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No proper conjugation for quasiconvex cocompact groups of Gromov hyperbolic spaces

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Abstract. We prove that, if a quasiconvex cocompact subgroup of the isometry group of a Gromov hyperbolic space has a conjugation into itself, then it is onto itself.

1. Introduction

Let G be an abstract group and $\theta : G \to G$ an injective homomorphism of G into itself. There have been various studies on the conditions under which θ is an automorphism of G, which is the so called co-Hopf problem. A variant of this problem can be formulated by restricting θ to a conjugation in an ambient group H containing G.

Definition 1.1. Let H be an arbitrary abstract group. For a subgroup G of H and an element α of H, if $\alpha G \alpha^{-1}$ is strictly contained in G, we say that G has a *proper conjugation* in H by α.

Here are some examples of groups which admit proper conjugation.

Example 1.2. (1) Let H be the Baumslag-Solitar group $B(m,n)$ for $m,n \in \mathbb{N}$:

$$B(m,n) = \langle g, h \mid g^m h = h g^n \rangle.$$

For instance, we consider $H = B(2,1)$ and its subgroup $G = \langle g \rangle$. Then $hGh^{-1} = \langle g^2 \rangle \subsetneqq G$, which means that G has a proper conjugation in H by h.

(2) Let H be the free group $F_2 = \langle \alpha, \beta \rangle$ of rank 2. We take a subgroup G generated by infinitely many elements:

$$G = \langle \alpha^n \beta \alpha^{-n} \rangle_{n \geq 0}.$$

Then $\alpha G \alpha^{-1} = \langle \alpha^n \beta \alpha^{-n} \rangle_{n \geq 1}$ does not contain β. Since $\beta \in G$, we see that G has a proper conjugation in H by α.

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The problem of proper conjugation has been studied for several special cases, in particular for the case where $H = \mathrm{Isom}(\mathbb{H}^{n+1})$ is the group of all orientation-preserving isometries of the hyperbolic space $\mathbb{H}^{n+1}$ of dimension $n+1$ and G is a discrete subgroup of H which is also known as a Kleinian group. We can refer a history of this problem to Ohshika and Potyagailo [**13**]. Examples of proper conjugation of Kleinian groups were given in Jørgensen, Marden and Pommerenke [**8**].

Our previous theorem in [**11**] gave the following necessary condition for G to have a proper conjugation. Note that this can be regarded as a generalization of the theorem by Heins [**4**] in the case of $n = 1$.

Theorem 1.3. *If a Kleinian group $G \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ is of divergence type, then G admits no proper conjugation in* $\mathrm{Isom}(\mathbb{H}^{n+1})$.

The divergence type means that the Poincaré series of G diverges at the critical exponent. If G is convex cocompact, or more generally geometrically finite, then G is of divergence type (Sullivan [**20**]). The proof utilized the Patterson-Sullivan measure on the limit set of G.

Here we mention the relationship between the divergence type condition and certain conditions known to be avoiding proper conjugation. A geometrically finite Kleinian group $G \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ has no proper conjugation, which was shown by Wang and Zhou [**23**]. Theorem 1.3 extends this result. In the case of $n = 2$, if $G \subset \mathrm{Isom}(\mathbb{H}^3)$ is topologically tame (which is equivalent to being finitely generated by the solution of the tameness conjecture), then G is of divergence type. Ohshika and Potyagailo [**13**] proved that a geometrically tame Kleinian group has no proper conjugation, which is also in the scope of Theorem 1.3. In arbitrary dimension n, it was also proved in [**13**] that, if a Kleinian group $G \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ is isomorphic to a geometrically finite group and does not split over virtually abelian subgroups, then G does not admit proper conjugation. We do not know whether such a Kleinian group G is always of divergence type or not. It is possible to construct infinitely generated Kleinian groups of divergence type. For example, a normal subgroup of a convex cocompact Kleinian group of infinite cyclic quotient is of divergence type by Rees [**16**]. However, when $n \geq 3$, we have no example of a finitely generated but geometrically infinite Kleinian group of divergence type.

In this paper, we will give a certain generalization of these proper conjugation theorems to isometry groups of *Gromov hyperbolic spaces*. Let $X = (X, d)$ be a δ-hyperbolic geodesic metric space for some $\delta \geq 0$ with distance d. Moreover, we assume that X is proper, in other words, X is complete and locally compact. When we call X a Gromov hyperbolic space, we assume that X satisfies all the above requirements. The group of all isometric automorphisms of X is denoted by $\mathrm{Isom}(X)$. We consider the problem on proper conjugation for a subgroup G in $H = \mathrm{Isom}(X)$ that acts properly discontinuously on X.

The limit set $\Lambda(G)$ of G is the set of all accumulation points of the orbit of G, which is in the boundary ∂X of X. The *hull* $Q(\Lambda(G))$ for the limit set is the union of all geodesic lines connecting any two points of $\Lambda(G)$. This is a G-invariant closed quasiconvex subset of X. In general, a subset A of X is called *quasiconvex* if any geodesic segment connecting any two points in A is within a uniformly bounded distance of A. If the quotient $Q(\Lambda(G))/G$ is non-empty and compact, we say that G is *quasiconvex cocompact* or more precisely G acts quasiconvex cocompactly on X. This is equivalent to the condition that the orbit $G(x)$ of some point x is quasiconvex in X. See Swenson [**21**] for this and other equivalent conditions for quasiconvex cocompactness.

As in the case of Kleinian groups, we can think of the critical exponent of the Poincaré series for G, which will be defined later. Then our main theorem is stated as follows.

THEOREM 1.4. *If* $G \subset \mathrm{Isom}(X)$ *is quasiconvex cocompact with finite critical exponent, then* G *has no proper conjugation in* $\mathrm{Isom}(X)$.

In the case where $\mathrm{Isom}(X)$ acts on X properly discontinuously or the conjugation of G is restricted in some subgroup $H \subset \mathrm{Isom}(X)$ that acts properly discontinuously, the statement of Theorem 1.4 has been already given by Ranjbar-Motlagh [**15**]. In this direction, Yang [**24**] recently proved that, if H is a discrete convergence group acting on some compact metrizable space and G is a non-parabolic dynamically quasiconvex subgroup of H, then G has no proper conjugation in H.

In Theorem 1.3, we have only to assume that the group $G \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ is of divergence type whereas $G \subset \mathrm{Isom}(X)$ need to be quasiconvex cocompact in Theorem 1.4. We want to extend Theorem 1.4 to the case where G is of divergence type, but it is not so easy to formulate uniqueness of Patterson-Sullivan measure in this setting. This is continued to our ongoing research.

Instead of pursuing the general result, we put a further assumption on X to ensure the uniqueness. For instance, if X is a tree, in other words, X is 0-hyperbolic, then the Patterson-Sullivan measure is unique for a divergence type group G, which has been proved by Coornaert [**3**]. More generally, when X is a $\mathrm{CAT}(-1)$ space, this property is also satisfied as in Burger and Mozes [**1**] and Roblin [**17**]. Then we have the following claim.

THEOREM 1.5. *For a* $\mathrm{CAT}(-1)$ *space* X*, if* $G \subset \mathrm{Isom}(X)$ *is uniformly properly discontinuous and is of divergence type, then* G *has no proper conjugation.*

Here, we say that G acts on X *uniformly properly discontinuously* if there are $r > 0$ and $N < \infty$ such that the number of elements $g \in G$ satisfying $g(U(x,r)) \cap U(x,r) \neq \emptyset$ is bounded by N for every $x \in X$. Here $U(x,r)$ denotes the open ball of radius r centered at x. This assumption is necessary to ensure that the geometric limit Γ_∞ defined in Section 4 acts properly discontinuously on X. Once we have Γ_∞ with desired properties, then the proof of Theorem 1.5 can be obtained by almost line-by-line replication of the arguments for the proof of Theorem 1.3 in [**11**]. We will omit the detail.

2. The action of isometry groups

A geodesic metric space (X,d) is called *δ-hyperbolic* for $\delta \geq 0$ if, for every geodesic triangle (α,β,γ) in X, any edge, say α is contained in the closed δ-neighborhood of the union $\beta \cup \gamma$ of the other edges. Let (X,d) be a *proper* δ-hyperbolic geodesic metric space for some $\delta \geq 0$ with a fixed base point $x_0 \in X$. We consider geodesic rays $\sigma : [0,\infty) \to X$ starting from $x_0 = \sigma(0)$ and regard σ_1 and σ_2 asymptotically equivalent if there is some constant $K < \infty$ such that $d(\sigma_1(t),\sigma_2(t)) \leq K$ for all $t \geq 0$. Then the space of all geodesic rays based at x_0 modulo the asymptotic equivalence defines a boundary ∂X of X, which gives the compactification $\overline{X} = X \cup \partial X$ by providing the compact-open topology on the space of geodesic rays. We see that $\overline{X}$ is a compact Hausdorff space satisfying the second countability axiom. Let $\mathrm{Isom}(X)$ denote the group of all isometric automorphisms of X. Then every element γ of $\mathrm{Isom}(X)$ extends to a self-homeomorphism of $\overline{X}$.

For a subgroup $G \subset \mathrm{Isom}(X)$ acting on X properly discontinuously, we define the *limit set* $\Lambda(G)$ of G as the set of all accumulation points of the orbit $G(x_0)$ in $\overline{X}$. Then $\Lambda(G)$ is a G-invariant closed subset in ∂X. If $\#\Lambda(G) \geq 3$, then we say that G is *non-elementary.* Let $\Omega(G)$ denote the complement of the limit set $\Lambda(G)$ in ∂X, which we call the *region of discontinuity* of G.

The isometry group $\mathrm{Isom}(X)$ acts on $\overline{X}$ as a *convergence group.* This has been shown by Tukia [**22**]. Note that $\overline{X}$ is a compact metrizable space. Then, by the convergence property, we see that a subgroup $G \subset \mathrm{Isom}(X)$ acting on X properly discontinuously satisfies similar properties to Kleinian groups concerning the limit set and the region of discontinuity. For instance, if G is non-elementary, then $\Lambda(G)$ is the smallest G-invariant closed subset of $\overline{X}$. As another nature, we see the following, which has been also shown by Coornaert [**2**] in a different way.

PROPOSITION 2.1. *If $G \subset \mathrm{Isom}(X)$ acts on X properly discontinuously, then it also acts on $X \cup \Omega(G)$ properly discontinuously.*

For a subgroup $G \subset \mathrm{Isom}(X)$ acting on X properly discontinuously and for a point x_0, we define a *Dirichlet domain* as

$$D_G(x_0) = \{x \in X \mid d(x, x_0) \leq d(x, gx_0) \text{ for all } g \in G\}.$$

In the case where no element of G fixes x_0, in other words, the stabilizer subgroup $\mathrm{Stab}(x_0)$ in G is trivial, $D_G(x_0)$ is a fundamental domain, but in general, $D_G(x_0)$ is the union of the images of a fundamental domain by $\mathrm{Stab}(x_0)$. Actually, $D_G(x_0) = \bigcap_{g \in G - \mathrm{Stab}(x_0)} D_g(x_0)$ for

$$D_g(x_0) = \{x \in X \mid d(x, x_0) \leq d(x, gx_0)\}.$$

Let G be a subgroup of $\mathrm{Isom}(X)$ that acts on X properly discontinuously. The *hull* $Q(\Lambda(G))$ of the limit set of G is the union of all geodesic lines connecting any two points in $\Lambda(G)$, which is a G-invariant quasiconvex closed set in X. If the quotient space $Q(\Lambda(G))/G$ is non-empty and compact, then G is called *quasiconvex cocompact.*

We use the following property of the Dirichlet domain for a quasiconvex cocompact group. This follows from a characterization of quasiconvexity by Swenson [**21**], but we give a rather direct proof for it here.

LEMMA 2.2. *If $G \subset \mathrm{Isom}(X)$ is quasiconvex cocompact, then the closure $\overline{D_G(x_0)} \subset \overline{X}$ of the Dirichlet domain $D_G(x_0) \subset X$ does not intersect the limit set $\Lambda(G)$.*

PROOF. Assume that there is a point ξ in $\overline{D_G(x_0)} \cap \Lambda(G)$. We choose a sequence $\{x_n\} \subset D_G(x_0)$ that converges to ξ. Then there is a geodesic segment $[x_0, x_n]$ for each $n \in \mathbb{N}$ such that $[x_0, x_n]$ converge to a geodesic ray $[x_0, \xi)$ as $n \to \infty$. Since G is quasiconvex cocompact, there is a constant $L < \infty$ such that, for every point y on $[x_0, \xi)$, there exists an element $g_y \in G$ with $d(y, g_y x_0) \leq L$. We choose $y \in [x_0, \xi)$ so that $d(y, x_0) > L + 2\delta$. Here we use the following fact.

Claim: For any distinct points a and a' in X, set

$$D = \{x \in X \mid d(x, a) \leq d(x, a')\}.$$

Then every geodesic segment $[x_1, x_2]$ with x_1 and x_2 in D is contained in the closed δ-neighborhood $\overline{N_\delta D}$ of D. Proof: We first note that any geodesic segment $[a, x_i]$

is contained in D for $i = 1, 2$. Indeed, for every point $z \in [a, x_i]$, we have

$$d(z, a) = d(x_i, a) - d(z, x_i) \leq d(x_i, a') - d(z, x_i) \leq d(z, a').$$

Next consider a triangle $\triangle a x_1 x_2$. The δ-hyperbolicity implies that $[x_1, x_2]$ is in the closed δ-neighborhood of $[a, x_1] \cup [a, x_2]$. Since $[a, x_1]$ and $[a, x_2]$ are contained in D, we have $[x_1, x_2] \subset \overline{N_\delta D}$.

Now, since x_n belongs to $D_G(x_0) \subset D_{g_y}(x_0)$, the above claim implies that the geodesic segment $[x_0, x_n]$ is contained in the closed δ-neighborhood of $D_{g_y}(x_0)$ for all n. Taking the limit as $n \to \infty$, we have $[x_0, \xi) \subset \overline{N_\delta D_{g_y}(x_0)}$. Since $y \in [x_0, \xi)$, we conclude

$$d(y, x_0) \leq d(y, g_y x_0) + 2\delta \leq L + 2\delta.$$

However, this contradicts the choice of y so that $d(y, x_0) > L + 2\delta$. □

By this lemma, we have an expected property of quasiconvex cocompact groups as follows.

Proposition 2.3. *If $G \subset \mathrm{Isom}(X)$ is quasiconvex cocompact, then it acts uniformly properly discontinuously on X.*

Proof. Suppose to the contrary that G does not act uniformly properly discontinuously on X. Then we can find a sequence of points $\{x_n\} \subset X$ such that

$$\#\{g \in G \mid g(U(x_n, 1/n)) \cap U(x_n, 1/n) \neq \emptyset\} \geq n.$$

We may assume that all x_n belong to some Dirichlet domain $D \subset X$ of G and x_n converge to some point x_∞ of the closure $\overline{D}$ taken in $\overline{X}$. However, G acts properly discontinuously at every point in $\overline{D}$ by Proposition 2.1 and Lemma 2.2. This implies that G acts uniformly properly discontinuously on some neighborhood of x_∞, which contradicts the property of the sequence $\{x_n\}$. □

For a sequence of subgroups $\{G_n\}$ of $\mathrm{Isom}(X)$, we define the *envelop* denoted by $\mathrm{Env}\{G_n\}$ to be the subgroup of $\mathrm{Isom}(X)$ consisting of all elements $g = \lim_{n\to\infty} g_n$ given for some sequence $g_n \in G_n$. For a sequence of closed subsets $\{\Lambda_n\}$ of ∂X, we define the *envelop* denoted by $\mathrm{Env}\{\Lambda_n\}$ to be the closed subset of ∂X consisting of all points $x = \lim_{n\to\infty} x_n$ given for some sequence $x_n \in \Lambda_n$.

Proposition 2.4. *Let $\{G_n\}$ be a sequence of subgroups of $\mathrm{Isom}(X)$ that act uniformly properly discontinuously on X where the uniformity is also independent of n. Then $\mathrm{Env}\{G_n\}$ also acts uniformly properly discontinuously on X.*

Proof. By assumption, there are constants $r > 0$ and $N < \infty$ such that

$$\#\{g \in G_n \mid g(U(x, r)) \cap U(x, r) \neq \emptyset\} \leq N$$

for every $x \in X$ and for every $n \in \mathbb{N}$. Then we will prove the uniform proper discontinuity of $\mathrm{Env}\{G_n\}$ for these constants r and N. Suppose that this is not true. Then there are some $x \in X$ and distinct elements

$$g^{(1)}, \ldots, g^{(N)}, g^{(N+1)} \in \mathrm{Env}\{G_n\}$$

such that $g^{(i)}(U(x, r)) \cap U(x, r) \neq \emptyset$ for all $i = 1, \ldots, N+1$. For each i, we choose a sequence $g_n^{(i)} \in G_n$ such that $\lim_{n\to\infty} g_n^{(i)} = g^{(i)}$. Then there is some $n_i \geq 1$ such that $g_n^{(i)}(U(x, r)) \cap U(x, r) \neq \emptyset$ for all $n \geq n_i$. However, considering G_n for $n = \max\{n_1, \ldots, n_{N+1}\}$, we have a contradiction to the assumption on the boundedness by N. □

PROPOSITION 2.5. *Let* $\{G_n\}$ *be a sequence of subgroups of* $\mathrm{Isom}(X)$ *acting properly discontinuously on* X *such that* $\mathrm{Env}\{G_n\}$ *is a non-elementary subgroup of* $\mathrm{Isom}(X)$ *acting properly discontinuously on* X. *Assume further that the limit sets* $\Lambda(G_n)$ *for all* n *together with* $\Lambda(\mathrm{Env}\{G_n\})$ *share a common limit point* x. *Then* $\Lambda(\mathrm{Env}\{G_n\})$ *is contained in* $\mathrm{Env}\{\Lambda(G_n)\}$.

PROOF. Since $\mathrm{Env}\{G_n\}$ is non-elementary, the limit set $\Lambda(\mathrm{Env}\{G_n\})$ coincides with the closure of the orbit of x under $\mathrm{Env}\{G_n\}$. Take any orbit point $g(x)$ given by $g \in \mathrm{Env}\{G_n\}$. We can choose a sequence of elements $g_n \in G_n$ with $\lim_{n\to\infty} g_n = g$. Then $g_n(x) \in \Lambda(G_n)$ converge to $g(x)$ as $n \to \infty$. This implies that $g(x)$ belongs to $\mathrm{Env}\{\Lambda(G_n)\}$, and hence $\Lambda(\mathrm{Env}\{G_n\})$ is contained in $\mathrm{Env}\{\Lambda(G_n)\}$. □

3. Quasiconformal measure on the boundary of hyperbolic space

In this section, we introduce the Patterson-Sullivan theory on the boundary of a Gromov hyperbolic space according to the pioneer work due to Coornaert [**3**].

Let (X, d) be a proper δ-hyperbolic geodesic metric space for some $\delta \geq 0$ with a fixed base point $x_0 \in X$. We choose a so-called visual parameter $a \in (1, a_0(\delta))$ where $a_0(\delta)$ is some constant depending only on δ. Then there is a visual metric d_a on $\overline{X} = X \cup \partial X$ with respect to x_0 and a which satisfies the following properties.

(1) The topology on $\overline{X}$ induced by the visual metric d_a coincides with the topology of the compactification of (X, d).
(2) There exists a constant $\lambda = \lambda(\delta, a) \geq 1$ such that, for any geodesic line (ξ, η) connecting any $\xi, \eta \in \partial X$,
$$\lambda^{-1}\, a^{-d(x_0,(\xi,\eta))} \leq d_a(\xi, \eta) \leq \lambda a^{-d(x_0,(\xi,\eta))}$$
is satisfied.

This is an analog of the euclidean metric for the ball model $(\mathbb{B}^{n+1}, d_h)$ of the hyperbolic space of constant curvature -1.

DEFINITION 3.1. For a subgroup $G \subset \mathrm{Isom}(X)$ acting on X properly discontinuously, let
$$n_y(R) = \#\{g \in G \mid d(gy, x_0) \leq R\}$$
be the number of orbits of some $y \in X$ within distance $R > 0$ from x_0. Then the *critical exponent* $e = e_a(G)$ of G with respect to the visual parameter a is defined to be
$$e_a(G) = \limsup_{R\to\infty} \frac{\log_a n_y(R)}{R}.$$
The *Poincaré series* $P^s_G(y, x_0)$ for G of dimension (or exponent) $s > 0$ is given by
$$P^s_G(y, x_0) = \sum_{g\in G} a^{-sd(gy,x_0)}.$$
This also yields the critical exponent of G by
$$e_a(G) = \inf \{s > 0 \mid P^s_G(y, x_0) < \infty\}.$$

For a Kleinian group $G \subset \mathrm{Isom}(\mathbb{B}^{n+1}, d_h)$, a positive finite Borel measure μ on the boundary $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$ at infinity is G*-conformal measure* of dimension $s > 0$ if
$$\frac{d(g^*\mu)}{d\mu}(\xi) = k(g^{-1}(0), \xi)^s$$

for every $g \in G$ and μ-a.e. $\xi \in \mathbb{S}^n$. Here $k(z,\xi) = (1-|z|^2)/|\xi - z|^2$ is the Poisson kernel, and in particular,

$$k(g^{-1}(0),\xi) = |g'(\xi)|_{\rm euc} = \exp(-d_\xi(g^{-1}(0),0)),$$

where $|g'(\xi)|_{\rm euc}$ is the linear stretching factor of the conformal map g with respect to the euclidean metric on $\overline{\mathbb{B}^{n+1}}$. Also, d_ξ is the *horospherical signed distance* at ξ defined as follows. Let $S_\xi(z)$ be the horosphere tangent at $\xi \in \mathbb{S}^n$ passing through $z \in \mathbb{B}^{n+1}$. Then $d_\xi(z,z') = d_h(S_\xi(z), S_\xi(z'))$ if $S_\xi(z)$ is outside of $S_\xi(z')$ and $d_\xi(z,z') = -d_h(S_\xi(z), S_\xi(z'))$ if $S_\xi(z)$ is inside of $S_\xi(z')$.

For a Gromov hyperbolic space X, we can define an analogue of horosphere as the level set of the Busemann function and hence the horospherical signed distance. For a given point $\xi \in \partial X$, let $\sigma : [0,\infty) \to X$ be a geodesic ray such that $\sigma(0) = x_0$ and $\sigma(\infty) = \lim_{t\to\infty} \sigma(t) = \xi$. Then the *Busemann function* at ξ is defined to be

$$h_\xi(x) = \lim_{t\to\infty}(d(x,\sigma(t)) - d(x_0,\sigma(t))).$$

This depends on the choice of the geodesic ray σ but the difference is uniformly bounded by some constant depending only on δ. The analogue of the linear stretching factor of $g \in {\rm Isom}(X)$ at $\xi \in \partial X$ is given by

$$j_g(\xi) = a^{-h_\xi(g^{-1}x_0)}.$$

DEFINITION 3.2. For a proper δ-hyperbolic geodesic metric space X, we fix a base point x_0 and a visual parameter a. Let G be a subgroup of ${\rm Isom}(X)$. A positive finite Borel measure μ on the boundary ∂X is called a *G-quasiconformal measure* of dimension $s > 0$ if there exists a constant $C \geq 1$ such that

$$C^{-1}\, j_g(\xi)^s \leq \frac{d(g^*\mu)}{d\mu}(\xi) \leq C\, j_g(\xi)^s$$

for every $g \in G$ and for μ-a.e. $\xi \in \partial X$.

Similarly to the case of Kleinian groups, a G-quasiconformal measure of the critical exponent $e_a(G)$ plays an important role.

DEFINITION 3.3. For a subgroup $G \subset {\rm Isom}(X)$ acting on a proper δ-hyperbolic geodesic metric space X properly discontinuously, a G-quasiconformal measure μ of the critical exponent $e_a(G)$ with support in the limit set $\Lambda(G)$ is called a *Patterson-Sullivan measure* for G. Here the support of μ, denoted by ${\rm supp}(\mu)$, refers to the smallest closed subset whose complement has null measure for μ.

The existence of Patterson-Sullivan measure is guaranteed in a similar manner to the Patterson construction for Kleinian groups. Also, as in the case of Kleinian groups, the lower bound of the dimensions of quasiconformal measures is equal to the critical exponent, which is a consequence of the shadow lemma. These results were proved by Coornaert [**3**] as follows.

THEOREM 3.4. *Assume that a subgroup $G \subset {\rm Isom}(X)$ acts on X properly discontinuously and the critical exponent $e_a(G)$ is finite. Then*

(1) *a Patterson-Sullivan measure for G exists;*
(2) *the exponent s of any G-quasiconformal measure is not less than $e_a(G)$.*

In the classical case, a Kleinian group of divergence type has a special property for its Patterson-Sullivan measure. We introduce this class also in our present case.

Definition 3.5. Let $G \subset \mathrm{Isom}(X)$ act on X properly discontinuously. If the critical exponent $e_a(G)$ is finite and the Poincaré series $P^s_G(y, x_0)$ of dimension $s = e_a(G)$ diverges, then G is said to be of *divergence type*.

As an application of the covering theorem, Coornaert [**3**] obtained the same consequence as the Kleinian case for the property of quasiconvex cocompact groups.

Theorem 3.6. *If $G \subset \mathrm{Isom}(X)$ is quasiconvex cocompact with $e_a(G) < \infty$, then it is of divergence type.*

Next we will see that, for G of divergence type, every G-quasiconformal measure μ of the critical exponent $e_a(G)$ is actually a Patterson-Sullivan measure. For this claim, we have only to show that μ has no mass on $\Omega(G)$, which implies that the support of μ is in $\Lambda(G)$. This is a new ingredient of this paper which supplements [**3**].

Lemma 3.7. *Assume that a subgroup $G \subset \mathrm{Isom}(X)$ is of divergence type. If μ is a G-quasiconformal measure of exponent $e = e_a(G)$, then the support of μ is in the limit set $\Lambda(G)$, which means that μ is a Patterson-Sullivan measure for G.*

Proof. Suppose to the contrary that μ has a positive measure on $\Omega(G) = \partial X - \Lambda(G)$. Then there is a compact subset $B \subset \Omega(G)$ with $\mu(B) > 0$. Since G acts on $\Omega(G)$ properly discontinuously by Proposition 2.1, the number M of elements $g \in G$ satisfying $g(B) \cap B \neq \emptyset$ is finite. This implies that

$$\sum_{g \in G} \mu(g(B)) \leq M\mu(\Omega(G)) < \infty.$$

On the other hand, for the constant $C \geq 1$ of the quasiconformality of μ, we have

$$\mu(g(B)) = \int_B d(g^*\mu)(\xi) \geq C^{-1} \int_B j_g(\xi)^e d\mu(\xi).$$

Plugging the second inequality in the first and exchanging the sum and the integral, we have $\int_B \sum_{g \in G} j_g(\xi)^e d\mu(\xi) < \infty$, and hence there is some $\xi \in B$ such that $\sum_{g \in G} j_g(\xi)^e < \infty$. Then, since

$$j_g(\xi) = a^{-h_\xi(g^{-1}x_0)} \geq a^{-d(g^{-1}x_0, x_0)},$$

we conclude that $\sum_{g \in G} a^{-ed(g^{-1}x_0, x_0)} < \infty$. However, this contradicts the assumption that G is of divergence type. □

For a G-quasiconformal measure μ of dimension s with G quasiconvex cocompact, Theorem 3.6 combined with Lemma 3.7 asserts that, if the dimension s is equal to $e_a(G)$, then μ is nothing but a Patterson-Sullivan measure.

Remark 3.8. We also see that, for G-quasiconformal measure μ of dimension s with G quasiconvex cocompact, if the support of μ is contained in $\Lambda(G)$, then μ must be a Patterson-Sullivan measure (that is, $s = e_a(G)$). This is given in [**3**].

4. Proof of the main theorem

Suppose that $G \subset \mathrm{Isom}(X)$ is quasiconvex cocompact with $e_a(G) < \infty$ and that there exists $\alpha \in \mathrm{Isom}(X)$ such that the conjugate $\Gamma = \alpha G \alpha^{-1}$ is contained in G. Set $\Gamma_n = \alpha^{-n} \Gamma \alpha^n$ for every integer $n \geq 0$. Then we have an increasing sequence of quasiconvex cocompact groups

$$\Gamma = \Gamma_0 \subset G = \Gamma_1 \subset \Gamma_2 \subset \cdots$$

with the same critical exponent $e = e_a(G)$.

Proposition 4.1. *The limit sets $\Lambda(\Gamma)$ and $\Lambda(G)$ coincides. Moreover, $\Lambda(\Gamma_n)$ are the same for all $n \geq 0$.*

Proof. We may assume that G is non-elementary, for otherwise the statement is clear. Let μ be the Patterson-Sullivan measure for G. In particular, the dimension of μ is $e = e_a(G)$ and $\mathrm{supp}(\mu)$ coincides with the limit set $\Lambda(G)$. Since μ is also Γ-quasiconformal measure of the exponent $e = e_a(\Gamma)$, Theorem 3.6 and Lemma 3.7 assert that $\mathrm{supp}(\mu) = \Lambda(\Gamma)$. Hence we have $\Lambda(\Gamma) = \Lambda(G)$. □

Next we consider $\Gamma_\infty = \bigcup_{n \geq 0} \Gamma_n = \lim_{n\to\infty} \Gamma_n$, which clearly contained in $\mathrm{Env}\{\Gamma_n\}$. For this limit, we have $\alpha^{-1}\Gamma_\infty \alpha = \Gamma_\infty$.

Lemma 4.2. *The subgroup Γ_∞ acts properly discontinuously on X and its limit set $\Lambda(\Gamma_\infty)$ coincides with $\Lambda(\Gamma_n)$ for all $n \geq 0$. Hence Γ_∞ is also quasiconvex cocompact.*

Proof. Since G is quasiconvex cocompact, it acts uniformly properly discontinuously on X by Proposition 2.3. Hence Γ_n for all $n \geq 0$ act uniformly properly discontinuously on X with the uniformity independent of n, and Proposition 2.4 asserts that so does $\Gamma_\infty \subset \mathrm{Env}\{\Gamma_n\}$. We have only to consider the case that Γ_∞ is non-elementary, for otherwise the statement of the lemma is easily seen.

On the other hand, since the limit sets $\Lambda(\Gamma_n)$ are the same Λ for all $n \geq 0$ by Proposition 4.1, they coincide with $\mathrm{Env}\{\Lambda(\Gamma_n)\}$. Then we can apply Proposition 2.5 to see that $\Lambda(\Gamma_\infty) \subset \Lambda$. Since the converse inclusion is clear, we have $\Lambda(\Gamma_\infty) = \Lambda$. This implies $Q(\Lambda(\Gamma_\infty)) = Q(\Lambda(\Gamma_n))$ for their hulls. Since Γ_∞ includes quasiconvex cocompact subgroups Γ_n, we see that Γ_∞ is also quasiconvex cocompact. □

Since Γ_∞ acts properly discontinuously on X by Lemma 4.2, Γ_∞ is discrete in $\mathrm{Isom}(X)$. This in particular implies that $\Gamma_\infty = \bigcup_{n \geq 0} \Gamma_n$ coincides with $\mathrm{Env}\{\Gamma_n\}$.

Remark 4.3. There is an alternative proof of the fact $\Lambda(\Gamma_\infty) = \Lambda$ through showing a claim that $e_a(\Gamma_\infty) = e$ as follows. We take a Patterson-Sullivan measure μ_n for each Γ_n with the normalization of total mass and consider a weak-$*$ limit μ of a subsequence of $\{\mu_n\}$. Then μ is a Γ_∞-quasiconformal measure of dimension e. By Theorem 3.4, we have $e \geq e_a(\Gamma_\infty)$ against the trivial inequality $e \leq e_a(\Gamma_\infty)$. Once we have $e_a(\Gamma_\infty) = e$, the coincidence of the limit sets $\Lambda(\Gamma_\infty) = \Lambda$ follows by considering the supports of their Patterson-Sullivan measures as in Proposition 4.1. Note that this proof makes no use of Proposition 2.5.

Conversely, $\Lambda(\Gamma_\infty) = \Lambda$ yields $e_a(\Gamma_\infty) = e$. Indeed, we take a Patterson-Sullivan measure μ for Γ_∞. The dimension of μ is $e_a(\Gamma_\infty)$ and $\mathrm{supp}(\mu)$ is in $\Lambda(\Gamma_\infty) = \Lambda$. Since μ is also Γ_n-quasiconformal measure with support in $\Lambda = \Lambda(\Gamma_n)$, Remark 3.8 gives $e_a(\Gamma_\infty) = e$.

Proof of Theorem 1.4. In order to prove that G has no proper conjugation, we will show that the conjugate $\Gamma = \alpha G \alpha^{-1} \subset G$ actually coincides with G. By Lemma 4.2, we see that Γ_∞ is quasiconvex cocompact with the same limit set as $\Gamma = \Gamma_0$. Hence the hull $Q(\Lambda(\Gamma_\infty))$ coincides with $Q(\Lambda(\Gamma))$ and the compact quotient $Q(\Lambda(\Gamma))/\Gamma$ finitely covers $Q(\Lambda(\Gamma_\infty))/\Gamma_\infty$. In particular, the index $[\Gamma_\infty : \Gamma]$ is finite. Since $\Lambda(G) = \Lambda(\Gamma)$ by Proposition 4.1, this is also true for $G = \Gamma_1$, namely, the index $l = [G : \Gamma]$ is finite. Then we have $[\Gamma_n : \Gamma] = l^n$ for any subgroup Γ_n of Γ_∞ for $n \geq 0$. However, since this is bounded by the finite index $[\Gamma_\infty : \Gamma]$ for every n, we see that $l = [G : \Gamma] = 1$, that is, $G = \Gamma$. □

5. Proper conjugation in hyperbolic groups

We give some remarks on proper conjugation in (word) hyperbolic groups. A finitely generated group H is called a *hyperbolic group* if the Cayley graph $C(H)$ of H with respect to some generating system is Gromov hyperbolic as a geodesic space of the word metric. The canonical action of H on $C(H)$ is isometric as well as properly discontinuous and cocompact. We regard H itself as a subgroup of $\mathrm{Isom}(C(H))$.

Let G be a subgroup of a hyperbolic group H. As is mentioned in the introduction, G is quasiconvex in H (i.e. the orbit of G on $C(H)$ is quasiconvex, or equivalently the vertices corresponding to the elements of G is quasiconvex in $C(H)$) if and only if G acts quasiconvex cocompactly on $C(H)$. It was first proved by Mihalik and Towle [**12**] that if G is quasiconvex in a hyperbolic group H then G has no proper conjugation in H. Since H acts properly discontinuously on $C(H)$, the result in Ranjbar-Motlagh [**15**] without extending to our Theorem 1.4 also implies this claim. It also follows from a more general result due to Gitik, Mitra, Rips and Sageev [**7**]. More generally, the aforementioned result in Yang [**24**] implies that, if H is a relatively hyperbolic group and G is relatively quasiconvex in H, then G has no proper conjugation in H. (The authors was informed of these literatures by anonymous reviewers of this paper.)

Next we consider when a finitely generated subgroup G is quasiconvex in a hyperbolic group H. In the case where H is the free group F_m of rank $m \geq 1$, Short [**19**] proved that $G \subset F_m$ is finitely generated if and only if G is quasiconvex in F_m. (See also Hersonsky and Hubbard [**5**] for the fact that every finitely generated subgroup G of F_m acts quasiconvex cocompactly.) Hence this implies that a finitely generated subgroup G of the free group F_m has no proper conjugation in F_m. On the other hand, a concrete example of a finitely generated subgroup G of a hyperbolic group H that has proper conjugation in H (which is not quasiconvex) was given by Kapovich and Wise [**10**].

Remark 5.1. In Example 1.2 of proper conjugation in the introduction, the Baumslag-Solitar group $H = B(m, n)$ in (1) is not hyperbolic and the subgroup G of $H = F_2$ in (2) is not finitely generated.

We define H to be *locally quasiconvex* if every finitely generated subgroup $G \subset H$ is quasiconvex in H. This is well-defined independently of the choice of the generating system of H. Free groups are locally quasiconvex. The above arguments can be summarized as follows.

Proposition 5.2. *Every finitely generated subgroup G of a locally quasiconvex hyperbolic group H has no proper conjugation in H.*

There are several characterizations for local quasiconvexity (cf. Gitik [**6**] and Kapovich [**9**]). In particular, a surface group H is locally quasiconvex and so is the free product H of locally quasiconvex groups. On the contrary, the fundamental group H of a mapping torus of a closed surface S by a pseudo-Anosov homeomorphism is not locally quasiconvex because $G = \pi_1(S)$ is not quasiconvex in H. (This fact was pointed out to the authors by K. Ohshika.)

Note that, if G itself is a hyperbolic group and if it is torsion-free and indecomposable to a non-trivial free product, then G has no proper conjugation no matter in what group H embedded is G. In fact, Sela [**18**] has shown that if a torsion-free hyperbolic group G is indecomposable then it has the co-Hopf property and vice versa. This is an opposite situation to the free group case but still prevents proper conjugation. For example, this is the case for any G isomorphic to a surface group. Potyagailo and Wang [**14**] investigated the case where G is isomorphic to a 3-manifold fundamental group. Further, Ohshika and Potyagailo [**13**] gave a sufficient condition for the co-Hopf property when G is isomorphic to the fundamental group of a hyperbolic manifold including higher dimensional cases.

Finally, we deal with a subgroup G which is not necessarily finitely generated. Since the Cayley graph $C(F_m)$ of the free group F_m with a free generating system is a tree, we can apply Theorem 1.5 to look at proper conjugation of a subgroup in a free group. Note that divergence type groups can be infinitely generated.

Corollary 5.3. *If a subgroup G of the free group F_m is of divergence type, then G has no proper conjugation in F_m.*

This corollary shows that the subgroup $G = \langle \alpha^n \beta \alpha^{-n} \rangle_{n \geq 0}$ of $H = F_2$ in Example 1.2 (2) is not of divergence type. On the other hand, it is possible to show that $\widetilde{G} = \langle \alpha^n \beta \alpha^{-n} \rangle_{n \in \mathbb{Z}}$ is of divergence type in spite of its being infinitely generated.

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