

The universal Teichmüller space and diffeomorphisms of the circle with Hölder continuous derivatives

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Abstract

After summarizing several concepts in the quasiconformal theory of the universal Teichmüller space, we introduce the Teichmüller space of diffeomorphisms of the circle with Hölder continuous derivatives as a subspace of the universal Teichmüller space. This can be done by characterizing such a diffeomorphism in terms of the complex dilatation of its quasiconformal extension and the Schwarzian derivative given by the Bers embedding.

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1 Introduction

In this article, we survey the complex analytic theory of the universal Teichmüller space and its subspaces. In particular, we define the Teichmüller space of diffeomorphisms of the circle with Hölder continuous derivatives and provide this space with a complex structure modeled on a certain complex Banach space.

The universal Teichmüller space T is an ambient space of all other Teichmüller spaces. We can regard T as a quotient space of certain self-homeomorphisms of the unit disk \mathbb{D} or the unit circle \mathbb{S} . For the Teichmüller space $T(R)$ of a hyperbolic Riemann surface R , we usually represent R by a Fuchsian group Γ acting on \mathbb{D} and take a subspace of T that consists of Γ -equivariant homeomorphisms. In this way, $T(R)$ is embedded into T . However, what we deal with here is a subspace which spreads in a different direction of T . We restrict the self-homeomorphisms of \mathbb{D} or \mathbb{S}

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in various manners and consider the space of elements satisfying such conditions. In this article, we mainly deal with the subspace of diffeomorphisms of \mathbb{S} with Hölder continuous derivatives. Along the same line, subspaces of quasiconformal automorphisms with integrable complex dilatations with respect to the hyperbolic metric have been studied by Cui [10] and Takhtajan and Teo [23]. They are closely related to our Teichmüller spaces, but we do not treat them here.

We give a complex structure to the subspace of the diffeomorphisms of \mathbb{S} in the framework of the quasiconformal theory of Teichmüller spaces. To this end, the first task we have is to characterize circle diffeomorphisms with Hölder continuous derivatives in terms of their quasiconformal extension to \mathbb{D} . This originated in the work of asymptotically conformal maps by Carleson [9]. In the present article, we review two of his important theorems in detail, which are both described via quasisymmetric quotients of boundary mappings. Later, Gardiner and Sullivan [16] developed the concept of symmetric automorphisms of \mathbb{S} , which are the boundary extension of asymptotically conformal automorphisms of \mathbb{D} , using results on quasiconformal extension and Schwarzian derivatives of univalent functions. This has been generalized to the theory of asymptotic Teichmüller spaces.

The subspace of circle diffeomorphisms with Hölder continuous derivatives is contained in the Teichmüller space T_0 of symmetric automorphisms of \mathbb{S} . A complex structure on this space is given by determining its image under the Bers embedding in the space of Schwarzian derivatives of the developing maps of projective structures. To put it briefly, the corresponding elements in the Bers embedding have the property that their decay order at the boundary is dominated by the order of the Hölder continuity. A more precise statement is the following.

Theorem 1.1. *The Teichmüller space T_0^α of circle diffeomorphisms with Hölder continuous derivatives of order α is homeomorphic to a domain in the Banach space of holomorphic functions on $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$ with supremum norm of weight $\rho_{\mathbb{D}^*}^{-2+\alpha}$ for the hyperbolic density $\rho_{\mathbb{D}^*}$.*

A brief outline of this article is as follows. In the first part (Sections 2, 3 and 4), we review the theory of the universal Teichmüller space T in order to understand how to put metric, group and complex structures on it. We begin with the definition of quasisymmetric automorphisms of \mathbb{S} and then look at their quasiconformal extension to \mathbb{D} . We also explain the conformally natural extension of a quasisymmetric automorphism, a notion due to Douady and Earle [11]. The most important step is to realize T in the Banach space of hyperbolically bounded holomorphic functions by the Bers embedding. We refer to the canonical local section of the Bers projection due to Ahlfors and Weill [2]. Quasisymmetric functions on the real line and their quasisymmetric quotients are discussed in order to introduce the classical quasiconformal extension of a quasisymmetric function due to Beurling and Ahlfors [8]. Fundamental results stated in this part can be found in standard monographs on Teichmüller spaces [15], [19] and [22].

In the second part (Sections 5 and 6), we focus on symmetric automorphisms of \mathbb{S} and the small subspace T_0 of T made of these elements. This will be a foundation for providing a complex structure for the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives. We follow the framework

due to Gardiner and Sullivan [16]. One of the aforementioned theorems due to Carleson on the asymptotically conformal extension of a symmetric function is given with an explicit estimate of the complex dilatation in terms of the quasisymmetric quotient. This is based on the quasiconformal extension by Beurling and Ahlfors. The Bers embedding of T_0 is contained in the Banach space of hyperbolically bounded holomorphic functions that vanish at the boundary and the converse is also true. This comes from the result on univalent functions and their quasiconformal extension due to Becker and Pommerenke [6].

Finally, in the last part (Sections 7 and 8), we introduce a complex structure on the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives as in Theorem 1.1 above. A crucial step is to give conditions for a diffeomorphism of \mathbb{S} to have an α -Hölder continuous derivative in terms of the decay order of the complex dilatation of its quasiconformal extension to \mathbb{D} and the decay order of the Schwarzian derivative of the developing map of the corresponding projective structure on \mathbb{D}^* . The other theorem of Carleson [9] plays an important role at this stage, giving a connection between the Hölder continuity of the derivatives and the quasisymmetric quotients. The equivalent conditions are given as follows.

Theorem 1.2. *Let α be a constant with $0 < \alpha < 1$. For a quasisymmetric automorphism $g : \mathbb{S} \rightarrow \mathbb{S}$, the following conditions are equivalent:*

- (1) *g is a diffeomorphism of \mathbb{S} with Hölder continuous derivative of order α ;*
- (2) *g extends continuously to a quasiconformal automorphism of \mathbb{D} whose complex dilatation $\mu(z)$ decays in the order of $O((1 - |z|)^\alpha)$ as $z \in \mathbb{D}$ tends to the boundary;*
- (3) *the Schwarzian derivative $\varphi(z)$ of the conformal homeomorphism of \mathbb{D}^* determined by g is of order $O((|z| - 1)^{-2+\alpha})$ as $z \in \mathbb{D}^*$ tends to the boundary.*

Certain directions of this equivalence were already known. For instance, by using a harmonic quasiconformal extension of a diffeomorphism of \mathbb{S} , Tam and Wan [24] proved that the Schwarzian derivative has the exact decay order given by the exponent α of the Hölder continuity. A complete proof of this equivalence is not given here since we need further technical arguments but will appear in a separate paper [21]. We also obtain certain quantitative estimates concerning the indices involved in the equivalent conditions. In particular, the Hölder constant for the exponent α dominates the weighted norms of the complex dilatation and the Schwarzian derivative which catch their decay of exponent α . As an application of this estimate and with the aid of our forthcoming result on the conjugation of a diffeomorphism group to a Möbius group, we prove a rigidity phenomenon of a group of circle diffeomorphisms.

The last part of this article is based on the author's talk given in the conference of Group Actions and Applications in Geometry, Topology and Analysis, held at Kunming University of Science and Technology on July 21-29, 2012. The first and the second parts are based on a part of the author's lecture in the graduate course at Waseda University in the Fall Semester 2012. In that lecture, the author also gave a summary on Teichmüller spaces of quasiconformal automorphisms of

integrable complex dilatations with respect to the hyperbolic metric, but this topic is not included in this article. The formulation of the Teichmüller space of diffeomorphisms with Hölder continuous derivatives was added recently.

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2 Quasisymmetric automorphisms of the circle

In this section, we define a quasisymmetric automorphism of the unit circle and observe that this is a boundary map of a quasiconformal automorphism of the unit disk. We also introduce the barycentric quasiconformal extension of a quasisymmetric automorphism to the unit disk. The quasisymmetry is defined by using a variation of the cross ratio of four points on the circle.

Definition. For distinct complex numbers $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the *cross ratio* is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

It takes its values in $\widehat{\mathbb{C}} - \{0, 1, \infty\}$. We call the following modification *alternative cross ratio*:

$$[z_1, z_2, z_3, z_4]_* = \frac{[z_2, z_3, z_4, z_1]}{[z_1, z_2, z_3, z_4]} = -\frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}.$$

It takes its values in $\widehat{\mathbb{C}} - \{-1, 0, \infty\}$.

For a positively ordered quadruple of distinct points z_1, z_2, z_3, z_4 on the unit circle $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$, the alternative cross ratio satisfies $[z_1, z_2, z_3, z_4]_* \in (0, \infty)$.

Definition. An orientation-preserving automorphism g of \mathbb{S} is defined to be *M-quasisymmetric* if there exists a constant $M \geq 1$ such that any positively ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \mathbb{S}$ with $[z_1, z_2, z_3, z_4]_* = 1$ satisfy

$$\frac{1}{M} \leq [g(z_1), g(z_2), g(z_3), g(z_4)]_* \leq M.$$

Remark. A cyclic permutation of the ordered points z_1, z_2, z_3, z_4 affects the alternative cross ratio in $[g(z_2), g(z_3), g(z_4), g(z_1)]_* = [g(z_1), g(z_2), g(z_3), g(z_4)]_*^{-1}$. Hence the upper estimate $[g(z_1), g(z_2), g(z_3), g(z_4)]_* \leq M$ is enough for g to be *M-quasisymmetric*.

We denote the set of all quasisymmetric automorphisms of \mathbb{S} by QS.

Definition. An orientation-preserving homeomorphism $f : D \rightarrow D'$ between domains D and D' of the Riemann sphere $\widehat{\mathbb{C}}$ is called *K-quasiconformal* if there

exists a constant $K \geq 1$ such that any quadrilateral $Q(z_1, z_2, z_3, z_4)$ in D with positively ordered vertices z_1, z_2, z_3, z_4 in ∂Q satisfies

$$\frac{1}{K} \leq \frac{\text{mod}(f(Q(z_1, z_2, z_3, z_4)))}{\text{mod}(Q(z_1, z_2, z_3, z_4))} \leq K.$$

Here $\text{mod}(Q(z_1, z_2, z_3, z_4))$ stands for the *conformal modulus* given by the ratio of the length of $[h(z_2), h(z_3)]$ to that of $[h(z_1), h(z_2)]$ under any conformal homeomorphism h of $Q(z_1, z_2, z_3, z_4)$ onto a rectangle $Q(h(z_1), h(z_2), h(z_3), h(z_4))$.

Remark. A cyclic permutation of the ordered points z_1, z_2, z_3, z_4 affects the conformal modulus in $\text{mod}(Q(z_2, z_3, z_4, z_1)) = \text{mod}(Q(z_1, z_2, z_3, z_4))^{-1}$. Hence the upper estimate of the ratio of the conformal moduli by K is enough for f to be K -quasiconformal. Also, the definition of quasiconformality should be the same even if we consider only quadrilaterals $Q(z_1, z_2, z_3, z_4)$ satisfying $\text{mod}(Q(z_1, z_2, z_3, z_4)) = 1$.

From this definition, it is easy to see that (1) a conformal homeomorphism is 1-quasiconformal; (2) the inverse map of a K -quasiconformal homeomorphism is also K -quasiconformal; and (3) the composition of K_1 -quasiconformal and K_2 -quasiconformal homeomorphisms is $K_1 K_2$ -quasiconformal.

We denote the set of all quasiconformal automorphisms of the unit disk \mathbb{D} by $\text{QC}(\mathbb{D})$. By the above properties, $\text{QC}(\mathbb{D})$ constitutes a group. Moreover, each quasiconformal automorphism $f \in \text{QC}(\mathbb{D})$ extends to the boundary \mathbb{S} as a self-homeomorphism and to a quasiconformal automorphism of $\widehat{\mathbb{C}}$ by reflection. For every $f \in \text{QC}(\mathbb{D})$, let $q(f)$ denote the orientation-preserving self-homeomorphism of \mathbb{S} given by the extension.

For positively ordered distinct points z_1, z_2, z_3, z_4 on \mathbb{S} , we can think of two moduli which are invariant under the conformal automorphisms $\text{Conf}(\mathbb{D})$ of \mathbb{D} , equivalently the Möbius transformations $\text{Möb}(\mathbb{D})$ preserving \mathbb{D} . One is the alternative cross-ratio and the other is the conformal modulus of \mathbb{D} with the prescribed four points. The distortion function $\lambda : (0, \infty) \rightarrow (0, \infty)$ defined by

$$\lambda(\text{mod}(\mathbb{D}(z_1, z_2, z_3, z_4))) = [z_1, z_2, z_3, z_4]_*$$

serves as a transition between these two values. This is a continuous and monotone increasing function satisfying $\lambda(1/t) = 1/\lambda(t)$ for every $t \in (0, \infty)$; in particular $\lambda(1) = 1$.

Proposition 2.1. *For a K -quasiconformal automorphism $f \in \text{QC}(\mathbb{D})$, the boundary extension $g = q(f)$ is an M -quasisymmetric automorphism of \mathbb{S} for $M = \lambda(K)$.*

Proof. For any positively ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \mathbb{S}$ with $[z_1, z_2, z_3, z_4]_* = 1$, we have

$$\begin{aligned} [g(z_1), g(z_2), g(z_3), g(z_4)]_* &= \lambda(\text{mod}(\mathbb{D}(f(z_1), f(z_2), f(z_3), f(z_4)))) \\ &\leq \lambda(K \text{mod}(\mathbb{D}(z_1, z_2, z_3, z_4))) \\ &= \lambda(K \lambda^{-1}([z_1, z_2, z_3, z_4]_*)) = \lambda(K). \end{aligned}$$

This shows that $g = q(f)$ is $\lambda(K)$ -quasisymmetric. \square

This proposition in particular implies that the boundary extension defines a map $q : \text{QC}(\mathbb{D}) \rightarrow \text{QS}$. Later, we will see that this is a homomorphism and that QS is a group.

Conversely, every quasisymmetric automorphism $g \in \text{QS}$ extends continuously to a quasiconformal automorphism of \mathbb{D} . In particular, $q : \text{QC}(\mathbb{D}) \rightarrow \text{QS}$ is surjective. The *conformally natural extension* defined below gives a canonical way of quasiconformal extension.

Let m be a probability measure on $\mathbb{S} \subset \mathbb{C}$. The (complex) average of m viewed at $0 \in \mathbb{D}$ is defined by

$$\xi_m(0) = \int_{\mathbb{S}} \zeta dm(\zeta).$$

For an arbitrary point $w \in \mathbb{D}$, we take a conformal automorphism

$$h_w(\zeta) = \frac{\zeta - w}{1 - \bar{w}\zeta} \in \text{Conf}(\mathbb{D}) = \text{Möb}(\mathbb{D}) \quad (h_w : w \mapsto 0),$$

and define the average of m viewed at w by reducing the situation to the case at the origin:

$$\xi_m(w) = h'_w(w)^{-1} \xi_{(h_w)_*m}(0) = (1 - |w|^2) \int_{\mathbb{S}} h_w(\zeta) dm(\zeta).$$

Here $(h_w)_*m$ is the push-forward of the measure m by h_w . If m has no point mass, it is known that there is a unique point $w \in \mathbb{D}$ such that $\xi_m(w) = 0$, which is called the barycenter of m .

For a probability measure m on \mathbb{S} , the conformal measure $\{m_z\}_{z \in \mathbb{D}}$ is a family of probability measures satisfying

$$\frac{dm_z}{dm}(\zeta) = |h'_z(\zeta)| = \frac{1 - |z|^2}{|\zeta - z|^2}.$$

If m has no point mass, by the correspondence of the barycenter $w(z) \in \mathbb{D}$ of m_z to each $z \in \mathbb{D}$, we have the barycentric map $w_m : \mathbb{D} \rightarrow \mathbb{D}$.

For a homeomorphism $g : \mathbb{S} \rightarrow \mathbb{S}$, we define a probability measure m_g as the push-forward of the normalized Lebesgue measure on \mathbb{S} ; $dm_g = g_*(d\theta/2\pi)$. Then the conformally natural extension of g is defined by $w_{m_g} : \mathbb{D} \rightarrow \mathbb{D}$. Douady and Earle [11] proved the following result especially for a quasisymmetric automorphism $g \in \text{QS}$.

Theorem 2.2. *For an M -quasisymmetric automorphism $g \in \text{QS}$, the conformally natural extension $w_{m_g} : \mathbb{D} \rightarrow \mathbb{D}$ is a K -quasiconformal automorphism of \mathbb{D} that is a continuous extension of g . Here $K = K(M)$ can be estimated by M and tends to 1 as $K \rightarrow 1$. Moreover, w_{m_g} is a bi-Lipschitz diffeomorphism with respect to the hyperbolic metric on \mathbb{D} .*

Let $e_{\text{DE}} : \text{QS} \rightarrow \text{QC}(\mathbb{D})$ be the map defined by $g \mapsto w_{m_g}$, which is a section for the boundary extension $q : \text{QC}(\mathbb{D}) \rightarrow \text{QS}$, that is, $q \circ e_{\text{DE}} = \text{id}_{\text{QS}}$. We also call this map the *conformally natural extension*. By the existence of sections and

Proposition 2.1, we see that if g_1 and g_2 are in QS, then so is $g_1 \circ g_2$. This implies that QS is a group and hence q is a surjective homomorphism. However, the conformally natural extension e_{DE} is not a homomorphism. Nevertheless, e_{DE} satisfies the following property due to the way of defining the barycentric map.

Let $\text{Möb}(\mathbb{S})$ denote the group of all Möbius transformations preserving \mathbb{S} , which is a subgroup of QS. Since every element of $\text{Möb}(\mathbb{S})$ is the restriction of an element of $\text{Möb}(\mathbb{D})$, we can identify $\text{Möb}(\mathbb{S})$ with $\text{Möb}(\mathbb{D})$. The conformally natural extension $e_{\text{DE}}(\phi)$ of $\phi \in \text{Möb}(\mathbb{S})$ is nothing but the corresponding element of $\text{Möb}(\mathbb{D})$ under this identification. Then e_{DE} satisfies

$$e_{\text{DE}}(\phi_1 \circ g \circ \phi_2) = e_{\text{DE}}(\phi_1) \circ e_{\text{DE}}(g) \circ e_{\text{DE}}(\phi_2)$$

for any $\phi_1, \phi_2 \in \text{Möb}(\mathbb{S})$ and any $g \in \text{QS}$.

Although $e_{\text{DE}}(g_1 \circ g_2^{-1}) \neq e_{\text{DE}}(g_1) \circ e_{\text{DE}}(g_2)^{-1}$ in general, the dependence of the maximal dilatation on the quasisymmetric constant is also proved in [11].

Proposition 2.3. *If $g_1 \circ g_2^{-1}$ is M -quasisymmetric for $g_1, g_2 \in \text{QS}$, then $e_{\text{DE}}(g_1) \circ e_{\text{DE}}(g_2)^{-1}$ is K -quasiconformal, where the constant $K = K(M)$ depends only on M and tends to 1 as $M \rightarrow 1$.*

3 The universal Teichmüller space

In this section, we define the universal Teichmüller space in terms of the group of quasisymmetric automorphisms of the circle and then introduce a topological and a complex structure on this space by using the quasiconformal theory, the Beltrami equation and the Schwarzian derivative. The universal Teichmüller space is realized as a bounded domain of a certain complex Banach space under the Bers embedding. Basic results are stated without proof. We can consult Lehto [19] for details.

Definition. The *universal Teichmüller space* T is defined to be the set of cosets $\text{Möb}(\mathbb{S}) \backslash \text{QS}$. We denote the coset of $g \in \text{QS}$ by $[g]$.

A *Beltrami coefficient* μ on a domain $D \subset \widehat{\mathbb{C}}$ is a measurable function with supremum norm $\|\mu\|_\infty$ less than 1. We denote the set of all Beltrami coefficients on D by

$$\text{Bel}(D) = \{\mu \in L^\infty(D) \mid \|\mu\|_\infty < 1\}.$$

It is known from the analytic definition of a quasiconformal map that every quasiconformal homeomorphism $f : D \rightarrow D'$ has partial derivatives ∂f and $\bar{\partial} f$ in the distribution sense, and the ratio $\mu_f(z) = \bar{\partial} f(z) / \partial f(z)$, called the *complex dilatation*, is a Beltrami coefficient on D . This property completely characterizes K -quasiconformal homeomorphisms and the *maximal dilatation* defined by

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

satisfies $K(f) \leq K$.

The following *measurable Riemann mapping theorem* due to Ahlfors and Bers [1] asserts that a Beltrami coefficient essentially determines a quasiconformal homeomorphism.

Theorem 3.1. *For every $\mu \in \text{Bel}(D)$, there exists a unique quasiconformal homeomorphism $f : D \rightarrow \widehat{\mathbb{C}}$ whose complex dilatation μ_f coincides with μ up to post-composition by a conformal homeomorphism.*

Applying this theorem to quasiconformal automorphisms of the unit disk \mathbb{D} , we see that $\text{Bel}(\mathbb{D})$ can be identified with the set of cosets $\text{Möb}(\mathbb{D}) \backslash \text{QC}(\mathbb{D})$. Then the boundary extension $q : \text{QC}(D) \rightarrow \text{QS}$ induces a surjective map $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ by taking the quotient of $\text{Möb}(\mathbb{D}) = \text{Möb}(\mathbb{S})$. This is called the *Teichmüller projection*. The topology of the universal Teichmüller space T is the quotient topology of the unit ball $\text{Bel}(\mathbb{D})$ of the Banach space $L^\infty(\mathbb{D})$ by the projection π so that π is continuous. Also, the conformally natural extension $e_{\text{DE}} : \text{QS} \rightarrow \text{QC}(\mathbb{D})$ induces a section $s_{\text{DE}} : T \rightarrow \text{Bel}(\mathbb{D})$ of π by taking the quotient of $\text{Möb}(\mathbb{S}) = \text{Möb}(\mathbb{D})$. Proposition 2.3 implies that s_{DE} is continuous.

Theorem 3.1 implies that for every $\nu \in \text{Bel}(\mathbb{D})$ there is a unique normalized quasiconformal automorphism $f \in \text{QC}(\mathbb{D})$ whose complex dilatation coincides with ν . Here the normalization is given, say, by fixing three boundary points 1, i and -1 on \mathbb{S} . We denote this normalized quasiconformal automorphism by f^ν . Given a normalization, we can define a group structure on $\text{Bel}(\mathbb{D})$ and T as follows.

For any $\nu_1, \nu_2 \in \text{Bel}(\mathbb{D})$, set $\nu_1 * \nu_2$ to be the complex dilatation of the composition $f^{\nu_1} \circ f^{\nu_2}$. Then $\text{Bel}(\mathbb{D})$ has a group structure with this operation $*$. We denote the inverse element of $\nu \in \text{Bel}(\mathbb{D})$ by ν^{-1} , that is, the complex dilatation of $(f^\nu)^{-1}$. The chain rule of partial differentials yields the formula

$$\nu_1 * \nu_2^{-1}(\zeta) = \frac{\nu_1(z) - \nu_2(z)}{1 - \nu_2(z)\nu_1(z)} \cdot \frac{\partial f^{\nu_2}(z)}{\partial f^{\nu_2}(z)} \quad (\zeta = f^{\nu_2}(z)).$$

The inverse image $\pi^{-1}([\text{id}])$ of the base point $[\text{id}]$ of T under $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ is represented by

$$\pi^{-1}([\text{id}]) = \{\nu \in \text{Bel}(\mathbb{D}) \mid q(f^\nu) = \text{id}\}.$$

This shows that $\pi^{-1}([\text{id}])$ is a normal subgroup of $\text{Bel}(\mathbb{D})$ since $q : \text{QC}(\mathbb{D}) \rightarrow \text{QS}$ is a homomorphism. Having $T = \text{Bel}(\mathbb{D})/\pi^{-1}([\text{id}])$, we see that T has a group structure with the operation $*$ defined by $\pi(\nu_1) * \pi(\nu_2) = \pi(\nu_1 * \nu_2)$. Then the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ is a surjective homomorphism.

Each $\nu \in \text{Bel}(\mathbb{D})$ induces a right translation map $r_\nu : \text{Bel}(\mathbb{D}) \rightarrow \text{Bel}(\mathbb{D})$ defined by $\mu \mapsto \mu * \nu^{-1}$. The projection under π yields a well-defined map $R_{\pi(\nu)} : T \rightarrow T$ by

$$\pi(\mu) \mapsto \pi(\mu * \nu^{-1}) = \pi(\mu) * \pi(\nu^{-1}).$$

In this way, for every point $\tau \in T$, we have the base point change map $R_\tau : T \rightarrow T$ sending τ to the base point. By the above formula, we see that r_ν and $(r_\nu)^{-1} = r_{\nu^{-1}}$ are continuous, hence r_ν is a homeomorphism onto $\text{Bel}(\mathbb{D})$. For an arbitrary

open subset $U \subset \text{Bel}(\mathbb{D})$, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{\nu \in \pi^{-1}([\text{id}])} r_\nu(U).$$

This shows that π is an open map. Since $R_{\pi(\nu)} = \pi \circ r_\nu \circ s_{\text{DE}}$ and $R_{\pi(\nu)}^{-1} = \pi \circ (r_\nu)^{-1} \circ s_{\text{DE}}$ are continuous, the base point change map of T is also a homeomorphism onto T .

A distance on the universal Teichmüller space T is given as follows. First we define a distance d_{Bel} on $\text{Bel}(\mathbb{D})$ by

$$d_{\text{Bel}}(\mu_1, \mu_2) = \log \frac{1 + \left\| \frac{\mu_1 - \mu_2}{1 - \mu_2 \mu_1} \right\|_\infty}{1 - \left\| \frac{\mu_1 - \mu_2}{1 - \mu_2 \mu_1} \right\|_\infty}.$$

Clearly the topology induced by this distance is the same as the topology on $\text{Bel}(\mathbb{D})$ induced by the norm $\|\cdot\|_\infty$. On the other hand, the right translation map r_ν is an isometric automorphism of $\text{Bel}(\mathbb{D})$.

The *Teichmüller distance* on T is the quotient distance induced from d_{Bel} by the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$. That is, for any $\tau_1, \tau_2 \in T$,

$$d_T(\tau_1, \tau_2) = \inf \{d_{\text{Bel}}(\mu_1, \mu_2) \mid \mu_1 \in \pi^{-1}(\tau_1), \mu_2 \in \pi^{-1}(\tau_2)\}.$$

In general, this gives a pseudo-distance. If we set $\tau_1 = [g_1]$ and $\tau_2 = [g_2]$ for $g_1, g_2 \in \text{QS}$, then this can be written as

$$\begin{aligned} d_T([g_1], [g_2]) &= \inf \{\log K(\tilde{g}_1 \circ \tilde{g}_2^{-1}) \mid \tilde{g}_1 \in q^{-1}(g_1), \tilde{g}_2 \in q^{-1}(g_2)\} \\ &= \inf \{\log K(\tilde{g}) \mid \tilde{g} \in q^{-1}(g_1 \circ g_2^{-1})\}, \end{aligned}$$

where $K(\cdot)$ stands for the maximal dilatation.

Due to the compactness property of normalized K -quasiconformal homeomorphisms, we see that the above infimum is actually attained and is a minimum. In addition, by the Weyl lemma which says that a 1-quasiconformal homeomorphism is conformal (holomorphic), we can prove the requirement for the pseudo-distance d_T to be a distance: $d_T(\tau_1, \tau_2) = 0$ implies $\tau_1 = \tau_2$. Furthermore, T is complete with respect to this Teichmüller distance d_T . Since the right translation map is an isometric automorphism of $\text{Bel}(\mathbb{D})$, the base point change map R_τ for $\tau \in T$ is an isometric automorphism of T .

Proposition 3.2. *The universal Teichmüller space (T, d_T) is a contractible metric space.*

Proof. The contractibility follows from the contractibility of the unit ball $\text{Bel}(\mathbb{D})$ of $L^\infty(\mathbb{D})$ and the continuity of the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ and its section $s_{\text{DE}} : T \rightarrow \text{Bel}(\mathbb{D})$. \square

The universal Teichmüller space has a complex structure modeled on a certain complex Banach space. This is done through the *Bers embedding* defined as follows.

For $\mu \in \text{Bel}(\mathbb{D})$, we extend $\mu(z)$ to $\widehat{\mathbb{C}}$ by setting $\mu(z) \equiv 0$ for $z \in \mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$. By Theorem 3.1, there exists a quasiconformal automorphism f_μ of $\widehat{\mathbb{C}}$ whose complex dilatation coincides with the extended Beltrami coefficient μ . Take the Schwarzian derivative

$$S_f(z) = \left\{ \frac{f''(z)}{f'(z)} \right\}' - \frac{1}{2} \left\{ \frac{f''(z)}{f'(z)} \right\}^2$$

of the conformal homeomorphism $f(z) = f_\mu|_{\mathbb{D}^*}(z)$ on \mathbb{D}^* . Note that f_μ is only defined up to post-composition by a Möbius transformation of $\widehat{\mathbb{C}}$, but taking the Schwarzian derivative ignores this ambiguity because of the identity $S_{h \circ f}(z) = S_f(z)$ for every $h \in \text{Möb}(\widehat{\mathbb{C}})$.

We equip the Banach space of holomorphic functions on \mathbb{D}^* with the finite hyperbolic supremum norm, that is:

$$B(\mathbb{D}^*) = \{ \varphi \in \text{Hol}(\mathbb{D}^*) \mid \|\varphi\|_\infty = \sup_{z \in \mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z) |\varphi(z)| < \infty \},$$

where $\rho_{\mathbb{D}^*}(z) = 2/(|z|^2 - 1)$ is the hyperbolic density on \mathbb{D}^* . The Nehari-Kraus theorem says that $\|\varphi\|_\infty \leq 3/2$ for the Schwarzian derivative $\varphi(z) = S_f(z)$ of any conformal homeomorphism f of \mathbb{D}^* . If $f = f_\mu|_{\mathbb{D}^*}$ as above, then the strict inequality $\|\varphi\|_\infty < 3/2$ holds. We have a map $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$ by the correspondence of $\mu \in \text{Bel}(\mathbb{D})$ to $S_{f_\mu|_{\mathbb{D}^*}}$, which is called the *Bers projection*. We will see later that the image of Φ is a bounded domain of $B(\mathbb{D}^*)$.

Now we have two projections from $\text{Bel}(\mathbb{D})$: the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ and the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$. It can be proved that $\pi(\mu_1) = \pi(\mu_2)$ if and only if $\Phi(\mu_1) = \Phi(\mu_2)$. Therefore we have a well-defined injection $\beta : T \rightarrow B(\mathbb{D}^*)$ that satisfies $\beta \circ \pi = \Phi$. This is called the *Bers embedding* of the universal Teichmüller space T .

The Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$ is continuous. Indeed, an improvement of the Nehari-Kraus theorem for a conformal homeomorphism that is quasiconformally extendable to the Riemann sphere gives an estimate $\|\Phi(\mu)\|_\infty \leq 3\|\mu\|_\infty/2$. For arbitrary two points $\mu, \nu \in \text{Bel}(\mathbb{D})$, applying the right translation map r_ν to μ , we transfer the situation to the previous case to obtain

$$\|\Phi(\mu) - \Phi(\nu)\|_\infty \leq 3 \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_\infty.$$

In fact, the Bers projection Φ is holomorphic. Once we have Φ is continuous, then the holomorphy is a consequence of the holomorphic dependence of the normalized solution $f_\mu(z)$ of the Beltrami equation on a Beltrami coefficient μ , which comes from the arguments for the measurable Riemann mapping theorem due to Ahlfors and Bers [1]. In addition, the following result was first proved by Bers [7].

Theorem 3.3. *The Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$ is a holomorphic submersion.*

The condition for Φ to be a holomorphic submersion is equivalent to the existence of a local holomorphic section σ for Φ at every $\varphi \in \Phi(\text{Bel}(\mathbb{D}))$ that maps φ

to an arbitrary $\mu \in \Phi^{-1}(\varphi)$. This implies that Φ is an open map and in particular the image $\Phi(\text{Bel}(\mathbb{D}))$ in $B(\mathbb{D}^*)$ is open (hence it is a bounded domain).

A holomorphic local section of Φ at the origin $0 \in B(\mathbb{D}^*)$ can be given explicitly in the following form by Ahlfors and Weill [2].

Theorem 3.4. *Let $V_0(1/2)$ be the open ball of the Banach space $B(\mathbb{D}^*)$ centered at the origin with radius $1/2$. For every $\varphi \in V_0(1/2)$, set*

$$\sigma_0(\varphi)(z) = -2\rho_{\mathbb{D}^*}^{-2}(z^*)(zz^*)^2\varphi(z^*).$$

Then $\mu(z) = \sigma_0(\varphi)(z)$ belongs to $\text{Bel}(\mathbb{D})$ and satisfies $\Phi(\mu) = \varphi$. Here $z^ = 1/\bar{z} \in \mathbb{D}^*$ is the reflection of $z \in \mathbb{D}$ with respect to \mathbb{S} . Hence $\sigma_0 : V_0(1/2) \rightarrow B(\mathbb{D}^*)$ is a holomorphic local section of Φ at 0.*

Since both π and Φ are continuous and have continuous sections, the Bers embedding $\beta = \Phi \circ \pi^{-1} : T \rightarrow B(\mathbb{D}^*)$ is a homeomorphism onto the image $\beta(T) = \Phi(\text{Bel}(\mathbb{D}))$. By identifying T with a bounded domain $\beta(T) \subset B(\mathbb{D}^*)$, we provide T with the complex structure. Then the base point change map R_τ for every $\tau \in T$ is a biholomorphic automorphism of T . Indeed, for an arbitrary point $\varphi \in \beta(T)$, take a local holomorphic section σ of Φ . Also, take $\nu \in \text{Bel}(\mathbb{D})$ such that $\pi(\nu) = \tau$. Represent R_τ at $\beta^{-1}(\varphi)$ by

$$R_\tau = \beta^{-1} \circ \Phi \circ r_\nu \circ \sigma \circ \beta.$$

Since $\Phi \circ r_\nu \circ \sigma$ is holomorphic, this shows that R_τ is holomorphic. By $R_\tau^{-1} = R_{\tau^{-1}}$, R_τ^{-1} is also holomorphic, namely, R_τ is biholomorphic.

4 Quasisymmetric functions on the real line

Quasisymmetry can be also defined by using quasisymmetric quotients. We introduce a quasisymmetric function on the real line having a uniformly bounded quasisymmetric quotient and show that the lift of our quasisymmetric automorphism of the circle to the real line is precisely a quasisymmetric function. We also give a relationship between certain quantities characterizing quasisymmetry. The advantage of considering quasisymmetric functions on the real line is to apply the canonical quasiconformal extension due to Beurling and Ahlfors to the upper half-plane, which is convenient for the computation of the complex dilatation.

Definition. An increasing (continuous) bijection $h : \mathbb{R} \rightarrow \mathbb{R}$ is called an \widetilde{M} -quasisymmetric function if there exists a constant $\widetilde{M} \geq 1$ such that

$$\frac{1}{\widetilde{M}} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq \widetilde{M}$$

holds for every $x \in \mathbb{R}$ and for every $t > 0$. The ratio in the middle term is called the *quasisymmetric quotient* of h and is denoted by $m_h(x, t)$.

By setting $h(\infty) = \infty$, we may regard a quasisymmetric function h as an orientation-preserving self-homeomorphism of the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. If h is an M -quasisymmetric automorphism of $\widehat{\mathbb{R}}$ in the previous sense, then h is an M -quasisymmetric function on \mathbb{R} since

$$[x-t, x, x+t, \infty]_* = 1; \quad [h(x-t), h(x), h(x+t), h(\infty)]_* = m_h(x, t).$$

Conversely, we will see later that every quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ is a quasisymmetric automorphism of $\widehat{\mathbb{R}}$ with $h(\infty) = \infty$.

Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be a K -quasiconformal automorphism of the upper half-plane \mathbb{H} . By Proposition 2.1 (applied after conjugation by a Möbius transformation $\mathbb{D} \rightarrow \mathbb{H}$), it extends to a $\lambda(K)$ -quasisymmetric automorphism \widehat{f} of $\widehat{\mathbb{R}}$. If $\widehat{f}(\infty) = \infty$, then $h = \widehat{f}|_{\mathbb{R}}$ is a $\lambda(K)$ -quasisymmetric function on \mathbb{R} by the same reason as above. Conversely, every quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ extends continuously to a quasiconformal automorphism of \mathbb{H} . This can be done as follows..

For a quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$, set

$$\alpha(x, y) = \int_0^1 h(x+ty)dt; \quad \beta(x, y) = \int_0^1 h(x-ty)dt$$

and define

$$F_h^{(r)}(z) = \frac{1}{2}\{\alpha(x, y) + \beta(x, y)\} + \frac{ir}{2}\{\alpha(x, y) - \beta(x, y)\}$$

for $z = x + iy \in \mathbb{H}$, which is called the *Beurling-Ahlfors extension* of h with parameter $r > 0$. The following result was originally given in [8] and the estimate for the maximal dilatation K has been improved in Lehtinen [18] among others (In fact, $K \leq \min\{\widetilde{M}^{3/2}, 2\widetilde{M} - 1\}$ is obtained).

Theorem 4.1. *For an \widetilde{M} -quasisymmetric function h of \mathbb{R} , its Beurling-Ahlfors extension $F_h^{(r)}$ is a K -quasiconformal automorphism of \mathbb{H} whose boundary extension is h . Here K depends only on \widetilde{M} and r , and it can be estimated as $K \leq \widetilde{M}^2$ for some suitable choice of r .*

This in particular shows that an \widetilde{M} -quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ is a $\lambda(\widetilde{M}^2)$ -quasisymmetric automorphism of $\widehat{\mathbb{R}}$ with $h(\infty) = \infty$. Also, conjugation by a Möbius transformation $\mathbb{D} \rightarrow \mathbb{H}$ yields the following consequence, which may have an advantage for an estimate of the maximal dilatation over the conformally natural extension.

Corollary 4.2. *Every M -quasisymmetric automorphism $g \in \text{QS}$ extends continuously to an M^2 -quasiconformal automorphism of \mathbb{D} .*

Proof. Conjugation of g by some Möbius transformation $\mathbb{D} \rightarrow \mathbb{H}$ yields an M -quasisymmetric automorphism h of $\widehat{\mathbb{R}}$ with $h(\infty) = \infty$. This is in particular an M -quasisymmetric function on \mathbb{R} . Then there is an M^2 -quasiconformal extension of h by Theorem 4.1, which turns out to be an M^2 -quasiconformal automorphism of \mathbb{D} by conjugation. \square

Let $g : \mathbb{S} \rightarrow \mathbb{S}$ be an orientation-preserving self-homeomorphism. Take a lift \tilde{g} of g to the universal cover $u : \mathbb{R} \rightarrow \mathbb{S}$ given by $u(x) = e^{2\pi ix}$, that is, $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ is the uniquely determined continuous function with $u \circ \tilde{g} = g \circ u$ up to additive constants. Clearly \tilde{g} satisfies $\tilde{g}(x+1) = \tilde{g}(x) + 1$.

Lemma 4.3. *For a quasisymmetric automorphism $g \in \text{QS}$, its lift $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ is a quasisymmetric function on \mathbb{R} . Moreover, if g is M -quasisymmetric, then there is a Möbius transformation $\phi \in \text{Möb}(\mathbb{S})$ such that the lift $\tilde{\phi \circ g}$ of $\phi \circ g$ is a $\lambda(M^2)$ -quasisymmetric automorphism of $\widehat{\mathbb{R}}$ with $\tilde{\phi \circ g}(\infty) = \infty$ and hence $\tilde{\phi \circ g}$ is a $\lambda(M^2)$ -quasisymmetric function on \mathbb{R} .*

Proof. By Corollary 4.2, an M -quasisymmetric automorphism $g \in \text{QS}$ extends continuously to an M^2 -quasiconformal automorphism f of \mathbb{D} . We modify f in the following two ways to make it fix the origin $0 \in \mathbb{D}$. (1) Take a quasiconformal extension $j \in \text{QC}(\mathbb{D})$ of the identity $\text{id}|_{\mathbb{S}}$ such that $j(f(0)) = 0$; (2) Take a Möbius transformation $\phi \in \text{Möb}(\mathbb{D})$ such that $\phi(f(0)) = 0$. In both cases, we consider the lifts \tilde{f}_1 and \tilde{f}_2 of the quasiconformal automorphisms $j \circ f$ and $\phi \circ f$ respectively under the universal cover $u : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$ defined by $u(z) = e^{2\pi iz}$. Note that since $\phi \circ f$ is M^2 -quasiconformal, \tilde{f}_2 is an M^2 -quasiconformal automorphism of \mathbb{H} . By Proposition 2.1, the boundary extensions \tilde{g}_1 and \tilde{g}_2 of \tilde{f}_1 and \tilde{f}_2 are quasisymmetric automorphisms of $\widehat{\mathbb{R}}$ and \tilde{g}_2 is $\lambda(M^2)$ -quasisymmetric. Here $\tilde{g}_1|_{\mathbb{R}}$ is the lift of g and $\tilde{g}_2|_{\mathbb{R}}$ is the lift of $\phi \circ g$ under the universal cover $u : \mathbb{R} \rightarrow \mathbb{S}$. \square

The converse statement of this lemma is also true and actually the quasisymmetry of the lift of $g : \mathbb{S} \rightarrow \mathbb{S}$ can characterize that of g itself.

Theorem 4.4. *An orientation-preserving self-homeomorphism $g : \mathbb{S} \rightarrow \mathbb{S}$ is quasisymmetric if and only if its lift $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ is quasisymmetric. Moreover, if \tilde{g} is M -quasisymmetric, then $\phi \circ g$ is $\lambda(M^2)$ -quasisymmetric for every $\phi \in \text{Möb}(\mathbb{S})$.*

Proof. For an \tilde{M} -quasisymmetric function \tilde{g} on \mathbb{R} satisfying $\tilde{g}(x+1) = \tilde{g}(x) + 1$, its Beurling-Ahlfors extension gives an \tilde{M}^2 -quasiconformal automorphism f of \mathbb{H} by Theorem 4.1. Moreover, it satisfies $f(z+1) = f(z) + 1$ due to the form of the Beurling-Ahlfors extension. Then f projects down by $u : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$ to an \tilde{M}^2 -quasiconformal automorphism f of $\mathbb{D} - \{0\}$. Since f extends continuously to 0 by $f(0) = 0$, f is actually an \tilde{M}^2 -quasiconformal automorphism of \mathbb{D} . Moreover, $\phi \circ f$ is also \tilde{M}^2 -quasiconformal for every Möbius transformation $\phi \in \text{Möb}(\mathbb{D})$. The boundary extension $q(\phi \circ f)$ is $\lambda(\tilde{M}^2)$ -quasisymmetric, which coincides with $\phi \circ g$. \square

As an application of this theorem, we characterize the quasisymmetry of $g : \mathbb{S} \rightarrow \mathbb{S}$ by using just three points on \mathbb{S} . Any positively ordered triple of distinct points $z_1, z_2, z_3 \in \mathbb{S}$ with an equal interval can be represented by $z_1 = e^{2\pi i(x-t)}$, $z_2 = e^{2\pi ix}$ and $z_3 = e^{2\pi i(x+t)}$ for some $x \in \mathbb{R}$ and $0 < t < 1/2$. Set

$$\ell_g(e^{2\pi ix}, t) = \frac{\ell(\overline{g(e^{2\pi ix})g(e^{2\pi i(x+t)})})}{\ell(\overline{g(e^{2\pi i(x-t)})g(e^{2\pi ix})})},$$

where $\ell(\overline{z_1 z_2})$ is the length of the arc in \mathbb{S} with initial point z_1 and terminal point z_2 . We consider such $g : \mathbb{S} \rightarrow \mathbb{S}$ that $\ell_g(e^{2\pi i x}, t)$ is uniformly bounded above and below. To be more precise, define

$$L(g) = \sup\{\ell_g(e^{2\pi i x}, t), \ell_g(e^{2\pi i x}, t)^{-1} \mid x \in \mathbb{R}, 0 < t < 1/2\}.$$

Correspondingly, for the quasisymmetric quotient $m_{\tilde{g}}(x, t)$ of the lift $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ of g , set

$$\widetilde{M}(\tilde{g}) = \sup\{m_{\tilde{g}}(x, t), m_{\tilde{g}}(x, t)^{-1} \mid x \in \mathbb{R}, t > 0\}.$$

Proposition 4.5. *A quasisymmetric automorphism g of \mathbb{S} and its lift \tilde{g} , which is a quasisymmetric function on \mathbb{R} , satisfy $L(g) = \widetilde{M}(\tilde{g})$.*

Proof. Since $m_{\tilde{g}}(x, t) = \ell_g(e^{2\pi i x}, t)$ for any $x \in \mathbb{R}$ and $0 < t < 1/2$, we have $\widetilde{M}(\tilde{g}) \geq L(g)$. For the converse inequality, we will prove that $m_{\tilde{g}}(x, t)^{\pm 1} \leq L(g)$ by induction.

Clearly $m_{\tilde{g}}(x, t)^{\pm 1} = \ell_g(e^{2\pi i x}, t)^{\pm 1} \leq L(g)$ for $0 < t < 1/2$. By taking the limit as $t \nearrow 1/2$, we also have $m_{\tilde{g}}(x, 1/2)^{\pm 1} \leq L(g)$. Hence $m_{\tilde{g}}(x, t)^{\pm 1} \leq L(g)$ is true for $0 < t \leq 1/2$. Assume that $m_{\tilde{g}}(x, t)^{\pm 1} \leq L(g)$ is true for $0 < t \leq 2^{n-1}$ with $n \in \mathbb{N} \cup \{0\}$. Note that \tilde{g} satisfies $\tilde{g}(x+1) = \tilde{g}(x) + 1$. Then, by using

$$\begin{aligned} \tilde{g}(x+t) - \tilde{g}(x) &= \{\tilde{g}(x+t) - \tilde{g}(x+2^{n-1})\} + \{\tilde{g}(x+2^{n-1}) - \tilde{g}(x)\} \\ &= \{\tilde{g}(x+t-2^n) - \tilde{g}(x-2^{n-1})\} + \{\tilde{g}(x+2^{n-1}) - \tilde{g}(x)\}; \\ \tilde{g}(x) - \tilde{g}(x-t) &= \{\tilde{g}(x) - \tilde{g}(x-2^{n-1})\} + \{\tilde{g}(x-2^{n-1}) - \tilde{g}(x-t)\}, \end{aligned}$$

we have

$$\begin{aligned} m_{\tilde{g}}(x, t) &= \frac{\{\tilde{g}(x+2^{n-1}) - \tilde{g}(x)\} + \{\tilde{g}((x-2^{n-1}) + (t-2^{n-1})) - \tilde{g}(x-2^{n-1})\}}{\{\tilde{g}(x) - \tilde{g}(x-2^{n-1})\} + \{\tilde{g}(x-2^{n-1}) - \tilde{g}((x-2^{n-1}) - (t-2^{n-1}))\}} \\ &\leq L(g) \end{aligned}$$

for any t with $2^{n-1} < t \leq 2^n$. The lower estimate of $m_{\tilde{g}}(x, t)$ is similar and hence $\widetilde{M}(\tilde{g}) \leq L(g)$ is obtained. \square

Remark. Suppose that $\ell_g(e^{2\pi i x}, t)^{\pm 1}$ for $0 < t < 1/2$ is dominated by some increasing function $1 + \varepsilon(t)$. Then $m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ holds for $t > 0$. Indeed, this is obvious for $0 < t < 1/2$. Since $L(g) = \widetilde{M}(\tilde{g})$ by Proposition 4.5, we have

$$m_{\tilde{g}}(x, t)^{\pm 1} \leq \widetilde{M}(\tilde{g}) = L(g) \leq 1 + \varepsilon(1/2) \quad (t > 0).$$

Therefore $m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ holds also for $t \geq 1/2$.

By Theorem 4.4 and Proposition 4.5, we see that an orientation-preserving self-homeomorphism g of \mathbb{S} is quasisymmetric if and only if $L(g) \geq 1$ is bounded. If we define the infimum of the constant $M \geq 1$ to be $M(g)$ for which any positively ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \mathbb{S}$ with $[z_1, z_2, z_3, z_4]_* = 1$ satisfies

$$\frac{1}{M} \leq [g(z_1), g(z_2), g(z_3), g(z_4)]_* \leq M,$$

then a quantitative version of the above characterization can be formulated as follows.

Proposition 4.6. *An orientation-preserving self-homeomorphism $g : \mathbb{S} \rightarrow \mathbb{S}$ satisfies*

$$\sqrt{\lambda^{-1}(M(g))} \leq \inf \{L(\phi \circ g) \mid \phi \in \text{Möb}(\mathbb{S})\} \leq \lambda(M(g)^2).$$

Proof. By Proposition 4.5, $L(g)$ can be replaced by $\widetilde{M}(\widetilde{g})$. Theorem 4.4 implies that $M(g) \leq \lambda(\widetilde{M}(\widetilde{g})^2)$. This gives the first inequality even if we take the infimum over all post-composition of $\text{Möb}(\mathbb{S})$. The second inequality follows from the latter statement of Lemma 4.3. \square

5 Symmetric automorphisms and functions

If the quasisymmetric quotient $m_h(x, t)$ of a quasisymmetric function h uniformly tends to 1 as $t \rightarrow 0$, we call h symmetric. As the corresponding concept for quasiconformal maps, there are asymptotically conformal maps whose complex dilatations vanish at the boundary. In this section, we review the relation of these two maps, symmetric and asymptotically conformal. Especially, a certain quantitative estimate of the complex dilatation of the quasiconformal extension in terms of the quasisymmetric quotient is given, which is originally due to Carleson [9].

Definition. A quasisymmetric automorphism $g \in \text{QS}$ is called *symmetric* if there exists a non-negative increasing function $\varepsilon(t)$ for $t > 0$ with $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ such that $\ell_g(e^{2\pi i x}, t)^{\pm 1} \leq 1 + \varepsilon(t)$ for all $x \in \mathbb{R}$. Equivalently, $m_{\widetilde{g}}(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ holds for the lift $\widetilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ of g . We denote the set of all symmetric automorphisms of \mathbb{S} by Sym .

We will see later that Sym is a subgroup of QS .

A quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ is also called symmetric if there exists a non-negative increasing function $\varepsilon(t)$ for $t > 0$ with $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ such that $m_h(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ for all $x \in \mathbb{R}$. We call such a function $\varepsilon(t)$ the gauge for h . Concerning the Beurling-Ahlfors extension of symmetric functions, there is a fundamental result by Carleson [9, Lemma 3]. We improve the result slightly by computing explicitly the constants involved.

Theorem 5.1. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric function such that $m_h(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ for the gauge function $\varepsilon(t)$. Let $f(z) = F_h^{(2)}(z)$ be the Beurling-Ahlfors extension with parameter $r = 2$, which is a quasiconformal automorphism of \mathbb{H} . Then the complex dilatation μ_f of f satisfies $|\mu_f(z)| \leq 4\varepsilon(y)$ for every $z = x + iy \in \mathbb{H}$.*

Proof. Let $H(t)$ be a primitive function of $h(t)$. Then

$$\begin{aligned} \alpha(x, y) &= \int_0^1 h(x + ty) dt = \frac{1}{y} \int_0^y h(x + s) ds = \frac{1}{y} (H(x + y) - H(x)); \\ \beta(x, y) &= \int_0^1 h(x - ty) dt = \frac{1}{y} \int_0^y h(x - s) ds = \frac{1}{y} (H(x) - H(x - y)). \end{aligned}$$

Their partial derivatives with respect to x are

$$\begin{aligned}\alpha_x(x, y) &= \frac{1}{y}(h(x+y) - h(x)); \\ \beta_x(x, y) &= \frac{1}{y}(h(x) - h(x-y)) = \alpha_x(x, y)m_h(x, y)^{-1}.\end{aligned}$$

Hence they satisfy

$$(1 - \varepsilon(y))\alpha_x(x, y) \leq \beta_x(x, y) \leq (1 + \varepsilon(y))\alpha_x(x, y).$$

On the other hand, concerning the partial derivative with respect to y , we have

$$\begin{aligned}\alpha_y(x, y) &= \frac{1}{y}h(x+y) - \frac{1}{y^2} \int_0^y h(x+s)ds \\ &= \frac{1}{y}(h(x+y) - h(x)) - \frac{1}{y} \int_0^1 (h(x+ty) - h(x))dt \\ &= \alpha_x(x, y) \left\{ 1 - \int_0^1 \psi(t)dt \right\},\end{aligned}$$

where we set

$$\psi(t) = \frac{1}{\alpha_x(x, y)y}(h(x+ty) - h(x)).$$

This is a continuous increasing function with $\psi(0) = 0$ and $\psi(1) = 1$. Moreover, the dyadic decomposition of $[0, 1]$ and the quasisymmetry of h yield

$$\left(\frac{1}{2} - \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^k \leq \psi((n+1)2^{-k}) - \psi(n2^{-k}) \leq \left(\frac{1}{2} + \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^k$$

for any $k \in \mathbb{N}$ and for any $n = 0, 1, \dots, 2^k - 1$.

We will show that the above function ψ satisfies

$$-\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right)^{-1} \leq \psi(t) - t \leq \varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right).$$

For each $t \in [0, 1]$, its dyadic expansion $t = \sum_{k=1}^{\infty} \tau_k 2^{-k}$ ($\tau_k = 0, 1$) and the above inequalities for ψ give

$$\sum_{k=1}^{\infty} \tau_k \left(\frac{1}{2} - \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^k \leq \psi(t) \leq \sum_{k=1}^{\infty} \tau_k \left(\frac{1}{2} + \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^k.$$

Then the upper estimate is obtained as follows:

$$\begin{aligned}\psi(t) - t &\leq \sum_{k=1}^{\infty} \tau_k \left\{ \left(\frac{1}{2} + \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^k - \left(\frac{1}{2} \right)^k \right\} \\ &\leq \sum_{k=1}^{\infty} \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \sum_{i+j=k-1} \left(\frac{1}{2} + \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \right)^i \left(\frac{1}{2} \right)^j \\ &\leq \frac{\varepsilon(y)}{2(2 + \varepsilon(y))} \sum_{k=1}^{\infty} k \left(\frac{1 + \varepsilon(y)}{2 + \varepsilon(y)} \right)^{k-1} = \varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right).\end{aligned}$$

Similarly, the lower estimate is obtained as follows:

$$\begin{aligned}
\psi(t) - t &\geq \sum_{k=1}^{\infty} \tau_k \left\{ \left(\frac{1}{2} - \frac{\varepsilon(y)}{2(2+\varepsilon(y))} \right)^k - \left(\frac{1}{2} \right)^k \right\} \\
&\geq \sum_{k=1}^{\infty} \frac{-\varepsilon(y)}{2(2+\varepsilon(y))} \sum_{i+j=k-1} \left(\frac{1}{2} - \frac{\varepsilon(y)}{2(2+\varepsilon(y))} \right)^i \left(\frac{1}{2} \right)^j \\
&\geq \frac{-\varepsilon(y)}{2(2+\varepsilon(y))} \sum_{k=1}^{\infty} k \left(\frac{1}{2} \right)^{k-1} = -\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right)^{-1}.
\end{aligned}$$

The integration of the above inequalities for $\psi(t) - t$ over $t \in [0, 1]$ yields

$$\frac{1}{2} - \varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right)^{-1} \leq \int_0^1 \psi(t) dt \leq \frac{1}{2} + \varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right).$$

Therefore

$$\begin{aligned}
\frac{1}{2} \alpha_x(x, y) \left(1 - 2\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right) \right) &\leq \alpha_y(x, y) \\
&\leq \frac{1}{2} \alpha_x(x, y) \left(1 + 2\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right)^{-1} \right).
\end{aligned}$$

The estimate for $-\beta_y(x, y)$ can be carried out in similar arguments. We represent

$$-\beta_y(x, y) = \beta_x(x, y) \left\{ 1 - \int_0^1 \bar{\psi}(t) dt \right\}; \quad \bar{\psi}(t) = \frac{1}{\beta_x(x, y)y} (h(x) - h(x - ty)).$$

Then we can conclude that

$$\begin{aligned}
\frac{1}{2} \beta_x(x, y) \left(1 - 2\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right) \right) &\leq -\beta_y(x, y) \\
&\leq \frac{1}{2} \beta_x(x, y) \left(1 + 2\varepsilon(y) \left(1 + \frac{\varepsilon(y)}{2} \right)^{-1} \right).
\end{aligned}$$

Finally, we will estimate the complex dilatation $\mu_f(z)$ of

$$f(z) = \frac{1}{2} \{ \alpha(x, y) + \beta(x, y) \} + i \{ \alpha(x, y) - \beta(x, y) \}.$$

By simple calculation,

$$\begin{aligned}
|\mu_f(z)| &= \left| \frac{\left\{ \frac{1}{2}(\alpha_x + \beta_x) - (\alpha_y - \beta_y) \right\} + i \left\{ (\alpha_x - \beta_x) + \frac{1}{2}(\alpha_y + \beta_y) \right\}}{\left\{ \frac{1}{2}(\alpha_x + \beta_x) + (\alpha_y - \beta_y) \right\} + i \left\{ (\alpha_x - \beta_x) - \frac{1}{2}(\alpha_y + \beta_y) \right\}} \right| \\
&\leq \sqrt{2} \frac{\max \left\{ \left| \frac{1}{2}(\alpha_x + \beta_x) - (\alpha_y - \beta_y) \right|, \left| (\alpha_x - \beta_x) + \frac{1}{2}(\alpha_y + \beta_y) \right| \right\}}{\max \left\{ \left| \frac{1}{2}(\alpha_x + \beta_x) + (\alpha_y - \beta_y) \right|, \left| (\alpha_x - \beta_x) - \frac{1}{2}(\alpha_y + \beta_y) \right| \right\}}.
\end{aligned}$$

Here we can replace $\alpha_y = \alpha_y(x, y)$, $\beta_x = \beta_x(x, y)$ and $\beta_y = \beta_y(x, y)$ with $\alpha_x = \alpha_x(x, y)$ by the following estimates when $\varepsilon(y) \leq 1/4$. Using $\beta_x(x, y) = \alpha_x(x, y)m_h(x, y)^{-1}$, we obtain

$$\frac{4}{5}\alpha_x(x, y) \leq \beta_x(x, y) \leq \frac{5}{4}\alpha_x(x, y).$$

Also, the above two inequalities turn out to be

$$\begin{aligned} \frac{1}{2}\alpha_x(x, y) \left(1 - \frac{9}{4}\varepsilon(y)\right) &\leq \alpha_y(x, y) \leq \frac{1}{2}\alpha_x(x, y) (1 + 2\varepsilon(y)); \\ \frac{1}{2}\beta_x(x, y) \left(1 - \frac{9}{4}\varepsilon(y)\right) &\leq -\beta_y(x, y) \leq \frac{1}{2}\beta_x(x, y) (1 + 2\varepsilon(y)). \end{aligned}$$

Thus, for $\varepsilon(y) \leq 1/4$, we have

$$|\mu_f(z)| \leq \sqrt{2} \left(\frac{81\alpha_x(x, y)\varepsilon(y)}{32} \right) / \left(\frac{207\alpha_x(x, y)}{160} \right) \leq 4\varepsilon(y).$$

Since $|\mu_f(z)| < 1$, we see that this is true even if $\varepsilon(y) > 1/4$. \square

In particular, this theorem shows that a symmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ extends continuously to a quasiconformal automorphism $f : \mathbb{H} \rightarrow \mathbb{H}$ with $f(\infty) = \infty$ whose complex dilatation $\mu_f(z)$ tends to 0 as $y \rightarrow 0$ independently of $x \in \mathbb{R}$. More precisely, this means that

$$\text{ess. sup}_{x \in \mathbb{R}, y \leq t} |\mu_f(z)| \rightarrow 0 \quad (t \rightarrow 0).$$

Conversely, such a quasiconformal automorphism f of \mathbb{H} extends to a symmetric function on \mathbb{R} .

Theorem 5.2. *If a quasiconformal automorphism $f : \mathbb{H} \rightarrow \mathbb{H}$ with $f(\infty) = \infty$ satisfies $\mu_f(z) \rightarrow 0$ as $y \rightarrow 0$ independently of $x \in \mathbb{R}$, then its boundary extension $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ is a symmetric function.*

Proof. Suppose that $|\mu_f(z)| \leq \varepsilon(y)$ for a non-negative increasing function $\varepsilon(t)$ with $\lim_{t \rightarrow 0} \varepsilon(t) = 0$. For each t with $0 < t < 1/2$, define a Beltrami coefficient $\mu_t(z)$ by letting $\mu_t(z) = \mu_f(z)$ on $\{z \in \mathbb{H} \mid y > \sqrt{t}\}$ and $\mu_t(z) \equiv 0$ elsewhere. Let f_t be the quasiconformal automorphism of \mathbb{H} with complex dilatation μ_t and with 0, 1 and ∞ fixed, and g_t the quasiconformal automorphism of \mathbb{H} such that $f = g_t \circ f_t$. Since $g_t = f \circ f_t^{-1}$, the complex dilatation of g_t satisfies

$$|\mu_{g_t}(z)| = \frac{|\mu_f(z) - \mu_t(z)|}{|1 - \overline{\mu_t(z)}\mu_f(z)|} \leq \frac{\varepsilon(\sqrt{t})}{1 - \|\mu_f\|_\infty^2}.$$

In particular, the maximal dilatation of g_t is estimated as $K_{g_t} \leq 1 + \varepsilon_1(t)$ for some non-negative increasing function $\varepsilon_1(t)$ with $\lim_{t \rightarrow 0} \varepsilon_1(t) = 0$.

By reflection with respect to \mathbb{R} , we may assume that f_t is a quasiconformal automorphism of \mathbb{C} . The restriction of f_t to the strip domain $\{z \in \mathbb{C} \mid |y| < \sqrt{t}\}$

is conformal. For each $x \in \mathbb{R}$, consider the ball of radius \sqrt{t} with center x and apply the Koebe distortion theorem to the conformal map f_t on this disk. Then, for every ξ in the interval $[x - t, x + t] \subset \mathbb{R}$, we have

$$\frac{1 - \sqrt{t}}{(1 + \sqrt{t})^3} \leq \frac{|f'_t(\xi)|}{|f'_t(x)|} \leq \frac{1 + \sqrt{t}}{(1 - \sqrt{t})^3}.$$

This leads us to the following estimate for the quasisymmetric quotient $m_{f_t}(x, t)$ of f_t restricted to \mathbb{R} :

$$\frac{1 - \sqrt{t}}{(1 + \sqrt{t})^3} \frac{(1 - \sqrt{t})^3}{1 + \sqrt{t}} \leq m_{f_t}(x, t) = \frac{f_t(x + t) - f_t(x)}{f_t(x) - f_t(x - t)} \leq \frac{1 + \sqrt{t}}{(1 - \sqrt{t})^3} \frac{(1 + \sqrt{t})^3}{1 - \sqrt{t}}.$$

In particular, we have a non-negative increasing function $\varepsilon_2(t)$ with $\lim_{t \rightarrow 0} \varepsilon_2(t) = 0$ such that $m_{f_t}(x, t)^{\pm 1} \leq 1 + \varepsilon_2(t)$.

Next, we apply the quasiconformal automorphism g_t to the points $f_t(x - t)$, $f_t(x)$ and $f_t(x + t)$, which are mapped to $f(x - t)$, $f(x)$ and $f(x + t)$ respectively. Note that the quasisymmetric quotients can be given by the conformal moduli through the alternative cross ratios:

$$\begin{aligned} m_{f_t}(x, t) &= [f_t(x - t), f_t(x), f_t(x + t), \infty]_* \\ &= \lambda(\text{mod } \mathbb{H}(f_t(x - t), f_t(x), f_t(x + t), \infty)); \\ m_f(x, t) &= [f(x - t), f(x), f(x + t), \infty]_* \\ &= \lambda(\text{mod } \mathbb{H}(f(x - t), f(x), f(x + t), \infty)). \end{aligned}$$

On the other hand, the ratio of the conformal moduli are bounded by the maximal dilatation $K_{g_t} \leq 1 + \varepsilon_1(t)$:

$$\frac{1}{K_{g_t}} \leq \frac{\text{mod } \mathbb{H}(f(x - t), f(x), f(x + t), \infty)}{\text{mod } \mathbb{H}(f_t(x - t), f_t(x), f_t(x + t), \infty)} \leq K_{g_t}.$$

Plugging the quasisymmetric quotients in this inequality gives

$$\begin{aligned} m_f(x, t) &= \lambda(\text{mod } \mathbb{H}(f(x - t), f(x), f(x + t), \infty)) \\ &\leq \lambda(K_{g_t} \text{mod } \mathbb{H}(f_t(x - t), f_t(x), f_t(x + t), \infty)) \\ &= \lambda(K_{g_t} \lambda^{-1}(m_{f_t}(x, t))) \\ &\leq \lambda\{(1 + \varepsilon_1(t)) \lambda^{-1}(1 + \varepsilon_2(t))\}. \end{aligned}$$

The last term can be represented as $1 + \tilde{\varepsilon}(t)$ for a non-negative increasing function $\tilde{\varepsilon}(t)$ with $\lim_{t \rightarrow 0} \tilde{\varepsilon}(t) = 0$. The lower bound is obtained similarly, which concludes that the boundary extension \bar{f} of f is a symmetric function with gauge function $\tilde{\varepsilon}(t)$. \square

Remark. Carleson [9, Lemma 2] proved that the gauge function $\tilde{\varepsilon}(t)$ can be taken as $\tilde{\varepsilon}(t) = O(\varepsilon(\sqrt{t}))$ ($t \rightarrow 0$). In the above proof, we see that $\varepsilon_1(t) = O(\varepsilon(\sqrt{t}))$ and $\varepsilon_2(t) = O(\sqrt{t})$. Hence, using the behavior of the distortion function $\lambda(K)$ near $K = 1$, we would have a similar estimate for $\tilde{\varepsilon}(t)$.

As in the arguments of the previous section, the universal cover $u : \mathbb{R} \rightarrow \mathbb{S}$ together with the universal cover $u : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$ can transform the symmetric functions on \mathbb{R} to the symmetric automorphisms of \mathbb{S} . We consider the converse. Note that for a given point z in \mathbb{D} , there is a quasiconformal automorphism j with $j(z) = 0$ and $q(j) = \text{id}|_{\mathbb{S}}$ whose complex dilatation vanishes outside some compact subset in \mathbb{D} . Post-composition of j makes the quasiconformal extension of a symmetric automorphism of \mathbb{S} a quasiconformal automorphism of $\mathbb{D} - \{0\}$ and lifts it to a quasiconformal automorphism of the universal cover \mathbb{H} without changing the property of vanishing on \mathbb{R} . Thus we have the following result attributed to Fehlmann [14]. See Gardiner and Sullivan [16].

Corollary 5.3. *A quasisymmetric automorphism $g \in \text{QS}$ is symmetric if and only if g extends continuously to a quasiconformal automorphism f of \mathbb{D} whose complex dilatation μ_f vanishes at the boundary, that is,*

$$\lim_{t \rightarrow 1} \text{ess. sup}_{t < |z|} |\mu_f(z)| = 0.$$

We say that a quasiconformal automorphism $f \in \text{QC}(\mathbb{D})$ is *asymptotically conformal* if the complex dilatation $\mu_f(z)$ vanishes at the boundary. We denote the subset of $\text{QC}(\mathbb{D})$ consisting of all asymptotically conformal automorphisms by $\text{AC}(\mathbb{D})$.

Remark. It was proved by Earle, Markovic and Saric [13] that the conformally natural extension $e_{\text{DE}}(g)$ of a symmetric automorphism $g \in \text{Sym}$ is asymptotically conformal.

By the chain rule of complex dilatations, the composition of asymptotically conformal automorphisms of \mathbb{D} is also asymptotically conformal. Hence $\text{AC}(\mathbb{D})$ is a subgroup of $\text{QC}(\mathbb{D})$. Accordingly, Corollary 5.3 shows that Sym is a subgroup of QS .

6 The small subspace

Before moving to the Teichmüller space of diffeomorphisms of \mathbb{S} , we review here the Teichmüller space of symmetric automorphisms, which is already well-known in the theory of asymptotic Teichmüller spaces. This will be a prototype for our construction of the Teichmüller space of circle diffeomorphisms.

Definition. The *small subspace* T_0 of $T = \text{Möb}(\mathbb{S}) \setminus \text{QS}$ (or the Teichmüller space of symmetric automorphisms) is defined to be $T_0 = \text{Möb}(\mathbb{S}) \setminus \text{Sym}$.

We define the subset $\text{Bel}_0(\mathbb{D})$ of $\text{Bel}(\mathbb{D})$ consisting of all Beltrami coefficients vanishing at the boundary. Since $\text{Möb}(\mathbb{D}) \setminus \text{AC}(\mathbb{D})$ can be identified with $\text{Bel}_0(\mathbb{D})$, Corollary 5.3 implies that the image of $\text{Bel}_0(\mathbb{D})$ under the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ is T_0 . This implies that its Bers embedding is $\beta(T_0) = \Phi(\text{Bel}_0(\mathbb{D}))$ for the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$. Under the group structure $*$ of $\text{Bel}(\mathbb{D})$, $\text{Bel}_0(\mathbb{D})$ is a subgroup. Correspondingly, T_0 is a subgroup of $(T, *)$.

As mentioned in the remark at the end of the previous section, it was shown in [13] that the conformally natural section $s_{DE} : T \rightarrow \text{Bel}(\mathbb{D})$ induced by e_{DE} sends T_0 into $\text{Bel}_0(\mathbb{D})$. Note that $\text{Bel}_0(\mathbb{D})$ is the unit ball of the Banach subspace $L_0^\infty(\mathbb{D}) \subset L^\infty(\mathbb{D})$ consisting of bounded measurable functions vanishing at the boundary. In particular, $\text{Bel}_0(\mathbb{D})$ is contractible. Therefore, T_0 is also contractible as a topological subspace of T (cf. Proposition 3.2).

To consider the complex structure of T_0 , we introduce the Banach subspace $B_0(\mathbb{D}^*)$ of $B(\mathbb{D}^*)$ as follows:

$$B_0(\mathbb{D}^*) = \{\varphi \in B(\mathbb{D}^*) \mid \lim_{|z| \rightarrow 1} \rho_{\mathbb{D}^*}^{-2}(z) |\varphi(z)| = 0\}.$$

An element in $B_0(\mathbb{D}^*)$ is also said to be vanishing at the boundary. The following theorem was given by Becker and Pommerenke [6]. The proof below contains an argument in Gardiner and Sullivan [16].

Theorem 6.1. *For the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$,*

$$\Phi(\text{Bel}_0(\mathbb{D})) = \beta(T) \cap B_0(\mathbb{D}^*)$$

is satisfied.

Proof. For the inclusion $\Phi(\text{Bel}_0(\mathbb{D})) \subset \beta(T) \cap B_0(\mathbb{D}^*)$, we consider a dense subspace $\text{Bel}_c(\mathbb{D})$ of $\text{Bel}_0(\mathbb{D})$ consisting of Beltrami coefficients whose essential support is in some compact subset of \mathbb{D} . We will show that $\Phi(\text{Bel}_c(\mathbb{D})) \subset B_0(\mathbb{D}^*)$. Then, by the continuity of Φ , we have $\Phi(\text{Bel}_0(\mathbb{D})) \subset B_0(\mathbb{D}^*)$.

For every $\mu \in \text{Bel}_c(\mathbb{D})$, take the quasiconformal automorphism f_μ of $\widehat{\mathbb{C}}$ as before. Then, there is some $t > 0$ such that f_μ is conformal on

$$\mathbb{D}_{1-t}^* = \{|z| > 1-t\} \cup \{\infty\}.$$

By the Nehari-Kraus theorem, the Schwarzian derivative of f_μ on \mathbb{D}_{1-t}^* satisfies

$$\rho_{\mathbb{D}_{1-t}^*}^{-2}(z) |S_{f_\mu}(z)| \leq \frac{3}{2}.$$

Here the hyperbolic density of \mathbb{D}_{1-t}^* is given by

$$\rho_{\mathbb{D}_{1-t}^*}(z) = \frac{2(1-t)}{|z|^2 - (1-t)^2} = \rho_{\mathbb{D}^*}(z) \cdot \frac{(|z|^2 - 1)(1-t)}{|z|^2 - (1-t)^2}.$$

Then we see that $\Phi(\mu) \in B_0(\mathbb{D}^*)$ from

$$\rho_{\mathbb{D}^*}^{-2}(z) |S_{f_\mu}(z)| \leq \frac{3}{2} \cdot \left\{ \frac{(|z|^2 - 1)(1-t)}{|z|^2 - (1-t)^2} \right\}^2 \rightarrow 0 \quad (|z| \rightarrow 1).$$

The other inclusion $\Phi(\text{Bel}_0(\mathbb{D})) \supset \beta(T) \cap B_0(\mathbb{D}^*)$ is a consequence of the next lemma, which we can also obtain from Becker [4, Theorem 5.4] and [5, Theorem 3]. Namely, for each $\varphi \in \beta(T) \cap B_0(\mathbb{D}^*)$, the quasiconformal automorphism of $\widehat{\mathbb{C}}$ defined in this lemma is conformal on \mathbb{D}^* with Schwarzian derivative φ and with complex dilatation in $\text{Bel}_0(\mathbb{D})$. \square

Lemma 6.2. *For an arbitrary element $\varphi \in \beta(T) \cap B_0(\mathbb{D}^*)$, take a quasiconformal automorphism f of $\widehat{\mathbb{C}}$ such that f is conformal on \mathbb{D}^* and $S_f(z) = \varphi(z)$ for $z \in \mathbb{D}^*$. Set*

$$F(z) = f(z^*) - \frac{(z^* - z)f'(z^*)}{1 + (z^* - z)f''(z^*)/(2f'(z^*))}$$

for $z \in \mathbb{D}$, where $z^* = 1/\bar{z}$ is the reflection of z with respect to \mathbb{S} . Then there is some $t > 0$ such that $f|_{\mathbb{D}^*}$ extends to a quasiconformal automorphism of $\widehat{\mathbb{C}}$ that coincides with F on the annulus $\{1-t < |z| < 1\}$ and which has complex dilatation

$$\mu_F(z) = \frac{\bar{\partial}F(z)}{\partial F(z)} = -2\rho_{\mathbb{D}^*}^{-2}(z^*)(zz^*)^2 S_f(z^*) = \sigma_0(\varphi)(z).$$

In particular, the complex dilatation of the quasiconformal extension of $f|_{\mathbb{D}^*}$ belongs to $\text{Bel}_0(\mathbb{D})$.

By this theorem, we have $\beta(T_0) = \beta(T) \cap B_0(\mathbb{D}^*)$. Hence T_0 is identified with a bounded contractible domain of the complex Banach space $B_0(\mathbb{D}^*)$.

Next, we consider how T_0 is mapped by the base point change map $R_\tau : T \rightarrow T$ for $\tau \in T$. Since R_τ is a biholomorphic automorphism of T , T_0 is mapped biholomorphically onto the image $R_\tau(T_0)$. Recall that T_0 is a subgroup of $(T, *)$ and R_τ is defined by $R_\tau(\sigma) = \sigma * \tau^{-1}$ for every $\sigma \in T$. By a property of subgroup, we have the following.

Proposition 6.3. *For each $\tau \in T_0$, R_τ is a biholomorphic automorphism of T_0 . Conversely, $R_\tau(T_0) = T_0$ for $\tau \in T$ implies $\tau \in T_0$. Moreover, T is represented as a disjoint union of mutually biholomorphically equivalent subspaces:*

$$T = \bigsqcup_{\tau \in T/T_0} R_\tau(T_0).$$

Remark. Although T is not a topological group since the left translation and the inverse operation are not continuous, the subgroup T_0 is a topological group. Actually, T_0 is a maximal subgroup having this property in a suitable sense. See [16].

The image of the decomposition $T = \bigsqcup_{\tau \in T/T_0} R_\tau(T_0)$ under the Bers embedding $\beta : T \rightarrow B(\mathbb{D}^*)$ gives a foliation of $\beta(T)$ by the family of Banach affine subspaces $\{\psi + B_0(\mathbb{D}^*)\}_\psi$. This is seen from the following theorem, which is a generalization of Theorem 6.1. The fact that $\beta(T) \cap \{\psi + B_0(\mathbb{D}^*)\}$ is contained in $\Phi \circ r_\nu(\text{Bel}_0(\mathbb{D}))$ can be obtained as a consequence of a property of the conformally natural section $s_{\text{DE}} : T \rightarrow \text{Bel}(\mathbb{D})$, which was given by Earle, Markovic and Saric [13].

Theorem 6.4. *For each $\nu \in \text{Bel}(\mathbb{D})$, set $\psi = \Phi(\nu^{-1}) \in B(\mathbb{D}^*)$. Then*

$$\Phi \circ r_\nu(\text{Bel}_0(\mathbb{D})) = \beta(T) \cap \{\psi + B_0(\mathbb{D}^*)\}.$$

Hence $\beta \circ R_\tau(T_0) = \beta(T) \cap \{\beta(\tau^{-1}) + B_0(\mathbb{D}^*)\}$ for every $\tau \in T$.

Proof. We only show that $\Phi \circ r_\nu(\text{Bel}_0(\mathbb{D}))$ is contained in $\psi + B_0(\mathbb{D}^*)$. It suffices to show that $\Phi(\mu * \nu^{-1}) \in \psi + B_0(\mathbb{D}^*)$ for every $\mu \in \text{Bel}_c(\mathbb{D})$ with compact support because $\text{Bel}_c(\mathbb{D})$ is dense in $\text{Bel}_0(\mathbb{D})$ and $\Phi \circ r_\nu$ is continuous. Hereafter we exchange ν and ν^{-1} for the simplicity of notation. Take the quasiconformal automorphisms f_ν and $f_{\mu * \nu}$ of $\widehat{\mathbb{C}}$ with normalization fixing 0, 1 and ∞ whose complex dilatations are 0 on \mathbb{D}^* , ν and $\mu * \nu$ on \mathbb{D} respectively. Set $\Omega = f_\nu(\mathbb{D})$ and $\Omega^* = f_\nu(\mathbb{D}^*)$. Then the quasiconformal automorphism $\widehat{f} = f_{\mu * \nu} \circ f_\nu^{-1}$ is conformal off some compact subset \overline{W} in Ω . Indeed, the complex dilatations $\mu * \nu$ and ν coincides off the compact support of μ . Then \widehat{f} restricted to $W^* = \widehat{\mathbb{C}} - \overline{W}$, which can be assumed to be a simply connected domain containing $\overline{\Omega^*}$, is conformal.

By the Nehari-Kraus theorem, the Schwarzian derivative of \widehat{f} satisfies

$$\rho_{W^*}^{-2}(\zeta) |S_{\widehat{f}}(\zeta)| \leq 3,$$

where ρ denotes the hyperbolic density on the domain in question. On the other hand, since $\partial\Omega^* \subset W^*$ we see that $\lim_{\zeta \rightarrow \partial\Omega^*} \rho_{W^*}(\zeta)/\rho_{\Omega^*}(\zeta) = 0$. Hence

$$\rho_{\Omega^*}^{-2}(\zeta) |S_{\widehat{f}}(\zeta)| = \frac{\rho_{\Omega^*}^{-2}(\zeta)}{\rho_{W^*}^{-2}(\zeta)} \rho_{W^*}^{-2}(\zeta) |S_{\widehat{f}}(\zeta)| \leq 3 \left(\frac{\rho_{W^*}(\zeta)}{\rho_{\Omega^*}(\zeta)} \right)^2 \rightarrow 0 \quad (\zeta \rightarrow \partial\Omega^*).$$

The Cayley identity for the Schwarzian derivative with $S_{f_\nu|_{\mathbb{D}^*}} = \Phi(\nu) = \psi$ gives

$$\Phi(\mu * \nu)(z) = S_{\widehat{f} \circ f_\nu}(z) = S_{\widehat{f}}(f_\nu(z)) f_\nu'(z)^2 + \psi(z)$$

for $z \in \mathbb{D}^*$. For $\varphi(z) = S_{\widehat{f}}(f_\nu(z)) f_\nu'(z)^2$, we have

$$\rho_{\mathbb{D}^*}^{-2}(z) |\varphi(z)| = \rho_{\Omega^*}^{-2}(\zeta) |S_{\widehat{f}}(\zeta)| \quad (\zeta = f_\nu(z)).$$

Thus $\varphi \in B_0(\mathbb{D}^*)$, which shows that $\Phi(\mu * \nu) \in \psi + B_0(\mathbb{D}^*)$. \square

Remark. For the converse inclusion, we propose the following method, which might give a direct construction by using quasiconformal reflection. The quasiconformal automorphism f_ν defines the quasiconformal reflection $\lambda : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with respect to the quasicircle $\partial\Omega = f_\nu(\mathbb{S})$ by $\lambda(\zeta) = f_\nu(f_\nu^{-1}(\zeta)^*)$. We may assume that f_ν is diffeomorphic on \mathbb{D} and hence λ is diffeomorphic off $\partial\Omega$. Take an arbitrary $\varphi \in B_0(\mathbb{D}^*)$ with $\psi + \varphi \in \beta(T)$. Then there is a quasiconformal automorphism f of $\widehat{\mathbb{C}}$ whose restriction to Ω^* is conformal and whose Schwarzian derivative satisfies $S_f(f_\nu(z)) f_\nu'(z)^2 = \varphi(z)$ for $z \in \mathbb{D}^*$. In this situation, as in Lemma 6.2, consider the function

$$F(\zeta) = f(\lambda(\zeta)) - \frac{(\lambda(\zeta) - \zeta) f'(\lambda(\zeta))}{1 + (\lambda(\zeta) - \zeta) f''(\lambda(\zeta)) / (2f'(\lambda(\zeta)))}$$

for $\zeta \in \Omega$, which is a continuous extension of $f|_{\Omega^*}$ beyond $\partial\Omega$. By an estimate of $|\lambda(\zeta) - \zeta|$ in terms of the hyperbolic density (see Lehto [19, Section II.4.1]) and by the vanishing of φ at the boundary, it can be shown that $\mu_F(\zeta) \rightarrow 0$ as $\zeta \rightarrow \partial\Omega$. We expect that F should be a quasiconformal homeomorphism in some neighborhood of $\partial\Omega$. Here is the problem. If this is true, then a quasiconformal extension of F to Ω gives a required extension of $f|_{\Omega^*}$.

By Theorem 6.4, we have the following decomposition of the Bers embedding:

$$\beta(T) = \bigsqcup_{\tau \in T/T_0} \beta \circ R_\tau(T_0) = \bigsqcup_{\psi \in B(\mathbb{D}^*)/B_0(\mathbb{D}^*)} \beta(T) \cap (\psi + B_0(\mathbb{D}^*)).$$

The base point change map R_τ preserves the affine foliation of the Bers embedding by $B_0(\mathbb{D}^*)$. Each component $\beta(T) \cap (\psi + B_0(\mathbb{D}^*))$ is biholomorphically equivalent to T_0 . In particular, it is contractible.

The *asymptotic universal Teichmüller space* AT is defined by $AT = \text{Sym} \setminus \text{QS}$, and it was first introduced by Gardiner and Sullivan [16]. Since $\text{Sym} \subset \text{Möb}(\mathbb{S})$, we have a projection $\alpha : T \rightarrow AT$. With the group structure of T , AT is represented by the set of cosets T/T_0 .

In correspondence with the quotient by Sym , we consider the unit ball $\widehat{\text{Bel}}(\mathbb{D})$ of the quotient Banach space $L^\infty(\mathbb{D})/L_0^\infty(\mathbb{D})$, which coincides with the set of cosets $\text{Bel}(\mathbb{D})/\text{Bel}_0(\mathbb{D})$. Then, since the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$ is a continuous open surjection compatible with the group structure (i.e. homomorphism), it induces a continuous open surjection $\widehat{\pi} : \widehat{\text{Bel}}(\mathbb{D}) \rightarrow AT$.

We also consider the quotient Banach space $\widehat{B}(\mathbb{D}^*) = B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$. Theorem 6.4 implies that the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$ is compatible with the foliated structures. Hence it induces a continuous open map $\widehat{\Phi} : \widehat{\text{Bel}}(\mathbb{D}) \rightarrow \widehat{B}(\mathbb{D}^*)$. Also, the Bers embedding $\beta : T \rightarrow \beta(T) \subset B(\mathbb{D}^*)$ induces a bijection $\widehat{\beta} : AT \rightarrow \widehat{\beta}(AT) \subset \widehat{B}(\mathbb{D}^*)$ satisfying $\widehat{\beta} \circ \widehat{\pi} = \widehat{\Phi}$. Since we see that $\widehat{\beta}$ is homeomorphic and $\widehat{\beta}(AT)$ is a domain in $\widehat{B}(\mathbb{D}^*)$, this provides AT with a complex structure. As the quotient map $B(\mathbb{D}^*) \rightarrow B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$ is holomorphic, the projection $\alpha : T \rightarrow AT$ is also holomorphic.

The base point change map $R_\tau : T \rightarrow T$ for $\tau \in T$ is projected down to the base point change map $R_{\alpha(\tau)} : AT \rightarrow AT$. Since $\alpha : T \rightarrow AT$ is continuous and open, it is easy to see that $R_{\alpha(\tau)}$ is a homeomorphic automorphism. In fact, $R_{\alpha(\tau)}$ is a biholomorphic automorphism. To see this, we regard the differential dR_τ as a complex linear map $B(\mathbb{D}^*) \rightarrow B(\mathbb{D}^*)$ between the tangent spaces of the Bers embedding $\beta(T) \subset B(\mathbb{D}^*)$. Since R_τ preserves the affine foliation of the Bers embedding by $B_0(\mathbb{D}^*)$, the differential dR_τ preserves $B_0(\mathbb{D}^*)$. Hence it descends to a linear map $\widehat{B}(\mathbb{D}^*) \rightarrow \widehat{B}(\mathbb{D}^*)$ which gives the differential of $R_{\alpha(\tau)}$. For details, see Earle, Gardiner and Lakic [12].

7 Diffeomorphisms of the circle with Hölder continuous derivatives

This section is devoted to a set-up for the Teichmüller space of diffeomorphisms of \mathbb{S} with Hölder continuous derivatives. We will establish the relationship among the following three indices: the power of Hölder continuity of the derivative of a circle diffeomorphism; the decay order of the complex dilatation of its quasiconformal extension; and the decay order of the norm of the Schwarzian derivative given by the Bers embedding.

Definition. An orientation-preserving diffeomorphism g of \mathbb{S} belongs to a class $\text{Diff}^{1+\alpha}(\mathbb{S})$ for some α ($0 < \alpha < 1$) if its derivative is α -Hölder continuous. This means that the lift $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ of g satisfies

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x - y|^\alpha \quad (x, y \in \mathbb{R})$$

for some $c \geq 0$.

It is well-known that $\text{Diff}^{1+\alpha}(\mathbb{S})$ is a subgroup of Sym . We will characterize an element g of $\text{Diff}^{1+\alpha}(\mathbb{S})$ in terms of the quasisymmetric quotient of g . An essential part was done by Carleson [9, Lemma 5] and his result can be arranged in the following statement. See also Gardiner and Sullivan [16, Section 9]. Remark that the Hölder continuity of g is defined for its lift \tilde{g} and hence the condition is given for symmetric functions on \mathbb{R} .

Theorem 7.1. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric function with gauge function $\varepsilon(t)$, that is, the quasisymmetric quotient satisfies $m_h(x, t)^{\pm 1} \leq 1 + \varepsilon(t)$ for every $x \in \mathbb{R}$ and for every $t > 0$. If*

$$a = \int_0^1 \frac{\varepsilon(t)}{t} dt < \infty,$$

then h is continuously differentiable. Moreover, if $h(x + 1) = h(x) + 1$ for every $x \in \mathbb{R}$, then the derivative $h'(x)$ is uniformly bounded away from 0 and $+\infty$ and satisfies

$$|h'(x) - h'(y)| \leq A \int_0^{|x-y|} \frac{\varepsilon(t)}{t} dt,$$

where $A > 0$ is a constant depending only on a .

Proof. Fix a constant $\delta > 0$. Take an interval $I_0 \subset \mathbb{R}$ with length $|I_0| = \delta$ arbitrarily. Set $\varepsilon_0(t) = \varepsilon(\delta t)$. Divide I_0 into two equal intervals denoted by I_1 . Continue this process to obtain 4 sub-intervals I_2 , 8 sub-intervals I_3 and so on. Then the quasisymmetry of h gives

$$|h(I_0)| \cdot \frac{1}{2} (1 + \varepsilon_0(1/2))^{-1} \leq |h(I_1)| \leq |h(I_0)| \cdot \frac{1}{2} (1 + \varepsilon_0(1/2)).$$

Similarly, we have

$$|h(I_2)| \leq |h(I_1)| \cdot \frac{1}{2} (1 + \varepsilon_0(1/2^2)) \leq |h(I_0)| \cdot \frac{1}{2} (1 + \varepsilon_0(1/2)) \cdot \frac{1}{2} (1 + \varepsilon_0(1/2^2)),$$

and also the lower estimate for $|h(I_2)|$. In general,

$$\begin{aligned} |h(I_n)| &\leq |h(I_0)| \cdot \frac{1}{2^n} \prod_{i=1}^n (1 + \varepsilon_0(1/2^i)) = \frac{|h(I_0)|}{\delta} |I_n| \cdot \prod_{i=1}^n (1 + \varepsilon_0(1/2^i)); \\ |h(I_n)| &\geq |h(I_0)| \cdot \frac{1}{2^n} \prod_{i=1}^n (1 + \varepsilon_0(1/2^i))^{-1} = \frac{|h(I_0)|}{\delta} |I_n| \cdot \prod_{i=1}^n (1 + \varepsilon_0(1/2^i))^{-1}. \end{aligned}$$

To estimate the last term in the above inequalities independently of n , we take the logarithm of the infinite product. Then

$$\begin{aligned} \log \prod_{i=1}^{\infty} (1 + \varepsilon_0 (1/2^i)) &= \sum_{i=1}^{\infty} \log (1 + \varepsilon_0 (1/2^i)) \\ &\leq \sum_{i=1}^{\infty} \varepsilon_0 (1/2^i) = \sum_{i=1}^{\infty} \frac{\delta}{2^i} \varepsilon (\delta/2^i) / \frac{\delta}{2^i} \\ &\leq \int_0^{\delta} \frac{\varepsilon(t)}{t} dt. \end{aligned}$$

We denote the last integral by $\tilde{\varepsilon}(\delta)$. This integral is finite and bounded by $a = \tilde{\varepsilon}(1)$ when $\delta \leq 1$. In particular, $\tilde{\varepsilon}(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. An estimate of the exponential yields

$$\prod_{i=1}^{\infty} (1 + \varepsilon_0 (1/2^i)) \leq 1 + e(\delta)\tilde{\varepsilon}(\delta),$$

where the constant $e(\delta) > 0$ is bounded by $e(1)$ when $\delta \leq 1$, which depends on a .

First we prove the differentiability of h . For an arbitrary $x \in \mathbb{R}$, we choose an interval I so that x is in the interior of I . Take $t > 0$ so that both the closed intervals $[x - t, x]$ and $[x, x + t]$ are in I . Set $|I| = \delta$. Representing $[x, x + t]$ by a union of some dyadic sub-intervals $\{I_n\}$ for I , we have

$$\frac{|h(I)|}{\delta} t(1 + e(\delta)\tilde{\varepsilon}(\delta))^{-1} \leq h(x + t) - h(x) \leq \frac{|h(I)|}{\delta} t(1 + e(\delta)\tilde{\varepsilon}(\delta)).$$

The argument for $[x - t, x]$ is similar and we omit it.

Fix $I = I_0$ and $\delta = \delta_0$. Dividing the above inequalities by t gives

$$\frac{|h(I_0)|}{\delta_0} (1 + e(\delta_0)\tilde{\varepsilon}(\delta_0))^{-1} \leq \frac{h(x + t) - h(x)}{t} \leq \frac{|h(I_0)|}{\delta_0} (1 + e(\delta_0)\tilde{\varepsilon}(\delta_0)).$$

Then by taking $\overline{\lim}$ and $\underline{\lim}$ of the middle term as $t \rightarrow 0$, we see that the upper derivative $\overline{h}'(x)$ and the lower derivative $\underline{h}'(x)$ exist and are bounded by the first and the last terms.

The same estimate holds for an arbitrary interval I with $|I| = \delta$ containing x as an interior point:

$$\frac{|h(I)|}{\delta} (1 + e(\delta)\tilde{\varepsilon}(\delta))^{-1} \leq \underline{h}'(x) \leq \overline{h}'(x) \leq \frac{|h(I)|}{\delta} (1 + e(\delta)\tilde{\varepsilon}(\delta)).$$

Keeping that condition, we move I so that $\delta = |I| \rightarrow 0$. Since $e(\delta)\tilde{\varepsilon}(\delta) \rightarrow 0$, we have

$$\overline{\lim}_{\delta \rightarrow 0} \frac{|h(I)|}{\delta} \leq \underline{h}'(x) \leq \overline{h}'(x) \leq \underline{\lim}_{\delta \rightarrow 0} \frac{|h(I)|}{\delta}.$$

This shows that the derivative $h'(x)$ exists and coincides with $\lim_{\delta \rightarrow 0} |h(I)|/\delta$ which also exists.

Next, we show the continuity of h' . Once we know the existence of the derivative of h , the above inequalities turn out to be

$$\frac{|h(I)|}{\delta}(1 + e(\delta)\tilde{\varepsilon}(\delta))^{-1} \leq h'(x), \quad h'(y) \leq \frac{|h(I)|}{\delta}(1 + e(\delta)\tilde{\varepsilon}(\delta))$$

for any interior points x and y of I with $|I| = \delta$. From this, we have

$$|h'(x) - h'(y)| \leq \frac{|h(I)|}{\delta}(1 + e(\delta)\tilde{\varepsilon}(\delta)) - \frac{|h(I)|}{\delta}(1 + e(\delta)\tilde{\varepsilon}(\delta))^{-1} \leq \frac{|h(I)|}{\delta} \cdot 2e(\delta)\tilde{\varepsilon}(\delta).$$

Fix $x \in \mathbb{R}$ arbitrarily. Take an interval I with $|I| = \delta$ having x as an interior point and then consider an interior point y of I . Then

$$\lim_{y \rightarrow x} |h'(x) - h'(y)| \leq \lim_{\delta \rightarrow 0} \frac{|h(I)|}{\delta} \cdot 2e(\delta)\tilde{\varepsilon}(\delta).$$

Since $\lim_{\delta \rightarrow 0} |h(I)|/\delta = h'(x)$ and $\lim_{\delta \rightarrow 0} e(\delta)\tilde{\varepsilon}(\delta) = 0$, this converges to 0. Therefore h' is continuous.

Finally, under the assumption that $h(x+1) = h(x) + 1$ for every $x \in \mathbb{R}$, we show the uniform boundedness of the derivative h' and we give the modulus of continuity of h' . For every $x \in \mathbb{R}$, take an interval $I = [\xi, \xi + 1]$ having x in its interior. In this case, $\delta = |I| = 1$. Also, the assumption implies $|h(I)| = 1$. The substitution of these values to the obtained inequalities together with $\tilde{\varepsilon}(1) = a$ yields

$$(1 + ae(1))^{-1} \leq h'(x) \leq 1 + ae(1),$$

which says the uniform boundedness of h' .

To verify the modulus of continuity of h' , we take $x, y \in \mathbb{R}$ arbitrarily. Note that $h'(y+1) = h'(y)$ by assumption, which reduces the argument to the case where $|x - y| \leq 1$. Approximating the interval $[x, y]$ by larger intervals I with $|x - y| < \delta = |I|$, we can derive the following inequality from the previous one:

$$|h'(x) - h'(y)| \leq \frac{|h(x) - h(y)|}{|x - y|} \cdot 2e(|x - y|)\tilde{\varepsilon}(|x - y|).$$

Here the mean value theorem shows that for some $\eta \in [x, y]$ we have

$$\frac{|h(x) - h(y)|}{|x - y|} = h'(\eta) \leq 1 + ae(1).$$

Hence

$$|h'(x) - h'(y)| \leq 2(1 + ae(1))e(1) \int_0^{|x-y|} \frac{\varepsilon(t)}{t} dt.$$

Setting $A = 2(1 + ae(1))e(1)$ completes the proof. \square

Corollary 7.2. Fix $\alpha \in (0, 1)$. Suppose that for some $b \geq 0$, the lift \tilde{g} of $g \in \text{Sym}$ satisfies

$$(1 + bt^\alpha)^{-1} \leq m_{\tilde{g}}(x, t) \leq 1 + bt^\alpha$$

for every $x \in \mathbb{R}$ and for every $t > 0$. Then g belongs to $\text{Diff}^{1+\alpha}(\mathbb{S})$. More precisely, \tilde{g} satisfies

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x - y|^\alpha$$

for any $x, y \in \mathbb{R}$, where the constant $c \geq 0$ depends only on b and tends to 0 as $b \rightarrow 0$.

Proof. We apply Theorem 7.1 for the lift \tilde{g} with the gauge function $\varepsilon(t) = bt^\alpha$. Note that \tilde{g} satisfies $\tilde{g}(x+1) = \tilde{g}(x) + 1$. Then

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq A \int_0^{|x-y|} \frac{bt^\alpha}{t} dt = \frac{Ab}{\alpha} |x - y|^\alpha.$$

Set $c = Ab/\alpha$. The proof of Theorem 7.1 shows that A depends only on α and b , and A is bounded when b tends to 0 with α fixed. Then the property of the constant c as in the statement follows. \square

Remark. We can replace the assumption of Corollary 7.2 with

$$(1 + bt^\alpha)^{-1} \leq \ell_g(e^{2\pi i x}, t) \leq 1 + bt^\alpha,$$

as is seen in the remark after Proposition 4.5.

Conversely, every element $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ ($0 < \alpha < 1$) is a symmetric automorphism of \mathbb{S} with gauge function of order $O(t^\alpha)$.

Proposition 7.3. Fix $\alpha \in (0, 1)$. Suppose that for some $c \geq 0$, the lift \tilde{g} of $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ satisfies

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x - y|^\alpha$$

for any $x, y \in \mathbb{R}$. Then there is some $b \geq 0$ such that

$$(1 + bt^\alpha)^{-1} \leq m_{\tilde{g}}(x, t) \leq 1 + bt^\alpha$$

for every $x \in \mathbb{R}$ and for every $t > 0$. Here the constant b can be taken to be dependent only on c when $c \leq 1$ and tends to 0 as $c \rightarrow 0$.

Proof. Since \tilde{g} satisfies $\int_0^1 \tilde{g}'(x) dx = 1$, there exists some $x_0 \in [0, 1]$ such that $\tilde{g}'(x_0) \geq 1$. The Hölder continuity of \tilde{g}' implies that

$$|\tilde{g}'(x) - \tilde{g}'(x_0)| \leq c|x - x_0|^\alpha \leq c(1/2)^\alpha$$

for every $x \in \mathbb{R}$ with $|x - x_0| \leq 1/2$. Then using the periodicity $\tilde{g}'(x+1) = \tilde{g}'(x)$, we have $\tilde{g}'(x) \geq 1 - c(1/2)^\alpha$ for every $x \in \mathbb{R}$. If $c \leq 1$ then $1 - c(1/2)^\alpha > 0$. Even if $1 - c(1/2)^\alpha \leq 0$, there is some $c_0 > 0$ depending on the circle diffeomorphism g such that $\tilde{g}'(x) \geq c_0$ for every $x \in \mathbb{R}$.

The mean value theorem says that there are ξ_+ and ξ_- such that

$$\begin{aligned} g(x+t) - g(x) &= tg'(\xi_+) & (x < \xi_+ < x+t); \\ g(x) - g(x-t) &= tg'(\xi_-) & (x-t < \xi_- < x). \end{aligned}$$

This gives

$$m_{\tilde{g}}(x, t) = 1 + \frac{\tilde{g}'(\xi_+) - \tilde{g}'(\xi_-)}{\tilde{g}'(\xi_-)}; \quad m_{\tilde{g}}(x, t)^{-1} = 1 + \frac{\tilde{g}'(\xi_-) - \tilde{g}'(\xi_+)}{\tilde{g}'(\xi_+)}.$$

Here we have

$$|\tilde{g}'(\xi_+) - \tilde{g}'(\xi_-)| \leq c|\xi_+ - \xi_-|^\alpha \leq c(2t)^\alpha$$

by the Hölder continuity of \tilde{g}' . Combined with the lower estimate of \tilde{g}' , this yields

$$m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + \frac{2^\alpha c}{\max\{1 - c(1/2)^\alpha, c_0\}} t^\alpha.$$

Setting the coefficient of t^α by b , we obtain the statement. \square

Remark. With the same reasoning, the upper estimate $\tilde{g}'(x) \leq 1 + c(1/2)^\alpha$ is obtained in the proof of Proposition 7.3. Now assume that both $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ and its inverse g^{-1} satisfy the α -Hölder continuous derivative condition for a constant $c \geq 0$. In this case, a lower bound of $|\tilde{g}'(x)|$ is given in terms of c without the assumption $c \leq 1$ because

$$|\tilde{g}'(x)| = \frac{1}{|(\tilde{g}^{-1})'(\tilde{g}(x))|} \geq \frac{1}{1 + c(1/2)^\alpha}.$$

Hence the constant b can always be taken depending only on c .

Next we characterize the elements of $\text{Diff}^{1+\alpha}(\mathbb{S})$ by their quasiconformal extension to \mathbb{D} through the order of gauge functions of quasisymmetric quotients. Recall that since $\text{Diff}^{1+\alpha}(\mathbb{S}) \subset \text{Sym}$, the quasiconformal extension is asymptotically conformal. We look at the decay order of its complex dilatation close to the boundary.

Proposition 7.4. *Suppose that for some $b \geq 0$, the lift \tilde{g} of $g \in \text{Sym}$ satisfies*

$$(1 + bt^\alpha)^{-1} \leq m_{\tilde{g}}(x, t) \leq 1 + bt^\alpha$$

for every $x \in \mathbb{R}$ and for every $t > 0$. Then there exists a quasiconformal extension $f \in \text{AC}(\mathbb{D})$ of g whose complex dilatation μ_f satisfies

$$|\mu_f(\zeta)| \leq \min \left\{ \frac{4b}{(2\pi)^\alpha} (-\log |\zeta|)^\alpha, 1 \right\}$$

for every $\zeta \in \mathbb{D}$.

Proof. By Theorem 5.1, the complex dilatation $\mu(z)$ of the Beurling-Ahlfors extension $F_{\tilde{g}}^{(2)}(z)$ of \tilde{g} satisfies $|\mu(z)| \leq 4by^\alpha$ for every $z = x + iy \in \mathbb{H}$. Since $\tilde{g}(x+1) = \tilde{g}(x) + 1$, $F_{\tilde{g}}^{(2)}$ also satisfies $F_{\tilde{g}}^{(2)}(z+1) = F_{\tilde{g}}^{(2)}(z) + 1$. Then the universal cover $u : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$ defined by $z \mapsto \zeta = e^{2\pi iz}$ descends $F_{\tilde{g}}^{(2)}$ to a quasiconformal automorphism f of $\mathbb{D} - \{0\}$. Note that f extends to a quasiconformal automorphism of \mathbb{D} and its boundary extension to \mathbb{S} coincides with g . The complex dilatation μ_f of f satisfies

$$|\mu_f(\zeta)| = |\mu(z)| = |\mu((\log \zeta)/(2\pi i))|.$$

Since $\text{Im}\{(\log \zeta)/(2\pi i)\} = -\log|\zeta|/(2\pi)$, the condition $|\mu(z)| \leq 4by^\alpha$ yields

$$|\mu_f(\zeta)| \leq \frac{4b}{(2\pi)^\alpha} (-\log|\zeta|)^\alpha$$

for every $\zeta \in \mathbb{D}$. □

Propositions 7.3 and 7.4 conclude the following.

Theorem 7.5. *For every $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$, there exists a quasiconformal extension $f \in \text{AC}(\mathbb{D})$ of g whose complex dilatation μ_f satisfies*

$$|\mu_f(\zeta)| \leq d(1 - |\zeta|)^\alpha$$

for some constant $d \geq 0$ and for every $\zeta \in \mathbb{D}$.

Proof. By Proposition 7.3, the lift $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ of g satisfies $m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + bt^\alpha$ for some $b \geq 0$. Then we choose the quasiconformal extension $f \in \text{AC}(\mathbb{D})$ of g as in Proposition 7.4. Since $-\log|\zeta|$ is comparable to $1 - |\zeta|$ near $|\zeta| = 1$, we can find a constant $d \geq 0$ such that $|\mu_f(\zeta)| \leq d(1 - |\zeta|)^\alpha$. □

We will investigate the converse of Theorem 7.5. In Theorem 5.2 and Corollary 5.3, we saw that an asymptotically conformal automorphism $f \in \text{AC}(\mathbb{D})$ extends to a symmetric automorphism $g \in \text{Sym}$ of \mathbb{S} and actually showed a certain estimate of the gauge function for g in terms of the decay order of the complex dilatation μ_f of f . The order of the gauge function and the Hölder continuity of the derivative are related to each other as in Corollary 7.2 and Proposition 7.3. The converse of Theorem 7.5 turns out to be a statement that if the complex dilatation of $f \in \text{AC}(\mathbb{D})$ satisfies

$$|\mu_f(\zeta)| = O((1 - |\zeta|)^\alpha)$$

then the boundary extension $g = q(f)$ belongs to $\text{Diff}^{1+\alpha}(\mathbb{S})$. If we use the result of Carleson [9, Lemma 2] mentioned in the remark after Theorem 5.2, we can obtain a weaker consequence, namely, $g \in \text{Diff}^{1+\alpha/2}(\mathbb{S})$ from the same assumption. To explain the arguments for the full converse statement, we begin with the following definition.

Definition. Let α be a fixed constant with $0 < \alpha < 1$. For a Beltrami coefficient $\mu \in \text{Bel}(\mathbb{D})$, we define a new norm by

$$\|\mu\|_{\infty, \alpha} = \text{ess. sup}_{z \in \mathbb{D}} \rho_{\mathbb{D}}^{\alpha}(z) |\mu(z)|.$$

The space of Beltrami coefficients with finite norm is denoted by

$$\text{Bel}_0^{\alpha}(\mathbb{D}) = \{\mu \in \text{Bel}(\mathbb{D}) \mid \|\mu\|_{\infty, \alpha} < \infty\} \subset \text{Bel}_0(\mathbb{D}).$$

Correspondingly, for a hyperbolically bounded holomorphic function $\varphi \in B(\mathbb{D}^*)$, we define a new norm by

$$\|\varphi\|_{\infty, \alpha} = \sup_{z \in \mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2+\alpha}(z) |\varphi(z)|.$$

The Banach space of hyperbolically bounded holomorphic functions with respect to this norm is

$$B_0^{\alpha}(\mathbb{D}^*) = \{\varphi \in B(\mathbb{D}^*) \mid \|\varphi\|_{\infty, \alpha} < \infty\} \subset B_0(\mathbb{D}^*).$$

Theorem 7.6. *Let α be a constant with $0 < \alpha < 1$. For a quasimetric automorphism $g \in \text{QS}$, the following conditions are equivalent:*

- (1) g belongs to $\text{Diff}^{1+\alpha}(\mathbb{S})$;
- (2) there is $\mu \in \text{Bel}_0^{\alpha}(\mathbb{D})$ such that $\pi(\mu) = [g] \in T$;
- (3) $\beta([g]) \in \beta(T)$ is in $B_0^{\alpha}(\mathbb{D}^*)$.

The implication (1) \Rightarrow (2) is a reformulation of the statement of Theorem 7.5. To give the converse (2) \Rightarrow (1), we will prove (2) \Rightarrow (3) and (3) \Rightarrow (1). Note that the implication (3) \Rightarrow (2) follows from Lemma 6.2 and (1) \Rightarrow (3) will be seen below. We present a sketch of the proof here. Complete arguments will appear elsewhere ([21]).

Outline of the proof of Theorem 7.6. For the implication (2) \Rightarrow (3), a weaker result has been obtained by Becker [5, Theorem 2], which asserts that $\Phi(\mu) \in B_0^{\alpha+\epsilon}(\mathbb{D}^*)$ for every $\mu \in \text{Bel}_0^{\alpha}(\mathbb{D})$ and for every $\epsilon > 0$. The elimination of the constant ϵ leads to our desired result since $\beta \circ \pi = \Phi$. To this end, we decompose a Beltrami coefficient $\mu \in \text{Bel}_0^{\alpha}(\mathbb{D})$ into a finite number of coefficients whose supports are in mutually disjoint concentric annular domains of \mathbb{D} . Then a computation of the Schwarzian derivative of the composition of the corresponding conformal homeomorphisms establishes the estimate.

For the implication (3) \Rightarrow (1), we represent g by conformal welding as follows. First of all, for $\varphi = \beta([g]) \in \beta(T)$, there is a conformal homeomorphism w of \mathbb{D}^* quasiconformally extendable to $\widehat{\mathbb{C}}$ with $S_w = \varphi$. Then a suitable choice of a Riemann map $w_* : \mathbb{D} \rightarrow w(\mathbb{D})$ quasiconformally extendable to $\widehat{\mathbb{C}}$ gives the conformal welding $g = w_*^{-1} \circ w$ on \mathbb{S} .

Lemma 6.2 shows that the assumption $\varphi \in B_0^\alpha(\mathbb{D}^*)$ implies that there is a quasiconformal extension of w whose complex dilatation μ on \mathbb{D} belongs to $\text{Bel}_0^\alpha(\mathbb{D})$. Then in a similar way to the proof of $S_w = \Phi(\mu) \in B_0^\alpha(\mathbb{D}^*)$ above, we have

$$\sup_{\zeta \in \mathbb{D}^*} \rho_{\mathbb{D}^*}^{-1+\alpha}(\zeta) |\{\log w'(\zeta)\}'| < \infty.$$

We can also show the corresponding result for w_* . Actually, this is defined by considering μ^{-1} . Then the complex derivatives $(w|_{\mathbb{D}^*})'$ and $(w_*|_{\mathbb{D}})'$ have non-vanishing continuous extensions to \mathbb{S} . Hence the derivative g' of g along \mathbb{S} is represented by

$$g'(e^{2\pi ix}) = w'(e^{2\pi ix})/w'_*(g(e^{2\pi ix})).$$

In particular, g is continuously differentiable.

According to arguments by Anderson, Becker and Lesley [3], the modulus of continuity of $|g'(e^{2\pi ix})| = \tilde{g}'(x)$ can be estimated as follows. Put

$$I(t; |g'|) = \sup_{|x-y| \leq t} |\tilde{g}'(x) - \tilde{g}'(y)|.$$

Since $\tilde{g}'(x)$ is bounded by some constant $L > 0$,

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq L|x - y|$$

is satisfied. Then it follows that

$$\begin{aligned} \frac{1}{L} I(t; |g'|) \leq I(t; \log |g'|) &\leq I(t; \log |w'|) + I(t; \log |w'_* \circ g|) \\ &\leq I(t; \log |w'|) + I(Lt; \log |w'_*|). \end{aligned}$$

By using the estimates for $\{\log w'\}'$ and $\{\log w'_*\}'$ as above, we have $I(t; \log |w'|) = O(t^\alpha)$ and $I(t; \log |w'_*|) = O(t^\alpha)$. Hence $I(t; |g'|) = O(t^\alpha)$, which implies that $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$. \square

As we have mentioned above, the implication (1) \Rightarrow (3) has been already proved in Tam and Wan [24] by using harmonic extension of diffeomorphisms of \mathbb{S} . A C^2 -map $f : \mathbb{D} \rightarrow \mathbb{D}$ is harmonic with respect to the hyperbolic metric if it satisfies the Euler-Lagrange equation

$$\partial \bar{\partial} f(z) + \partial_\zeta (\log \rho_{\mathbb{D}}(\zeta)) \circ f(z) \cdot \partial f(z) \bar{\partial} f(z) \equiv 0.$$

In this case, the Hopf differential $\rho_{\mathbb{D}}(f(z)) \partial f(z) \bar{\partial} f(z)$ of f is holomorphic. Conversely, for every $\phi \in B(\mathbb{D})$ there exists a unique harmonic quasiconformal diffeomorphism f of \mathbb{D} whose Hopf differential is ϕ . The boundary extension of f to \mathbb{S} yields a quasisymmetric automorphism $g \in \text{QS}$. However, it is not known whether every $g \in \text{QS}$ is obtained in this way, or conversely, whether every $g \in \text{QS}$ has a harmonic quasiconformal extension to \mathbb{D} . The Schoen conjecture asserts that this should be always true and this is affirmative if g is a diffeomorphism of \mathbb{S} . In fact, in the case where $g \in \text{Sym}$, this has been solved by Markovic [20].

For every $g \in \text{Sym}$, we choose a harmonic quasiconformal diffeomorphism f of \mathbb{D} and set $\mu = \mu_f$ in $\text{Bel}(\mathbb{D})$, which can be described by its Hopf differential.

Then take the quasiconformal diffeomorphism f_μ of $\widehat{\mathbb{C}}$ that is conformal on \mathbb{D}^* . For $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$, Tam and Wan [24] gave an estimate of the decay order of the Schwarzian derivative $S_{f_\mu|_{\mathbb{D}^*}}$ in terms of the regularity of g and in particular obtained the following.

Proposition 7.7. *If $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ then $\beta([g]) \in B_0^\alpha(\mathbb{D}^*)$.*

8 The Teichmüller space of circle diffeomorphisms

Now we are ready to realize the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives as a subspace of the universal Teichmüller space. Then we will give some application of the structure of this space at the end of this section.

Definition. For a constant $\alpha \in (0, 1)$, the Teichmüller space of diffeomorphisms of \mathbb{S} with Hölder continuous derivatives is defined by $T_0^\alpha = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S})$.

Theorem 7.6 implies that, for the Teichmüller projection $\pi : \text{Bel}(\mathbb{D}) \rightarrow T$, we have

$$\pi(\text{Bel}_0^\alpha(\mathbb{D})) = T_0^\alpha,$$

and for the Bers embedding $\beta : T \rightarrow B(\mathbb{D}^*)$, we have

$$\beta(T_0^\alpha) = \beta(T) \cap B_0^\alpha(\mathbb{D}^*),$$

which coincides with $\Phi(\text{Bel}_0^\alpha(\mathbb{D}))$ for the Bers projection $\Phi : \text{Bel}(\mathbb{D}) \rightarrow B(\mathbb{D}^*)$. Here, we see that $\beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is an open subset of the Banach space $B_0^\alpha(\mathbb{D}^*)$. Indeed, this follows from the fact that $\beta(T)$ is open in $B(\mathbb{D}^*)$ and from the norm inequality $\|\varphi\|_\infty \leq \|\varphi\|_{\infty, \alpha}$ for $\varphi \in B_0^\alpha(\mathbb{D}^*)$.

We restrict π , Φ and β to the above spaces and consider continuity and openness of these maps. We provided T_0^α with the quotient topology of $\text{Bel}_0^\alpha(\mathbb{D})$ by π , which is so defined that π is continuous. Then, from the facts listed below, we are able to prove the following.

Theorem 8.1. *The Bers embedding $\beta : T_0^\alpha \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is a homeomorphism.*

Hence the Teichmüller space T_0^α of circle diffeomorphisms with Hölder continuous derivative is equipped with a complex structure modeled on the complex Banach space $B_0^\alpha(\mathbb{D}^*)$.

For the proof of this theorem, it suffices to show the following:

- (1) $\pi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow T_0^\alpha$ is open;
- (2) $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is continuous;
- (3) $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ has a continuous local section.

Details will be given elsewhere ([21]). Here we only mention an outline of the arguments. We need some distortion estimate of quasiconformal automorphisms of \mathbb{D} as in the next proposition, which is a variant of the Mori theorem.

Proposition 8.2. *Let f be a quasiconformal automorphism of \mathbb{D} with $f(0) = 0$ and with complex dilatation μ in $\text{Bel}_0^\alpha(\mathbb{D})$. Then there is a constant $A \geq 1$ depending only on α and $\|\mu\|_{\infty, \alpha}$ such that*

$$\frac{1}{A}(1 - |z|) \leq 1 - |f(z)| \leq A(1 - |z|)$$

for every $z \in \mathbb{D}$.

From this estimate, we see that for any μ and ν in $\text{Bel}_0^\alpha(\mathbb{D})$, $\mu * \nu^{-1}$ also belongs to $\text{Bel}_0^\alpha(\mathbb{D})$; we apply Proposition 8.2 to $\zeta = f^\nu(z)$ in the formula

$$\mu * \nu^{-1}(\zeta) = \frac{\mu(z) - \nu(z)}{1 - \overline{\nu(z)}\mu(z)} \cdot \frac{\partial f^\nu(z)}{\partial \overline{f^\nu(z)}}.$$

Hence $\text{Bel}_0^\alpha(\mathbb{D})$ is a subgroup of $\text{Bel}(\mathbb{D})$. Moreover, the right translation map $r_\nu : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \text{Bel}_0^\alpha(\mathbb{D})$ for $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ defined by $\mu \mapsto \mu * \nu^{-1}$ is a homeomorphism with respect to the topology induced by the norm $\|\cdot\|_{\infty, \alpha}$.

(1) The openness of $\pi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow T_0^\alpha$ is proved as follows. Take an open subset $U \subset \text{Bel}_0^\alpha(\mathbb{D})$. To see that $\pi(U)$ is open, we consider

$$\pi^{-1}(\pi(U)) = \bigcup_{\nu \in \text{Ker } \pi \cap \text{Bel}_0^\alpha(\mathbb{D})} r_\nu(U).$$

This is also open in $\text{Bel}_0^\alpha(\mathbb{D})$ and hence $\pi(U)$ is open. Note that the right translation map r_ν for $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ projects down to the base point change map $R_{\pi(\nu)} : T_0^\alpha \rightarrow T_0^\alpha$ and then the openness of π guarantees that $R_{\pi(\nu)}$ is a homeomorphism.

(2) The continuity of $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ can be proved in the following assertion. Proposition 8.2 is also necessary for the proof of this result.

Lemma 8.3. *For any μ and ν in $\text{Bel}_0^\alpha(\mathbb{D})$,*

$$\|\Phi(\mu) - \Phi(\nu)\|_{\infty, \alpha} \leq C\|\mu - \nu\|_{\infty, \alpha}$$

is satisfied for a constant $C > 0$ depending on α , $\|\mu\|_\infty$, $\|\nu\|_\infty$ and $\|\nu\|_{\infty, \alpha}$.

Remark. Similarly, for any μ and ν in $\text{Bel}_0^\alpha(\mathbb{D})$,

$$\|\Phi(r_\nu(\mu))\|_{\infty, \alpha} \leq C'\|\mu - \nu\|_{\infty, \alpha}$$

is satisfied for a constant $C' > 0$ depending on α , $\|\mu\|_\infty$, $\|\nu\|_\infty$ and $\|\nu\|_{\infty, \alpha}$.

To consider the norm $\|\Phi(\mu)\|_{\infty, \alpha}$ of the Schwarzian derivative $\Phi(\mu) = S_{f_\mu|_{\mathbb{D}^*}}$, we need an estimate of the derivative of the conformal homeomorphism f_μ of \mathbb{D}^* defined by $\mu \in \text{Bel}_0^\alpha(\mathbb{D})$. We use the following distortion result for this purpose.

Proposition 8.4. *Let f be a conformal homeomorphism of \mathbb{D}^* with $f(\infty) = \infty$ and with $\lim_{z \rightarrow \infty} f'(z) = 1$ whose quasiconformal extension to \mathbb{D} has complex dilatation μ in $\text{Bel}_0^\alpha(\mathbb{D})$. Then there is a constant $B \geq 1$ depending only on α and $\|\mu\|_{\infty, \alpha}$ such that*

$$\frac{1}{B} \leq |f'(z)| \leq B$$

for every $z \in \mathbb{D}^*$.

(3) The existence of a local continuous section for $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is verified by using a local continuous section for the original Bers projection Φ defined by quasiconformal reflection. For each $\psi \in \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$, take $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ such that $\Phi(\nu) = \psi$. Let f_ν be the quasiconformal automorphism of $\widehat{\mathbb{C}}$ with $S_{f_\nu|_{\mathbb{D}^*}} = \psi$. We may assume that $f_\nu|_{\mathbb{D}}$ is diffeomorphic. The quasiconformal reflection $\lambda : f_\nu(\mathbb{D}) \rightarrow f_\nu(\mathbb{D}^*)$ with respect to the quasicircle $f_\nu(\mathbb{S})$ is defined by $\lambda(\zeta) = f_\nu(f_\nu^{-1}(\zeta)^*)$. Then we generalize the Ahlfors-Weill local section as follows.

There is a constant $\varepsilon > 0$ depending on $\|\nu\|_\infty$ such that if $\varphi \in B_0^\alpha(\mathbb{D}^*)$ satisfies $\|\varphi\|_\infty \leq \|\varphi\|_{\infty, \alpha} < \varepsilon$ then there is a quasiconformal automorphism \widehat{f} of $\widehat{\mathbb{C}}$ conformal on $f_\nu(\mathbb{D}^*)$ such that $S_{\widehat{f} \circ f_\nu|_{\mathbb{D}^*}} = \psi + \varphi$. Here the Beltrami coefficient $\mu_{\widehat{f}}$ of \widehat{f} is given by

$$\mu_{\widehat{f}}(\zeta) = \frac{\varphi(f_\nu^{-1}(\lambda(\zeta)))\{(f_\nu^{-1})'(\lambda(\zeta))\}^2(\zeta - \lambda(\zeta))^2\bar{\partial}\lambda(\zeta)}{2 + \varphi(f_\nu^{-1}(\lambda(\zeta)))\{(f_\nu^{-1})'(\lambda(\zeta))\}^2(\zeta - \lambda(\zeta))^2\partial\lambda(\zeta)}$$

for $\zeta \in f_\nu(\mathbb{D})$. See Lehto [19, Section II.4.2]. The arguments there with the aid of Propositions 8.2 and 8.4 can be used to prove that the complex dilatation of $\widehat{f} \circ f_\nu$ on \mathbb{D} belongs to $\text{Bel}_0^\alpha(\mathbb{D})$, which we may denote by $\mu * \nu$. Then we have $\Phi(\mu * \nu) = \psi + \varphi$. By the correspondence $\psi + \varphi \mapsto \mu * \nu$, we have a local section of Φ at ψ . By the formula of $\mu_{\widehat{f}}$ in terms of φ , we see that the local section is continuous at ψ .

Now we have obtained the continuity of π , Φ , β and β^{-1} restricted to the related spaces to T_0^α with respect to the norm $\|\cdot\|_{\infty, \alpha}$. Also, we know that these maps are holomorphic on the larger spaces with respect to the norm $\|\cdot\|_\infty$. Once we are in this situation, to see that these maps are actually holomorphic is a matter of general argument. Hence the Bers projection $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is holomorphic and the base point change map $R_\tau : T_0^\alpha \rightarrow T_0^\alpha$ for $\tau \in T_0^\alpha$ is biholomorphic, too.

We have defined the Teichmüller space T_0^α as a parameter space of $\text{Diff}^{1+\alpha}(\mathbb{S})$. For this parametrization, the quasiconformal extension of an element of $\text{Diff}^{1+\alpha}(\mathbb{S})$ plays an essential role. Here we consider the relationship between the topology of T_0^α and the following quantity for an element of $\text{Diff}^{1+\alpha}(\mathbb{S})$ regarded as a mapping of \mathbb{S} itself.

Definition. For $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$, we define

$$c_\alpha(g) = \inf \{c \geq 0 \mid |\widetilde{g}'(x) - \widetilde{g}'(y)| \leq c|x - y|^\alpha\},$$

and for $[g] \in T_0^\alpha = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S})$,

$$c_\alpha([g]) = \inf \{c_\alpha(\phi \circ g) \mid \phi \in \text{Möb}(\mathbb{S})\}.$$

Proposition 8.5. *If $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ satisfies $c_\alpha([g]) < 1$, then there exists $\mu(g) \in \text{Bel}_0^\alpha(\mathbb{D})$ with $\pi(\mu(g)) = [g]$ such that $\|\mu(g)\|_{\infty, \alpha}$ is bounded and $1 - \|\mu(g)\|_\infty$ is bounded away from zero by constants depending only on $c_\alpha([g])$. Moreover, $\|\beta([g])\|_{\infty, \alpha} = \|\Phi(\mu(g))\|_{\infty, \alpha}$ is bounded by a constant depending only on $c_\alpha([g])$.*

Proof. By post-composition of an element of $\text{Möb}(\mathbb{S})$, we may assume that the lift \tilde{g} of g satisfies $|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x - y|^\alpha$ for some $c \geq c_\alpha([g])$ arbitrarily close to $c_\alpha([g]) < 1$. By Proposition 7.3, there is a constant $b \geq 0$ depending only on c such that $m_{\tilde{g}}(x, t)^\pm \leq 1 + bt^\alpha$ for every $x \in \mathbb{R}$ and for every $t > 0$. In particular, $\tilde{M} = \tilde{M}(\tilde{g}) \leq 1 + b(1/2)^\alpha$ by Proposition 4.5 and the remark after that.

We take the complex dilatation of the quasiconformal extension $f \in \text{AC}(\mathbb{D})$ of g as $\mu(g)$ by using Proposition 7.4. Then $\|\mu(g)\|_{\infty, \alpha}$ is bounded by a constant depending only on b . On the other hand, Theorem 4.1 implies that $1 - \|\mu(g)\|_\infty$ is bounded away from zero by a constant depending only on \tilde{M} . Hence both norms are estimated by $c_\alpha([g])$. Moreover, Lemma 8.3 in the case where $\nu = 0$ shows that

$$\|\beta([g])\|_{\infty, \alpha} = \|\Phi(\mu(g))\|_{\infty, \alpha} \leq C\|\mu(g)\|_{\infty, \alpha}$$

is satisfied for a constant $C > 0$ depending only on $\|\mu(g)\|_\infty$. Thus we see that $\|\beta([g])\|_{\infty, \alpha}$ is bounded by a constant depending only on $c_\alpha([g])$. \square

As an application of this result, we consider an infinite non-abelian subgroup G of $\text{Diff}^{1+\alpha}(\mathbb{S})$ such that $c_\alpha([g])$ are sufficiently small for all $g \in G$. With the aid of the fixed point theorem proved in the forthcoming paper [21], we can conclude that such a subgroup G must be in $\text{Möb}(\mathbb{S})$.

Theorem 8.6. *Let G be an infinite non-abelian subgroup of $\text{Diff}^{1+\alpha}(\mathbb{S})$. There is a constant $\delta > 0$ depending only on $\alpha \in (0, 1)$ such that if $c_\alpha([g]) < \delta$ for all $g \in G$, then G is a subgroup of $\text{Möb}(\mathbb{S})$.*

Proof. Choose $p > 1/\alpha$ and consider the Banach space $A^p(\mathbb{D}^*)$ of p -integrable holomorphic functions on \mathbb{D}^* with respect to the hyperbolic metric:

$$A^p(\mathbb{D}^*) = \{\varphi \in B(\mathbb{D}^*) \mid \|\varphi\|_p^p = \int_{\mathbb{D}^*} \rho_{\mathbb{D}^*}^{2-2p}(z) |\varphi(z)|^p dx dy < \infty\}.$$

We will show that for every $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$, $\beta([g])$ belongs to $A^p(\mathbb{D}^*)$ and $\|\beta([g])\|_p$ is estimated in terms of $c_\alpha([g])$. As in the proof of Proposition 8.5, we take the complex dilatation $\mu(g)$ of the quasiconformal extension of g . Proposition 7.4 asserts that

$$|\mu(g)(\zeta)| \leq \min \left\{ \frac{4b}{(2\pi)^\alpha} (-\log |\zeta|)^\alpha, 1 \right\}$$

for every $\zeta \in \mathbb{D}$. Here the constant b tends to 0 as $c_\alpha([g]) \rightarrow 0$ by Proposition 7.3. Moreover, $1 - \|\mu(g)\|_\infty$ is bounded away from zero by a constant depending only on $c_\alpha([g])$.

We consider the p -integrable norm of $\mu \in \text{Bel}(\mathbb{D})$ with respect to the hyperbolic metric:

$$\|\mu\|_p^p = \int_{\mathbb{D}} |\mu(\zeta)|^p \rho_{\mathbb{D}}^2(\zeta) d\xi d\eta.$$

For the complex dilatation $\mu(g)$ as above, this norm satisfies

$$\|\mu(g)\|_p \leq \frac{4b}{(2\pi)^\alpha} \left(\int_{\mathbb{D}} (-\log |\zeta|)^{p\alpha} \rho_{\mathbb{D}}^2(\zeta) d\xi d\eta \right)^{1/p},$$

where the integral is finite due to the condition $p\alpha > 1$. Hence $\|\mu(g)\|_p \rightarrow 0$ when $c_\alpha([g]) \rightarrow 0$. It was proved by Cui [10] and Guo [17] that for each $p \geq 2$ there is a constant $C > 0$ depending only on $\|\mu\|_\infty$ such that

$$\|\Phi(\mu)\|_p \leq C\|\mu\|_p.$$

Applying this inequality for $\Phi(\mu(g)) = \beta([g])$, we see that $\|\beta([g])\|_p$ tends to 0 as $c_\alpha([g]) \rightarrow 0$.

The following result will be proved in [21]. There is a constant $\varepsilon > 0$ depending only on p such that if $\|\beta([g])\|_p < \varepsilon$ for all $g \in G$, then there is $h \in \text{Diff}^{1+\alpha}(\mathbb{S})$ such that the conjugate $\widehat{G} = hGh^{-1}$ belongs to $\text{Möb}(\mathbb{S})$. Note that h corresponds to some $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ with $\pi(\nu) = [h]$ by Theorem 7.6. For this ε , the above argument shows the existence of a constant δ with $0 < \delta < 1$ such that if $c_\alpha([g]) < \delta$ then $\|\beta([g])\|_p < \varepsilon$. Since we choose p according to α , the constant δ can be taken to be dependent only on α .

For every $\phi \in \text{Möb}(\mathbb{D}^*)$ and for every $\varphi \in B(\mathbb{D}^*)$, we set

$$(\phi^*\varphi)(z) = \varphi(\phi(z))\phi'(z)^2.$$

Then

$$\rho_{\mathbb{D}^*}^{-2}(\phi(z))|\varphi(\phi(z))| = \rho_{\mathbb{D}^*}^{-2}(z)|(\phi^*\varphi)(z)|.$$

Moreover, the Cayley identity for the Schwarzian derivative implies $\phi^*\beta([g]) = \beta([g \circ \phi])$. For each $g \in G$, we set $\varphi_g = \beta([g \circ h^{-1}])$ and $\widehat{g} = hgh^{-1} \in \text{Möb}(\mathbb{D}^*)$, which satisfy $\widehat{g}^*\varphi_{\text{id}} = \varphi_g$. This gives an equality

$$\begin{aligned} \rho_{\mathbb{D}^*}^{-2+\alpha}(\widehat{g}^{-1}(z))|\varphi_g(\widehat{g}^{-1}(z))| &= \rho_{\mathbb{D}^*}^\alpha(\widehat{g}^{-1}(z))\rho_{\mathbb{D}^*}^{-2}(\widehat{g}^{-1}(z))|(\widehat{g}^*\varphi_{\text{id}})(\widehat{g}^{-1}(z))| \\ &= \rho_{\mathbb{D}^*}^\alpha(\widehat{g}^{-1}(z))\rho_{\mathbb{D}^*}^{-2}(z)|\varphi_{\text{id}}(z)|. \end{aligned}$$

We will show that $\varphi_{\text{id}} = 0$. Then $[h^{-1}] = [\text{id}]$, which implies that $G \subset \text{Möb}(\mathbb{S})$. Suppose to the contrary that $\varphi_{\text{id}} \neq 0$; there is some $z_0 \in \mathbb{D}^*$ such that $\varphi_{\text{id}}(z_0) \neq 0$. Since $\widehat{G} \subset \text{Möb}(\mathbb{D}^*)$ is infinite non-abelian, there is a sequence of elements \widehat{g}_n in \widehat{G} such that $|\widehat{g}_n^{-1}(z_0)| \rightarrow 1$ as $n \rightarrow \infty$. Then the above equality in particular yields

$$\|\varphi_{g_n}\|_{\infty, \alpha} \geq \rho_{\mathbb{D}^*}^\alpha(\widehat{g}_n^{-1}(z_0))\rho_{\mathbb{D}^*}^{-2}(z_0)|\varphi_{\text{id}}(z_0)|,$$

whereas the second term tends to ∞ as $n \rightarrow \infty$.

On the other hand, if $c_\alpha([g]) < \delta$ for all $g \in G$ by assumption, then Proposition 8.5 asserts that $\|\mu(g)\|_{\infty, \alpha}$ are uniformly bounded and $1 - \|\mu(g)\|_\infty$ are uniformly bounded away from zero. Under these conditions, we consider the norm of

$$\varphi_g = \beta([g \circ h^{-1}]) = \Phi(\mu(g) * \nu^{-1}) = \Phi(r_\nu(\mu(g))).$$

The remark after Lemma 8.3 shows that

$$\|\varphi_g\|_{\infty,\alpha} = \|\Phi(r_\nu(\mu(g)))\|_{\infty,\alpha} \leq C' \|\mu(g) - \nu\|_{\infty,\alpha} \leq C'(\|\mu(g)\|_{\infty,\alpha} + \|\nu\|_{\infty,\alpha}).$$

Taking the dependence of the constant C' into account, we see that these norms are uniformly bounded for all $g \in G$. This is a contradiction and we conclude that $\varphi_{\text{id}} = 0$. \square

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