APPENDIX D

Kleinian groups by Katsuhiko Matsuzaki

Katsuhiko Matsuzaki
Department of Mathematics, School of Education
Waseda University, Japan

A study of the dynamics for a discontinuous group of linear fractional transformations on the Riemann sphere, often called Kleinian groups, was begun by Robert Fricke and Felix Klein in [**1911**]. Fricke and Klein studied the set of accumulation points (the limit set) of the orbit under the action of a discontinuous group. Its complement, where the action is discontinuous, is called the ordinary set. In a recent monograph [**2002**] — *Indra's Pearls, The vision of Felix Klein* — by Mumford, Series and Wright, many beautiful fractal pictures of limit sets are exhibited with detailed expositions including Fricke and Klein's original ones.

The properties of the limit set and the ordinary set, for Kleinian groups, have strong similarities with those for the Julia set and the Fatou set, for the iterations of a rational map. For example, chaotic behavior and density of the orbit on the limit (Julia) set are the same. Actually, by means of the concept of normal family introduced by Montel, these two sets can be defined in a unified way. The ordinary (Fatou) set is a set of points in a neighborhood of which the linear fractional transformations $\{g_n\}$ in a discontinuous group G, or the iterated maps $\{f^n\}$, of a rational map f, constitute a normal family. Hence several properties of these holomorphic dynamics can be explained simultaneously by the normal family arguments.

The theory of discontinuous groups developed in complex analysis as a branch of the theory of Teichmüller spaces. A related powerful tool is the quasi-conformal map which gives deformation on the dynamics. One of the monumental results in this field was the Ahlfors finiteness theorem, obtained in the middle 1960s. About two decades later, a new revolution in the theory of iteration of rational maps was brought about by Dennis Sullivan, who imported quasi-conformal maps from discontinuous group theories and proved the non-wandering domain theorem with a method similar to that used in the proof of the Ahlfors finiteness theorem.

After this success, the similarity or the analogy between discontinuous groups and the iteration of rational maps was recognized again through more sophisticated analytic concepts, and it provided a sort of principle for new researches in these fields. A list of the correspondence of concepts and theorems between the two holomorphic dynamics is often called *Sullivan's dictionary*, Sullivan [**1985**]. The number of items in this dictionary is still increasing.

APPENDIX G

The Denjoy-Wolff Theorem by Katsuhiko Matsuzaki

Katsuhiko Matsuzaki
Department of Mathematics, School of Education
Waseda University, Japan

The Denjoy-Wolff Theorem considers the iteration of a holomorphic function $f : \Delta \to \Delta$ on the unit disk and asserts that, excluding a trivial exception where f is the identity or an elliptic automorphism of Δ, there exists a unique point ξ in the closure $\widehat{\Delta}$ of the unit disk such that the orbit of every point $z \in \Delta$ under the n-times iteration $f^n(z) \to \xi$, as $n \to \infty$. Since f is contracting with respect to the hyperbolic metric, as the Schwarz-Pick Lemma states, if f has a fixed point in Δ, the statement is easily obtained by taking the fixed point of f as $\xi \in \Delta$. Hence, the problem lies in the treatment of the case where f has no fixed points and especially the uniqueness of the point ξ on the boundary $\partial\Delta = \widehat{\Delta} - \Delta$ is not so obvious.

The works of Fatou and Julia classify the periodic components D for the iteration of rational maps g into several types and if D is simply connected and m is the period of g for D, then either:

1. $f = g^m$ has an attracting fixed point in D;
2. f has a rationally indifferent fixed point on the boundary ∂D;
3. or f is a rotation of D with the fixed point inside.

By the Riemann Mapping Theorem, the situation can be transferred to the unit disk Δ, and these three possibilities correspond to the above mentioned cases for the iteration of a holomorphic function $f : \Delta \to \Delta$. However, the rational map g is originally defined on the Riemann sphere, and $f = g^m$ is of course extendable to the closure $\bar{D}$. Hence, the existence of the fixed point of f is easily seen, and the behavior of the orbit as in the assertion of the Denjoy-Wolff Theorem is then clear. In other words, the Denjoy-Wolff Theorem localizes the problem on simply connected periodic components of rational maps and generalizes the arguments of Fatou and Julia. In fact, the statement of the theorem first appeared in Wolff's paper [**1926a**] under the assumption that $f : \Delta \to \Delta$ is continuously extendable to the boundary.

Soon afterwards, Denjoy [**1926**] and Wolff [**1926b**] independently succeeded in removing this assumption. Wolff utilized a general fact due to Fatou that a bounded holomorphic function on Δ has a non-tangential limit at almost every point on ∂D and proved that the full statement of the theorem follows even from this weaker continuity condition. In contrast, Denjoy's approach is more geometric: f is approximated by $(1 - \varepsilon)f$ with $\varepsilon \to 0$ in order to appeal to the stronger contraction property for $(1 - \varepsilon)f$. This method later extended to an argument given by Wolff [**1926c**] proving the existence of a horoball tangent at $\partial\Delta$ that is mapped by f into itself.

A historical survey of the proofs of Denjoy-Wolff Theorem as well as a concise exposition on the related topics can be found in an article by Burckel [**1981**]. The second proof due to Wolff [**1926c**] was refined by Beardon [**1990**] in a very simple geometric form. In a standard textbook on complex dynamics written by Carleson and Gamelin [**1993**], the Denjoy-Wolff Theorem was introduced with this proof attributed to Beardon; however, its origin is in the 1926 papers by Denjoy and Wolff. Beardon's paper also recognized the crucial points of the Denjoy-Wolff Theorem as the investigation on the contraction and the ideal boundary of metric spaces, and proposed the way to generalize the theorem to assertions for metric spaces regardless of the analyticity of the mappings. The references for other such generalizations, such as, to multiply connected domains, Riemann surfaces, higher dimensional manifolds, and so on, can be found in the bibliography of Burckel [**1981**] and Beardon [**1990**]. See also a survey given by Reich and Shoikhet [**1997b**].

The Denjoy-Wolff theorem was a prototype of hyperbolic dynamics and contributed much to modern studies on dynamical systems even if the relationships are not explicitly mentioned. Among them, we introduce two recent results on 1-dimensional holomorphic complex dynamics, which are directly related to the Denjoy-Wolff Theorem.

The first one is the so called *Snail Lemma*, which asserts that the Denjoy-Wolff Theorem forces an indifferent fixed point $\xi \in \partial D$ where the orbit converges, to be rationally indifferent for a holomorphic map $f : \bar{D} \to \bar{D}$ on a general domain D. This can be used to prove the classification of Fatou components for rational maps as well as to treat irrationally indifferent, non-linearizable fixed points. See, for example, Pérez-Marco's paper [**1997**].

The second one is a characterization of certain geometric properties of subdomains D in the unit disk Δ in terms of the convergence of the compositions of holomorphic functions in a family $F = \{f | f : \Delta \to D\}$. The Denjoy-Wolff Theorem states that the iteration of a single function $f : \Delta \to \Delta$ (non-elliptic) always converges to a constant function. Moreover, this happens for any sequence of compositions taken from the family F if the range D is degenerate in some sense. One can review classical and recent results on this problem in an article by Lorentzen [**1999**].