1, c'_1 > 0$ such that $c_1 < L_\Omega(\alpha_n) < c'_1$ for all $n \ge 1$. Moreover, since $\lim_{n\to\infty} \cosh L_\Omega(\lambda_n) = 1$, we see that $\sin \phi_n$ is uniformly bounded away from 0 and $\pi/2$.

We also estimate the length of γ'_{n+1}. Again the formula for the quadrilateral gives that

$$\frac{\cosh l_n}{\cosh L_\Omega(\gamma'_{n+1})} = \frac{\cosh L_\Omega(\alpha_n)}{\cosh L_\Omega(\lambda_n)} \,.$$

Since $\sqrt{2} \le \lim_{n\to\infty} \cosh L_\Omega(\alpha_n) \le \sqrt{5}$, we see that the value of this ratio is uniformly bounded from above and below. This implies that there are constants $c_2, c'_2 > 0$ such that $c_2 < l_n - L_\Omega(\gamma'_{n+1}) < c'_2$ for all $n \ge 1$. In fact, for the two right-angled triangles made of γ_n^+, α_n and γ'_{n+1}, we see that l_n is asymptotically equal to $L_\Omega(\alpha_n) + L_\Omega(\gamma'_{n+1})$ as $n \to \infty$, and thus

$$\lim_{n\to\infty} \frac{l_n - L_\Omega(\gamma'_{n+1})}{L_\Omega(\alpha_n)} = 1 \,.$$

We give certain amount of twists along the fundamental geodesics of Ω. The twist along each γ_{n+1} for $n \ge 1$ is given in such a way that the vertex $q_{n+1} \in Y_{n+1}$ meets the point $q'_{n+1} \in Y_n^+$. Namely, the length of the twist along γ_{n+1} is $t_{n+1} = l_{n+1} - L_\Omega(\gamma'_{n+1})$. The resulting surface is denoted by Ω'.

The sequence of the simple closed geodesics $\{\gamma_n\}_{n\ge1}$ escapes from any compact subset of Ω'. Otherwise, it would converge to a geodesic line that is the boundary of a half-disk. To see this fact, we apply [41, Lemma 2]. Since $\sum_{n=1}^\infty L_{\Omega'}(\alpha_n) = \sum_{n=1}^\infty L_\Omega(\alpha_n) = \infty$, we can show that there is no such geodesic line as a limit of the sequence $\{\gamma_n\}$. In particular, for any fixed $n_0 \geq 1$, the distance $d_{\Omega'}(\gamma_n, \gamma_{n_0})$ between γ_n and γ_{n_0} grows to ∞ as $n \to \infty$. In the case where $l_n = an$ for some $a > 0$, we expect that $d_{\Omega'}(\gamma_n, \gamma_{n_0})$ grows linearly with respect to n (or faster than n^{ε} for some ε with $0 < \varepsilon < 1$). We hope that this would be verified by showing that $d_{\Omega'}(\gamma_n, \gamma_{n_0})$ is comparable with $\sum_{i=n_0}^{n-1} L_{\Omega'}(\alpha_i)$ for sufficiently large n.

Since there is no half-disk in Ω', we see that either Ω' is represented by $\mathbb{D}$ minus countably many points which are accumulated on the entire circle $\partial\mathbb{D}$ or Ω' is represented by $\mathbb{C}$ from which countably many points are removed.

Now we formulate our problem. This would give us another example showing that hyperbolicity is not preserved by twists if it were verified.

Conjecture 7.1. If $l_n = an$ for a constant $a > 0$, then the twisted surface Ω' is not hyperbolic, whereas the original tight train Ω is hyperbolic.

To see this claim, we may consider the order of divergence for non-asymptotic geodesic rays on Ω'. Furthermore, we may ask whether Ω' is represented by $\mathbb{C}$ from which countably many points are removed.

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