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The Petersson series vanishes at infinity

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Abstract. The Petersson series with respect to a simple closed geodesic c on a hyperbolic Riemann surface R is the relative Poincaré series of the canonical holomorphic quadratic differential on the annular cover of R and it defines a holomorphic quadratic differential $\varphi_c(z)dz^2$ on R. For the hyperbolic metric $\rho(z)|dz|$ on R, we give an upper estimate of $\rho^{-2}(z(p))|\varphi_c(z(p))|$ in terms of the hyperbolic length of c and the distance of $p \in R$ from c.

1. Introduction

Let Γ be a torsion-free Fuchsian group acting on a upper half-plane model $\mathbb{H} = \{\zeta = \xi + i\eta \mid \eta > 0\}$ of the hyperbolic plane. Throughout this paper, we always assume that a Riemann surface R is represented by $\mathbb{H}/\Gamma$. A holomorphic quadratic differential $\varphi(z)dz^2$ on R can be identified with a holomorphic function $\varphi(\zeta)$ on $\mathbb{H}$ that satisfies $\varphi(\gamma(\zeta)) = \varphi(\zeta)\gamma'(\zeta)^2$ for every $\gamma \in \Gamma$. We call such a holomorphic function $(2,0)$-*automorphic form* for Γ. A holomorphic $(2,0)$-automorphic form $\varphi(\zeta)$ is *integrable* if the integral of $|\varphi(\zeta)|$ over a fundamental domain of Γ is finite. This is equivalent to saying that the integral $\int_R |\varphi(z)|dxdy$ is finite. We denote the space of all integrable holomorphic $(2,0)$-automorphic form on $\mathbb{H}$ for Γ by $Q^1(\mathbb{H},\Gamma)$. This can be identified with the space of all integrable holomorphic quadratic differentials on R which is a complex Banach space with the norm $\|\varphi\|_1 = \int_R |\varphi(z)|dxdy$. If Γ is the trivial group 1, then $Q^1(\mathbb{H},1)$ is nothing but the Banach space of all integrable holomorphic functions on $\mathbb{H}$.

An integrable holomorphic $(2,0)$-automorphic form for Γ is produced from an integrable holomorphic function f by the *Poincaré series*

$$\Theta_\Gamma(f(\zeta)) = \sum_{\gamma\in\Gamma} f(\gamma(\zeta))\gamma'(\zeta)^2.$$

It is known that $\Theta_\Gamma : Q^1(\mathbb{H},1) \to Q^1(\mathbb{H},\Gamma)$ is a surjective bounded linear operator with the operator norm not greater than 1 for every Fuchsian group Γ. See Kra [**7**] for details on automorphic forms and the Poincaré series.

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Let $\rho(\zeta)=1/\mathrm{Im}\,\zeta$ be the hyperbolic density on $\mathbb{H}$. It induces the hyperbolic metric $\rho(z)|dz|$ on a Riemann surface $R=\mathbb{H}/\Gamma$. For a hyperbolic element $\gamma_c\in\Gamma$ corresponding to a simple closed geodesic c on the hyperbolic Riemann surface R, we consider the annulus $A=\mathbb{H}/\langle\gamma_c\rangle$ which covers R. We may assume that $\gamma_c(\zeta)=e^{\ell(c)}\zeta$ where $\ell(c)$ denotes the hyperbolic length of c.

For an integrable holomorphic $(2,0)$-automorphic form ϕ for $\langle\gamma_c\rangle$, the *relative Poincaré series*

$$\Theta_{\langle\gamma_c\rangle\backslash\Gamma}(\phi(\zeta))=\sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma}\phi(\gamma(\zeta)\gamma'(\zeta)^2$$

also defines an integrable holomorphic $(2,0)$-automorphic form for Γ. Here the sum is taken over all representatives of the cosets $\langle\gamma_c\rangle\backslash\Gamma$. Then $\Theta_{\langle\gamma_c\rangle\backslash\Gamma}:Q^1(\mathbb{H},\langle\gamma_c\rangle)\to Q^1(\mathbb{H},\Gamma)$ is also a surjective bounded linear operator with norm not greater than 1.

We choose $\phi(\zeta)=\zeta^{-2}$, which is an integrable holomorphic $(2,0)$-automorphic form for $\langle\gamma_c\rangle$. The polar coordinates $(l,t)\in\mathbb{R}_{>0}\times(0,\pi)$ for $\zeta=\exp(l+it)\in\mathbb{H}$ induce an euclidean metric $\sqrt{dl^2+dt^2}$ on the annulus $A=\mathbb{H}/\langle\gamma_c\rangle$ and this coincides with the euclidean metric $|d\zeta/\zeta|$ induced by the holomorphic quadratic differential on A corresponding to $\phi(\zeta)=\zeta^{-2}$. In particular, the area form $|\phi(\zeta)|d\xi d\eta$ is equal to $dldt$ and hence $\int_A|\phi(\zeta)|d\xi d\eta=\pi\ell(c)$. The relative Poincaré series

$$\varphi_c(\zeta)=\Theta_{\langle\gamma_c\rangle\backslash\Gamma}(\phi(\zeta))=\sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma}\frac{\gamma'(\zeta)^2}{\gamma(\zeta)^2}$$

is called the *Petersson series* with respect to c, which defines the holomorphic quadratic differential $\varphi_c(z)dz^2$ on R. The norm $\|\varphi_c\|_1$ is bounded by $\|\phi\|_1=\pi\ell(c)$. This plays an important role on the variation of the hyperbolic length $\ell(c)$ under a quasiconformal deformation of R (cf. Gardiner [**5**]) and the Weil-Petersson geometry on Teichmüller spaces (cf. Wolpert [**15**]).

For a quadratic differential $\varphi(z)dz^2$ on R, $\rho^{-2}(z(p))|\varphi(z(p)|$ is well-defined for $p\in R$ independent of a local parameter z around p and hence $\rho^{-2}|\varphi|$ gives a function on R. For the $(2,0)$-automorphic form $\varphi(\zeta)$ and for a point $\zeta\in\mathbb{H}$ over $p\in R$, the function $\rho^{-2}(\zeta)|\varphi(\zeta)|$ is the lift of $\rho^{-2}|\varphi|$ to the universal cover $\mathbb{H}$. We provide the supremum norm $\|\varphi\|_\infty=\sup_{\zeta\in\mathbb{H}}\rho^{-2}(\zeta)|\varphi(\zeta)|$ for a holomorphic $(2,0)$-automorphic form $\varphi(\zeta)$ for Γ (and for a holomorphic quadratic differential) and call it *bounded* if $\|\varphi\|_\infty$ is finite. The space of all bounded holomorphic $(2,0)$-automorphic forms for Γ is denoted by $Q^\infty(\mathbb{H},\Gamma)$. This is a complex Banach space with the norm $\|\varphi\|_\infty$.

In this paper, we will give an estimate of the function $\rho^{-2}|\varphi_c|$ of $p\in R$ for $\varphi_c(z)dz^2$ defined by the Petersson series with respect to a simple closed geodesic c on R in terms of the hyperbolic distance $d(p,c)$ of p from c. Our main theorem can be stated as follows.

THE MAIN THEOREM. *Let $\varphi_c(z)dz^2$ be a holomorphic quadratic differential on a hyperbolic Riemann surface R given by the Petersson series with respect to a simple closed geodesic c on R. Then, for a sufficiently small $r_0>0$, there is a positive constant B depending only on r_0 such that*

$$\rho(z(p))^{-2}|\varphi_c(z(p))|\le B\,\ell(c)e^{-d(p,c)/3}$$

for every $p\in R$ with $d(p,c)>r_0$ such that there is no closed curve based at p and freely homotopic to c with length less than $2r_0$. In particular, $\varphi_c(z)dz^2$ is bounded and it vanishes at infinity.

Here we say that a holomorphic quadratic differential $\varphi(z)dz^2$ on R *vanishes at infinity* if, for every $\varepsilon > 0$, there is a compact subset V of R such that $\sup_{p\in R-V}\rho(z(p))^{-2}|\varphi(z(p))| < \varepsilon$. The corresponding holomorphic $(2,0)$-automorphic form on $\mathbb{H}$ is called similarly. We denote the subspace of $Q^\infty(\mathbb{H},\Gamma)$ consisting of all holomorphic $(2,0)$-automorphic forms vanishing at infinity by $Q_0^\infty(\mathbb{H},\Gamma)$. This space has an importance in the theory of asymptotic Teichmüller spaces developed by Earle, Gardiner and Lakic (see [**6**] and [**3**]).

The assumption on the point $p \in R$ in the statement of the Main Theorem eliminates the case where c is very short and p is in a collar neighborhood of c. An estimate of $\rho(z(p))^{-2}|\varphi_c(z(p))|$ in this case has been given in [**9**].

We remark that, if the injectivity radii of R are uniformly bounded away from zero, then the conclusion of the Main Theorem easily follows from a basic estimate given in the next section. For a point p on R, the *injectivity radius* $r(p)$ is defined to be the radius of a maximal hyperbolic open disk centered at p that is embedded in R. However, the existence of a cusp does not make the problem difficult even if $r(p)$ tends to zero as p gets closer to a cusp; the essential problem occurs in the case where R has a sequence of simple closed geodesics whose lengths tend to zero.

Note that, it has been proved by Niebur and Sheingorn [**10**] that $Q^1(\mathbb{H},\Gamma)$ is contained in $Q^\infty(\mathbb{H},\Gamma)$ if and only if $R=\mathbb{H}/\Gamma$ has no such sequence of short simple closed geodesics whose lengths tend to zero. Moreover, it is shown in [**8**] that the operator norm of the inclusion map $Q^1(\mathbb{H},\Gamma)\hookrightarrow Q^\infty(\mathbb{H},\Gamma)$ is given in terms of the infimum of the lengths of simple closed geodesics on R (see also Sugawa [**14**]). On the other hand, when R has a sequence of simple closed geodesics whose lengths tend to zero, examples of integrable but not bounded holomorphic quadratic differentials have been constructed in Pommerenke [**12**] and Ohsawa [**11**] as well as in [**9**].

We further remark that, only to show that $\varphi_c(z)dz^2$ vanishes at infinity in the Main Theorem, there is a simpler argument. This can be done by transferring the Petersson series to the unit disk $\mathbb{D}$ by biholomorphic conjugation and relying on a technique due to Ahlfors [**1**]. These arguments as well as the density of $Q_0^\infty(\mathbb{H},\Gamma)$ in $Q^1(\mathbb{H},\Gamma)$ will be discussed in the last section.

2. Basic estimate

We will review an integral estimate of the hyperbolic supremum norm of a holomorphic function and apply it to the Poincaré series. This also shows that injectivity radius is the issue that we should manage.

Proposition 2.1. *Let $\varphi(z)dz^2$ be a holomorphic quadratic differential on a hyperbolic Riemann surface R, $r(p)$ the injectivity radius at $p\in R$ and $U(p,r(p))$ the hyperbolic disk of radius $r(p)$ centered at p. Then*

$$\rho^{-2}(z(p))|\varphi(z(p))| \le \frac{1}{4\pi\tanh^2(r(p)/2)}\int_{U(p,r(p))}|\varphi(z)|\,dxdy$$

for a local coordinate $z=x+iy$ around p.

Proof. By lifting $\varphi(z)dz^2$ to the unit disk $\mathbb{D}$, we have a holomorphic $(2,0)$-automorphic form $\varphi(\zeta)$ on $\mathbb{D}$. We may assume that $p\in R$ corresponds to the origin $0\in\mathbb{D}$, that is, $\zeta=\xi+i\eta$ gives a local coordinate such that $\zeta(p)=0$. Let

$\rho_{\mathbb{D}}(\zeta) = 2/(1-|\zeta|^2)$ denote the hyperbolic density on $\mathbb{D}$. Then

$$\rho^{-2}(z(p))|\varphi(z(p))| = \rho_{\mathbb{D}}^{-2}(0)|\varphi(0)| = \frac{|\varphi(0)|}{4}$$

and

$$\varphi(0) = \frac{1}{\pi a^2} \int_{|\zeta| \le a} \varphi(\zeta)\, d\xi d\eta,$$

where $U(p, r(p))$ lifts to the euclidean disk $\{|\zeta| \le a\}$ of radius $a = \tanh(r(p)/2)$. Hence

$$|\varphi(0)| \le \frac{1}{\pi a^2} \int_{|\zeta| \le a} |\varphi(\zeta)|\, d\xi d\eta = \frac{1}{\pi \tanh^2(r(p)/2)} \int_{U(p,r(p))} |\varphi(z)|\, dxdy,$$

which yields the desired inequality. □

It is well known that there is a constant $r_0 > 0$ (related to the Margulis constant) independent of the choice of a hyperbolic Riemann surface R such that if $r(p) < r_0$ then the disk neighborhood $U(p, r(p))$ of p is entirely contained either in the canonical cusp neighborhood or in the canonical collar of a short simple closed geodesic on R. Here the canonical cusp neighborhood is a horocyclic cusp neighborhood of hyperbolic area 2 and the canonical collar of a simple closed geodesic α is its neighborhood of width

$$\omega = \operatorname{arcsinh} \frac{1}{\sinh(\ell(\alpha)/2)}.$$

Note that, in this latter case, $\omega \ge r(p)$ and $2r(p) \ge \ell(\alpha)$ are satisfied. From these conditions, the upper bound of the hyperbolic length of α is known as $\ell(\alpha) \le 2 \operatorname{arcsinh} 1$.

Fix such a constant $r_0 > 0$. We define the cut-off injectivity radius at $p \in R$ as $\underline{r}(p) = \min\{r(p), r_0\}$. Then Proposition 2.1 implies that

$$\rho^{-2}(z(p))|\varphi(z(p))| \le \frac{r_0^2}{4\pi \tanh^2(r_0/2)\, \underline{r}(p)^2} \int_{U(p,\underline{r}(p))} |\varphi(z)|\, dxdy$$

for any holomorphic quadratic differential $\varphi(z)dz^2$ on R. We apply this formula for the quadratic differential $\varphi_c(z)dz^2$ on R induced by the Petersson series with respect to a simple closed geodesic c. By setting $b(r_0) = r_0^2/\{4\tanh^2(r_0/2)\}$, we have

$$\begin{aligned} \rho^{-2}(z(p))|\varphi_c(z(p))| &\le \frac{b(r_0)}{\pi \underline{r}(p)^2} \sum_{[\gamma] \in \langle \gamma_c \rangle \backslash \Gamma} \int_{U(\zeta(p),\underline{r}(p))} \frac{|\gamma'(\zeta)^2|}{|\gamma(\zeta)^2|}\, d\xi d\eta \\ &= \frac{b(r_0)}{\pi \underline{r}(p)^2} \sum_{[\gamma] \in \langle \gamma_c \rangle \backslash \Gamma} \int_{\gamma(U(\zeta(p),\underline{r}(p)))} \frac{1}{|\zeta^2|}\, d\xi d\eta. \end{aligned}$$

LEMMA 2.2. *For every p with $d(p,c) > r_0$,*

$$\rho^{-2}(z(p))|\varphi_c(z(p))| \le \frac{2e^{r_0} b(r_0)}{\underline{r}(p)^2} \ell(c)\, e^{-d(p,c)}$$

is satisfied.

PROOF. Since $d(p,c) > r_0 \geq \underline{r}(p)$, we see in the previous inequality that $\gamma(U(\zeta(p),\underline{r}(p)))$ are away from the imaginary axis by $d(p,c) - r_0$. This distance corresponds to the angle $t = \arctan(\sinh\{d(p,c) - r_0\})$ from the imaginary axis. Then

$$\sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma} \int_{\gamma(U(\zeta(p),\underline{r}(p)))} \frac{1}{|\zeta^2|}\, d\xi d\eta \leq \ell(c)[\pi - 2\arctan(\sinh\{d(p,c) - r_0\})].$$

Finally we use an inequality

$$\pi e^{-x} \leq \pi - 2\arctan(\sinh x) \leq 4e^{-x}$$

for $x \geq 0$ to obtain the required inequality. □

Suppose that the point $p \in R$ satisfies $r(p) \geq r_0$. Then, by $\underline{r}(p) = r_0$, Lemma 2.2 immediately shows that

$$\rho^{-2}(z(p))|\varphi_c(z(p))| \leq \frac{2e^{r_0} b(r_0)}{r_0^2} \ell(c)\, e^{-d(p,c)}.$$

Hence the Main Theorem is verified in this case.

Now we investigate the case where $r(p) < r_0$. Then $\underline{r}(p) = r(p)$ and p is either in the canonical cusp neighborhood or in the canonical collar. For the moment, suppose that p is in the canonical cusp neighborhood $\Omega \subset R$. Note that Ω is disjoint from c. We can represent Ω as a quotient space of $\{\zeta \in \mathbb{H} \mid \operatorname{Im}\zeta > 1/2\}$ by the parabolic element $\zeta \mapsto \zeta + 1$; we may assume that Γ contains this element. Then $\Omega = \{0 < |w| < e^{-\pi}\}$ by using the local parameter $w = \exp(2\pi i\zeta)$. Also the hyperbolic density is given by $\rho(w) = (-|w|\log|w|)^{-1}$. It is known that a larger punctured disk $\widetilde{\Omega} = \{0 < |w| < e^{-\pi/2}\}$ is also embedded in R (see Seppälä and Sorvali [**13**]).

PROPOSITION 2.3. *Let $\varphi(z)dz^2$ be an integrable holomorphic quadratic differential on R and p a point in the canonical cusp neighborhood $\Omega \subset R$ with the local parameter $w = \exp(2\pi i\zeta)$. Then*

$$\rho^{-2}(w(p))|\varphi(w(p))| \leq \frac{2e^{\pi}|w(p)|(\log|w(p)|)^2}{\pi}\|\varphi\|_1$$

is satisfied.

PROOF. It is easy to see that $\varphi(w)$ has at most a simple pole at the puncture $w = 0$. Hence $w\varphi(w)$ is a holomorphic function of $w = u + iv$ and satisfies

$$w(p)\varphi(w(p)) = \frac{1}{\pi a^2}\int_{|w-w(p)|\leq a} w\varphi(w)\, dudv$$

for $a = e^{-\pi}$. Then

$$\begin{aligned}\rho^{-2}(w(p))|\varphi(w(p))| &\leq \frac{|w(p)|(\log|w(p)|)^2}{\pi a^2}\int_{|w-w(p)|\leq a} |w||\varphi(w)|\, dudv \\ &\leq \frac{2|w(p)|(\log|w(p)|)^2}{\pi a}\int_R |\varphi(z)| dxdy,\end{aligned}$$

which is the desired inequality. □

Assume that $p \in \Omega$ is at distance $d \geq d(p,c)$ from the boundary $\partial\Omega$. Then $\operatorname{Im}\zeta(p) = e^d/2$ and hence

$$|w(p)| = \exp(-\pi e^d) \leq \exp(-\pi(1+d)).$$

Recall that the quadratic differential $\varphi_c(z)dz^2$ on R determined by the Petersson series satisfies $\|\varphi_c\|_1 \leq \pi\ell(c)$. From Proposition 2.3, we have

$$\rho^{-2}(w(p))|\varphi_c(w(p))| \leq 2\pi^2\ell(c)\exp(\pi + 2d - \pi(1+d)).$$

In particular,

$$\rho^{-2}(w(p))|\varphi_c(w(p))| \leq 2\pi^2\ell(c)e^{-d(p,c)},$$

which satisfies the condition of the Main Theorem. This means that we do not have to take care of the case where p with $r(p) < r_0$ is in the canonical cusp neighborhood.

3. Comparison of euclidean areas

In what follows, we investigate the case where the point p satisfying $r(p) < r_0$ is in the canonical collar of some short simple closed geodesic α. Recall that $\ell(\alpha) \leq 2\operatorname{arcsinh} 1$ is satisfied in this case. Since we assume in the Main Theorem that there is no closed curve based at p that is freely homotopic to c with its length less than $2r_0$, we know that α is distinct from c. Moreover, we see that α is disjoint from c. Indeed, if not, then every point of injectivity radius less than r_0 in the collar of α is within distance r_0 from c, but this violates the assumption $d(p,c) > r_0$.

Since we assume that Γ contains the element $\gamma_c(\zeta) = e^{\ell(c)}\zeta$ corresponding to c, every element $\gamma_\alpha \in \Gamma$ corresponding to a simple closed geodesic α different from c has the axis $\tilde{\alpha}$ in $\mathbb{H}$ whose end points are on the real axis $\mathbb{R}$. We take the neighborhood $\widetilde{C}(\tilde{\alpha})$ of $\tilde{\alpha}$ that is the lift of the canonical collar $C(\alpha)$ of α and consider a part of $\widetilde{C}(\tilde{\alpha})$ that contains the lifts of $U(p, r(p))$. In this section, we compare the euclidean areas of these regions as subsets of $\mathbb{R}^2$. To describe a signed distance from $\tilde{\alpha}$, we use an angle parameter $\theta \in (-\pi/2, \pi/2)$ representing the sector angle, which is given by $\theta = \arctan\sinh\omega$ for the signed distance ω from $\tilde{\alpha}$.

Proposition 3.1. *Let $\tilde{\alpha}$ be a hyperbolic geodesic line in $\mathbb{H}$ which is a semicircle of euclidean radius $h > 0$. Then the signed euclidean area of the one-sided neighborhood of $\tilde{\alpha}$ within angle $\theta \in (-\pi/2, \pi/2)$ is given by*

$$S(\theta) = h^2\left\{\frac{\pi}{2}\tan^2\theta + \theta\tan^2\theta + \theta + \tan\theta\right\}.$$

Here we assume that the one-sided neighborhood is outside the semicircle and its area is positive if $\theta > 0$ and it is inside the semicircle and its area is negative if $\theta < 0$.

Proof. We assume $\theta > 0$. The one-sided neighborhood of $\tilde{\alpha}$ in question is the crescent-shaped region in the euclidean disk D of radius $h/\cos\theta$ as in Figure 1. The area of the sector in D with angle $\pi + 2\theta$ is $(h/\cos\theta)^2(\pi + 2\theta)/2$ and the area of the triangle with base length $2h$ is $h^2\tan\theta$. Since $S(\theta)$ is the area of the chordal region in D over $\mathbb{R}$ minus the area $\pi h^2/2$ of the semi-disk of radius h, we have

$$S(\theta) = \left(\frac{h}{\cos\theta}\right)^2\left(\frac{\pi}{2} + \theta\right) + h^2\tan\theta - \frac{\pi h^2}{2}.$$

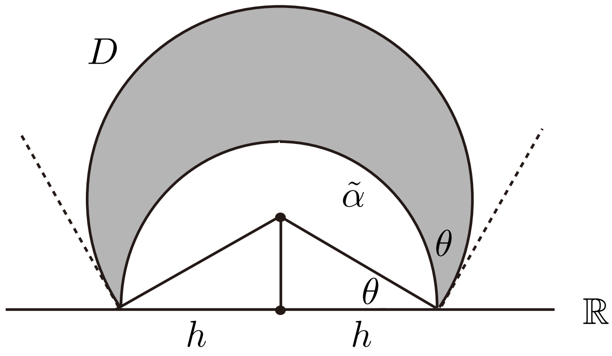

FIGURE 1. Crescent

This is equivalent to the required formula above. The case where $\theta < 0$ can be treated similarly and we obtain the same formula. □

An easy computation (omitted) also gives the derivative of $S(\theta)$ as follows.

PROPOSITION 3.2. *The derivative of the function $S(\theta)$ is given by*

$$S'(\theta) = \frac{h^2}{\cos^3\theta}\{(\pi + 2\theta)\sin\theta + 2\cos\theta\},$$

which satisfies

$$0 < S'(\theta) < \frac{2\pi h^2}{\cos^3\theta}$$

for $-\pi/2 < \theta < \pi/2$.

We are dealing with the case where $r(p) < r_0$ and $U(p, r(p))$ is contained in the canonical collar $C(\alpha)$ of some simple closed geodesic α of R. The width of $C(\alpha)$ is $\operatorname{arcsinh}(\sinh(\ell(\alpha)/2))^{-1}$, which is represented by an angle

$$\bar\theta = \arctan\frac{1}{\sinh(\ell(\alpha)/2)} > 0.$$

Then a connected component of the inverse image of $C(\alpha)$ under the universal cover $\mathbb{H} \to R$ is the two-sided neighborhood $\widetilde{C}(\tilde\alpha)$ of a geodesic line $\tilde\alpha$ within the angle $\bar\theta$. By Proposition 3.1, its euclidean area is given by

$$S(\bar\theta) - S(-\bar\theta) = 2h^2(\bar\theta\tan^2\bar\theta + \tan\bar\theta + \bar\theta),$$

where h is the euclidean radius of the semicircle $\tilde\alpha$. Here, we note that the condition $\ell(\alpha) \le 2\operatorname{arcsinh} 1$ is equivalent to $\bar\theta \ge \pi/4$. Then the euclidean area of $\widetilde{C}(\tilde\alpha)$ is estimated from below by

$$2h^2(\bar\theta\tan^2\bar\theta + \tan\bar\theta + \bar\theta) \ge 2h^2(\pi/4)\tan^2\bar\theta = \frac{\pi h^2}{2}\frac{1}{\sinh^2(\ell(\alpha)/2)}.$$

Assume that the point p is on the level curve of angle θ_0 in the collar $C(\alpha)$ and $U(p, r(p))$ is between θ_1 and θ_2 for $\theta_1 < \theta_0 < \theta_2$. Since $U(p, r(p))$ is contained in $C(\alpha)$, we have $-\bar\theta \le \theta_1$ and $\theta_2 \le \bar\theta$. Lifting $C(\alpha)$ to $\mathbb{H}$, we consider a subregion

$\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$ of $\widetilde{C}(\tilde{\alpha})$ between the angles θ_1 and θ_2 and estimate its euclidean area $S(\theta_2) - S(\theta_1)$ from above. By Proposition 3.2, we have

$$S(\theta_2) - S(\theta_1) = \int_{\theta_1}^{\theta_2} S'(\theta)\, d\theta \leq 2\pi h^2 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^3\theta}.$$

We assume that $\theta_0 \geq 0$ for the sake of simplicity. The case where $\theta_0 < 0$ can be treated similarly. Since $\cos\theta_1 \geq \cos\theta_2$ under this assumption, we have

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^3\theta} \leq \frac{1}{\cos^2\theta_2} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos\theta} = \frac{2r(p)}{\cos^2\theta_2}.$$

Here the last equality is a consequence from the following formula between the hyperbolic distance ω from the core geodesic α and the angle parameter θ:

$$\omega = \operatorname{arcsinh}(\tan\theta) = \int_0^{\theta} \frac{d\theta}{\cos\theta}.$$

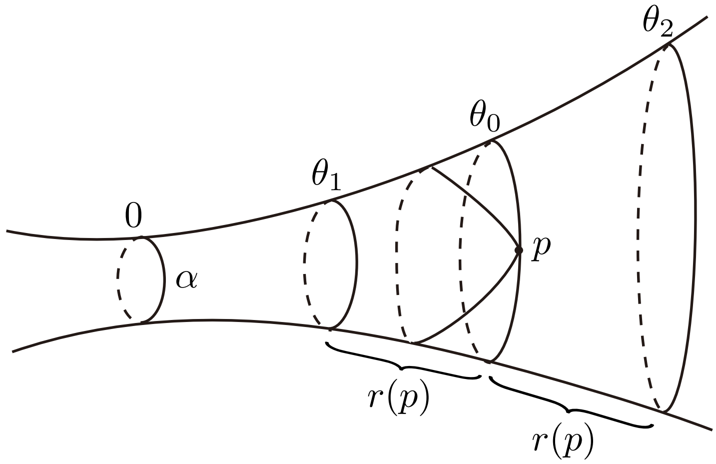

FIGURE 2. Level curves

To proceed the estimate of the area of $\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$, we use the following:

CLAIM. $\dfrac{1}{\cos\theta_2} \leq \dfrac{6r(p)}{\ell(\alpha)}$.

PROOF. Set $\theta_* = \max\{\theta_1, 0\}$. Then we have

$$\frac{\ell(\alpha)}{\cos\theta_2} \leq 2\int_{\theta_*}^{\theta_2} \frac{d\theta}{\cos\theta} + \frac{\ell(\alpha)}{\cos\theta_*}.$$

Indeed, consider a function

$$g(\theta) = 2\int_{\theta_*}^{\theta} \frac{d\theta}{\cos\theta} - \ell(\alpha)\left(\frac{1}{\cos\theta} - \frac{1}{\cos\theta_*}\right)$$

for $\theta_* \leq \theta \leq \bar{\theta}$. Then $g(\theta_*) = 0$ and $g'(\theta) = 2/\cos\theta - \ell(\alpha)\tan\theta/\cos\theta$. By using

$$\ell(\alpha)\tan\theta \leq \ell(\alpha)\tan\bar{\theta} = \frac{\ell(\alpha)}{\sinh(\ell(\alpha)/2)} \leq 2,$$

we have $g'(\theta) \geq 0$ and hence $g(\theta) \geq 0$. In particular, $g(\theta_2) \geq 0$, which yields the above inequality.

If $\theta_1 \geq 0$, then $\ell(\alpha)/\cos\theta_*$ is the length of the level curve of angle θ_1, which is bounded by $2r(p)$. Indeed, since $U(p, r(p))$ is located outside the level curve of angle θ_1, there is a length decreasing homeomorphism from the shortest closed curve of length $2r(p)$ based at p freely homotopic to α onto the level curve of angle θ_1. See Figure 2. If $\theta_1 \leq 0$, then $\ell(\alpha)/\cos\theta_* = \ell(\alpha)$, which is also bounded by $2r(p)$. Therefore we have

$$2\int_{\theta_*}^{\theta_2} \frac{d\theta}{\cos\theta} + \frac{\ell(\alpha)}{\cos\theta_*} \leq 2\int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos\theta} + 2r(p) = 6r(p),$$

from which the claimed inequality follows. □

As a consequence, we see that the euclidean area $S(\theta_2) - S(\theta_1)$ of $\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$ is bounded above by $144\,\pi h^2 r(p)^3/\ell(\alpha)^2$. Recall that we have already obtained the estimate of the euclidean area of $\widetilde{C}(\tilde{\alpha})$ from below.

Proposition 3.3. *The ratio of the euclidean area of the region $\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$ to the euclidean area of $\widetilde{C}(\tilde{\alpha})$ is bounded above by $288\,r(p)^3$ if $\ell(\alpha) \leq 2\,\mathrm{arcsinh}\,1$.*

Proof. The two estimates above yield

$$\frac{S(\theta_2) - S(\theta_1)}{S(\bar{\theta}) - S(-\bar{\theta})} \leq \frac{144\,\pi h^2 r(p)^3/\ell(\alpha)^2}{\pi h^2/\{2\sinh^2(\ell(\alpha)/2)\}} = 288\,r(p)^3 \left(\frac{\sinh(\ell(\alpha)/2)}{\ell(\alpha)}\right)^2.$$

If $\ell(\alpha) \leq 2\,\mathrm{arcsinh}\,1$, then $\sinh(\ell(\alpha)/2)/\ell(\alpha) \leq 1/(2\,\mathrm{arcsinh}\,1) < 1$. Hence the last term of the above inequality is bounded by $288\,r(p)^3$. □

4. Proof of the main theorem

Let $\gamma_c(\zeta) = e^{\ell(c)}\zeta$ and consider the annulus $A = \mathbb{H}/\langle\gamma_c\rangle$. The euclidean metric on A is the projection of the euclidean metric on the universal cover $\mathbb{H}$ defined by the polar coordinates (l, t) with $0 < l < \infty$ and $0 < t < \pi$ satisfying $\xi + i\eta = \exp(l + it)$ in $\mathbb{H}$. Then the Jacobian matrix of the coordinate change map $(l, t) \mapsto (\xi, \eta)$ is

$$\frac{\partial(\xi,\eta)}{\partial(l,t)} = \begin{pmatrix} e^l\cos t & -e^l\sin t \\ e^l\sin t & e^l\cos t \end{pmatrix},$$

and its determinant is $J(\zeta) = e^{2l} = |\zeta^2|$ for $\zeta = \xi + i\eta \in \mathbb{H}$. This shows that $d\xi d\eta/|\zeta^2| = dldt$.

By Proposition 3.3, we have an estimate of the ratio of areas of $\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$ and $\widetilde{C}(\tilde{\alpha})$ measured by the euclidean area element $d\xi d\eta$. Next we consider the ratio of areas of their projections onto the annulus A which are measured by the euclidean area element $dldt$. Since the Jacobian is $|\zeta^2|$, we have only to look at the minimal and maximal distances m and M of $\widetilde{C}(\tilde{\alpha})$ from the origin 0. Since $d(p, c) > r_0 > r(p)$, the simple closed geodesic α is disjoint from c. This implies that the neighborhood $\widetilde{C}(\tilde{\alpha})$ of the geodesic line $\tilde{\alpha}$ is disjoint from the imaginary axis in $\mathbb{H}$. Note also that the angle of $\widetilde{C}(\tilde{\alpha})$ is not less than $\pi/4$. Then Figure 3 illustrates the extremal situation where the ratio M/m should be the largest, and an elementary geometric calculus gives that $M/m = (\sqrt{3}+\sqrt{2})^2$ in this case. From this observation, we see that

$$\frac{\max_{\zeta\in\widetilde{C}(\tilde{\alpha})} J(\zeta)}{\min_{\zeta\in\widetilde{C}(\tilde{\alpha})} J(\zeta)} \leq (\sqrt{3}+\sqrt{2})^4.$$

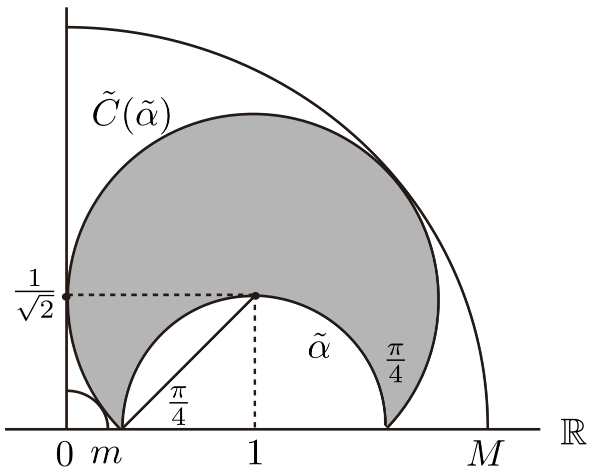

FIGURE 3. Extremal situation

Merging the above arguments into Proposition 3.3, we summarize a claim for proving the Main Theorem.

LEMMA 4.1. *The ratio of the area of* $\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha})$ *to the area of* $\widetilde{C}(\tilde{\alpha})$ *measured by the euclidean metric with respect to the polar coordinates* (l,t) *is bounded above by* $Kr(p)^3$ *for* $K = 288\,(\sqrt{3}+\sqrt{2})^4$ *if* $\alpha \cap c = \emptyset$ *and if the width of* $\widetilde{C}(\tilde{\alpha})$ *measured by the angle* θ *is not less than* $\pi/4$.

Now we are ready to complete our arguments.

Proof of the Main Theorem. We have only to consider the case where $r(p) < r_0$ and p is in the canonical collar of some short simple closed geodesic α. The other cases have been already discussed in Section 2. By Lemma 4.1, we have

$$\begin{aligned}
\sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma} \int_{\gamma(U(\zeta(p),r(p)))} \frac{1}{|\zeta^2|}\,d\xi d\eta &\leq \sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma} \int_{\gamma(\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha}))} \frac{1}{|\zeta^2|}\,d\xi d\eta \\
&= \sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma} \mathrm{Area}(\gamma(\widetilde{C}_{[\theta_1,\theta_2]}(\tilde{\alpha}))) \\
&\leq Kr(p)^3 \sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma} \mathrm{Area}(\gamma(\widetilde{C}(\tilde{\alpha}))) \\
&\leq Kr(p)^3\,\mathrm{Area}(A) = K\pi\ell(c)r(p)^3.
\end{aligned}$$

This yields one inequality

$$\begin{aligned}\rho^{-2}(z(p))|\varphi_c(z(p))| &\leq \frac{b(r_0)}{\pi r(p)^2}\sum_{[\gamma]\in\langle\gamma_c\rangle\backslash\Gamma}\int_{\gamma(U(\zeta(p),r(p)))}\frac{1}{|\zeta^2|}\,d\xi d\eta \\ &\leq Kb(r_0)\ell(c)r(p).\end{aligned}$$

On the other hand, Lemma 2.2 gives another inequality

$$\rho^{-2}(z(p))|\varphi_c(z(p))| \leq 2e^{r_0}b(r_0)\ell(c)e^{-d(p,c)}r(p)^{-2}.$$

We have obtained two estimates as

$$\rho^{-2}(z(p))|\varphi_c(z(p))| \leq \begin{cases} Kb(r_0)\ell(c)r(p) \\ 2e^{r_0}b(r_0)\ell(c)e^{-d(p,c)}r(p)^{-2}. \end{cases}$$

Now we consider the maximum of the smaller one of these values when $r(p)$ varies in $(0, r_0]$:

$$\begin{aligned}&\max_{r(p)\in(0,r_0]}\min\left\{Kr(p), 2e^{r_0}e^{-d(p,c)}r(p)^{-2}\right\}b(r_0)\ell(c) \\ &\leq K^{2/3}(2e^{r_0})^{1/3}b(r_0)\ell(c)e^{-d(p,c)/3}.\end{aligned}$$

This eliminates $r(p)$ from the formula. By setting $B = K^{2/3}(2e^{r_0})^{1/3}b(r_0)$, we have

$$\rho^{-2}(z(p))|\varphi_c(z(p))| \leq B\ell(c)e^{-d(p,c)/3},$$

which completes the proof of the Main Theorem. □

5. Application to the variation of length functions

For a Beltrami differential $\mu = \mu(z)d\bar{z}/dz$ on a hyperbolic Riemann surface R, consider a quasiconformal deformation R_μ of R given by μ and denote the geodesic length of the free homotopy class of c on R_μ by $\ell_\mu(c)$. Then a variational formula due to Gardiner [**5**] asserts that

$$\left.\frac{d\ell_{t\mu}(c)}{dt}\right|_{t=0} = \frac{2}{\pi}\,\mathrm{Re}\int_R \mu(z)\varphi_c(z)\,dxdy.$$

The Main Theorem can be applied to an estimate of the derivative $d\ell_{t\mu}(c)/dt|_{t=0}$ through this formula.

We say that a Beltrami differential $\mu(z)d\bar{z}/dz$ on R vanishes at infinity if, for every $\varepsilon > 0$, there exists a compact subset V of R such that $|\mu(z(p))| < \varepsilon$ for almost every $p \in R - V$. A quasiconformal homeomorphism f of R whose complex dilatation is a Beltrami differential vanishing at infinity is called *asymptotically conformal*.

THEOREM 5.1. *Let $\mu(z)d\bar{z}/dz$ be a Beltrami differential on a hyperbolic Riemann surface R that vanishes at infinity. Let $\{c_n\}_{n=1}^\infty$ be a sequence of simple closed geodesics on R escaping to the infinity. Then*

$$\frac{1}{\ell(c_n)}\cdot\left.\frac{d\ell_{t\mu}(c_n)}{dt}\right|_{t=0} \longrightarrow 0$$

as $n \to \infty$.

PROOF. For arbitrary $\varepsilon > 0$, we take a compact subset V of R such that $|\mu(z(p))| < \varepsilon$ for almost every $p \in R - V$. Let Area(V) be the hyperbolic area of V and $d(V, c_n)$ the hyperbolic distance between V and c_n. Then, by using the Main Theorem for the integral on V, we have

$$\begin{aligned}\int_R |\mu(z)\varphi_{c_n}(z)|\,dxdy &= \int_{R-V} |\mu(z)\varphi_{c_n}(z)|\,dxdy + \int_V |\mu(z)\varphi_{c_n}(z)|\,dxdy \\ &< \varepsilon\|\varphi_{c_n}\|_1 + \mathrm{Area}(V)\,\|\mu\|_\infty B\ell(c_n)e^{-d(V,c_n)/3} \\ &\leq \ell(c_n)\{\varepsilon\pi + \mathrm{Area}(V)\,Be^{-d(V,c_n)/3}\}.\end{aligned}$$

Since $d(V, c_n) \to \infty$ as $n \to \infty$, this inequality shows that

$$\frac{1}{\ell(c_n)} \int_R |\mu(z)\varphi_{c_n}(z)|\,dxdy \to 0$$

as $n \to \infty$. Then the Gardiner variation formula yields the statement of the theorem. □

Note that it has been shown by Earle, Markovic and Saric [**3**] that an asymptotically conformal homeomorphism f of R with the complex dilatation $\mu(z)d\bar{z}/dz$ has an asymptotically isometric homeomorphism in its homotopy class. In particular, the ratios $\ell_\mu(c_n)/\ell(c_n)$ for a sequence of simple closed geodesics $\{c_n\}_{n=1}^\infty$ escaping to the infinity tend to 1 as $n \to \infty$. See also [**4**]. Theorem 5.1 can be regarded as an infinitesimal version of this property.

6. Remarks on vanishing at infinity

It was noticed by Drasin and Earle [**2**] that, for an arbitrary Fuchsian group Γ, the Banach space $Q^1(\mathbb{H}, \Gamma)$ of the integrable holomorphic $(2,0)$-automorphic forms has a dense linear subspace consisting of bounded holomorphic $(2,0)$-automorphic forms in $Q^\infty(\mathbb{H}, \Gamma)$. Actually, this claim was given for holomorphic $(2,0)$-automorphic forms for a Fuchsian group G on the unit disk $\mathbb{D}$ by using the fact that polynomials $\{f(z)\}$ are dense in the Banach space $Q^1(\mathbb{D}, 1)$ of all integrable holomorphic functions on $\mathbb{D}$. Then the surjectivity of the Poincaré series operator $\Theta_G : Q^1(\mathbb{D}, 1) \to Q^1(\mathbb{D}, G)$ yields that $\{\Theta_G(f(z))\}$ are dense in $Q^1(\mathbb{D}, G)$. Also, the technique introduced by Ahlfors [**1**] proves that $\Theta_G(z^n)$ for all $n \geq 0$ are bounded holomorphic $(2,0)$-automorphic forms in $Q^\infty(\mathbb{D}, G)$.

In fact, Ahlfors' argument further shows that $\Theta_G(z^n)$ are vanishing at infinity, namely, they belong to $Q_0^\infty(\mathbb{D}, G)$. We will explain this method below. Then, after the conjugation to the upper half-plane $\mathbb{H}$, we can summarize the result as follows.

PROPOSITION 6.1. *For every Fuchsian group Γ, $Q_0^\infty(\mathbb{H}, \Gamma) \cap Q^1(\mathbb{H}, \Gamma)$ is dense in the Banach space $Q^1(\mathbb{H}, \Gamma)$ with the integrable norm.*

For a Fuchsian group G acting on $\mathbb{D}$, we consider

$$J(z) = \rho_{\mathbb{D}}^{-2}(z) \sum_{g \in G} |g'(z)|^2 = \frac{1}{4} \sum_{g \in G} (1 - |g(z)|^2)^2,$$

where $\rho_{\mathbb{D}}(z) = 2/(1 - |z|^2)$ is the hyperbolic density on $\mathbb{D}$. Then, as in [**1**], $J(z)$ is a subharmonic function outside the images of a certain disk under G. Also this is an automorphic function for G and thus regarded as a function on the Riemann surface $R = \mathbb{D}/G$. Because of the subharmonicity, the function J on R vanishes at infinity. See [**9**]. However, this method does not always tell the order of its decay in terms of the hyperbolic distance.

Let $f(z)$ be an integrable holomorphic function on $\mathbb{D}$ with $|f(z)| \leq M$ for some positive constant M. Its Poincaré series satisfies

$$\rho_{\mathbb{D}}^{-2}(z)|\Theta_G(f(z))| \leq MJ(z).$$

We apply this estimate for $f(z) = z^n$. Then we see that $\Theta_G(z^n)$ vanishes at infinity.

In addition, we look at the holomorphic $(2,0)$-automorphic form $\phi(\zeta) = 1/\zeta^2$ for $\langle \gamma_c \rangle$ on $\mathbb{H}$, where $\gamma_c(\zeta) = e^{\ell(c)}\zeta$ is a hyperbolic element of a Fuchsian group Γ. By a biholomorphic map $\mathbb{D} \to \mathbb{H}$, we pull back ϕ to $\mathbb{D}$, which we denote by $\tilde{\phi}(z)$. This also gives the conjugation of Γ with γ_c to a Fuchsian group G with the corresponding element g_c acting on $\mathbb{D}$. We can verify that there is a positive constant L depending on $\ell(c)$ such that

$$|\tilde{\phi}(z)| \leq L\ell(c) \sum_{n \in \mathbb{Z}} |(g_c^n)'(z)|^2.$$

See [**9**]. Then

$$\rho_{\mathbb{D}}^{-2}(z)|\Theta_{\langle g_c \rangle \backslash G}(\tilde{\phi}(z))| \leq L\ell(c)J(z).$$

This implies that the pull-back $\Theta_{\langle g_c \rangle \backslash G}(\tilde{\phi})$ of the Petersson series vanishes at infinity and so does the Petersson series $\varphi_c = \Theta_{\langle \gamma_c \rangle \backslash \Gamma}(\phi)$. Further arguments are necessary to obtain a quantitative estimate of the decay order for φ_c.

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