RECURRENT AND PERIODIC POINTS FOR ISOMETRIES OF L^∞ SPACES

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Abstract. We study the action of isometries on metric spaces. In particular, we consider the recurrent set of the bilateral shift operator on the Banach space $L^\infty(\mathbb{Z})$, and prove that the set of periodic points is not dense in the recurrent set. Then we apply this result to investigating the dynamics of Teichmüller mod ular groups acting on infinite dimensional Teichmüller spaces as well as composition operators acting on Hardy spaces.

1. Introduction

This paper is concerned with dynamical systems of isometric linear operators acting on certain function spaces. A relationship between the sets of recurrent and periodic points are mainly considered. A recurrent point is a point whose orbit returns to an arbitrary small neighborhood of itself. For an isometric automorphism, it is the same as a limit point.

First we recall the following two examples, though they are not systems of isometries. Let ℓ be the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that each symbol x_n is either 0 or 1. We provide product topology for ℓ and consider the symbolic dynamics caused by the shift operator $\sigma : (x_n) \mapsto (x_{n+1})$. Then there exists a point $x \in \ell$ whose orbit is dense in ℓ. It can be also proved that the set of periodic points for σ is dense in ℓ. See Devaney [3]. As another example, let H be a topological linear space consisting of all entire functions on the complex plane $\mathbb{C}$ having compact open topology, and let $C : H \to H$ be a linear operator defined by $\phi(z) \mapsto \phi(z + 1)$. Then there exists an entire function $\phi \in H$ such that the orbit of ϕ under C is dense in H. This was a striking fact discovered by Birkhoff [1].

In these examples, one can understand that the topology is weak enough for the limit set to be coincident with the entire space. Of course this does not always happen, and what we are interested in is a non-separable Banach space of certain holomorphic functions with supremum norm, for which the recurrent set of an isometric linear automorphism is different from the entire space. Hence we restrict its action to the recurrent set and consider its behavior there. In particular, we focus on a problem asking whether the periodic points are dense in the recurrent set or not.

This observation has certain applications to analyzing dynamics of Teichmüller modular groups acting on infinite dimensional Teichmüller spaces as well as composition operators acting on Hardy spaces. A study on the dynamics of the Teichmüller modular groups is recently developed in a series of our papers [7], [8], [9] and [15], where properties of the limit set are investigated analogously to the theory of Kleinian groups. The composition operators on the Hardy spaces are studied in an interacting field between the operator theory and complex analysis and a problem on hypercyclicity of composition operators developed by Bourdon and Shapiro [2] is related to our work.

This paper is organized as follows. Section 2 concerns metric spaces in general. For a group G of isometric automorphisms acting on a complete metric space X, we define the limit set $\Lambda(G)$, the recurrent set $\mathrm{Rec}(G)$ and the set of periodic points $\mathrm{Per}(G)$ and list up their fundamental properties. In particular, the nature of isometry is emphasized. In Section 3, we assume that X is a Banach space and that G is a group of isometric linear automorphisms.

In Section 4, as a prototype of our function spaces, we deal with the Banach space $L^\infty(\mathbb{Z})$ of the bilateral infinite sequences of complex numbers with supremum norm and the bilateral shift operator $\sigma : (x_n) \to (x_{n+1})$. An element $\xi = (\xi_n)_{n\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$ constructed for claiming Lemma 4.4 plays the most important role in the remainder of this paper, which satisfies

$$\xi \in \mathrm{Rec}(\sigma) - \overline{\mathrm{Per}(\sigma)}.$$

The construction of this ξ relies on a structure of the infinite cyclic group. Hence, by considering the Banach space $L^\infty(G)$ on a discrete group G in general having the canonical action of G itself, we expect that the property $\mathrm{Rec}(G) = \overline{\mathrm{Per}(G)}$ can characterize the group structure of G in some sense. In Section 5, we give a condition for G to satisfy this property.

Applications of the above mentioned results to Teichmüller spaces and Hardy spaces are demonstrated in the second half of this paper, from Sections 6 to 8. In Section 6, we realize a family of sequences $(x_n)_{n\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$ geometrically as a family of Riemann surfaces having prescribed punctures by $(x_n)_{n\in\mathbb{Z}}$. In this way, we transfer the dynamical system of the shift operator σ on $L^\infty(\mathbb{Z})$ into that of a modular transformation ω_* on the Teichmüller space $T(R)$ of a Riemann surface R. The Teichmüller space is a kind of a function space consisting of quasiconformal homeomorphisms, and a supremum norm induces a distance on $T(R)$. In Corollary 6.2, we prove that $\mathrm{Rec}(\omega_*) - \overline{\mathrm{Per}(\omega_*)} \neq \emptyset$. This gives certain information on the limit set for the Teichmüller modular transformation. Next, by the Bers embedding, we embed $T(R)$ into a Banach space $B(\Gamma)$ of the holomorphic functions on the upper half-plane satisfying a certain automorphic condition for a Fuchsian group Γ and finiteness of the hyperbolic supremum norm. Then we obtain the dynamical system of the isometric linear automorphism ω_* acting on the Banach space $B(\Gamma)$, which holds the same property as above.

In Section 7, we consider the Hardy space H^∞ of all bounded holomorphic functions f on the unit disk Δ. The isometric linear automorphism ω_* of $B(\Gamma)$ is transferred to an isometric composition operator $C_h : f \mapsto f \circ h$ induced by a Möbius transformation h of Δ. Theorem 7.6 gives a complete answer to the problem asking the difference between $\mathrm{Rec}(C_h)$ and $\overline{\mathrm{Per}(C_h)}$ for any Möbius composition operator C_h. Finally in Section 8 (Appendix), we compare our results for H^∞ with the corresponding results for other Hardy spaces H^p with finite p. Hypercyclicity of Möbius composition operators C_h is already characterized in H^p. Recently, researches on the density of periodic points of C_h in H^p are also developed. Although composition operators C_h on H^p are not isometric for finite p, our investigation of the relationship between $\mathrm{Rec}(C_h)$ and $\overline{\mathrm{Per}(C_h)}$ in H^∞ meets here with the results on H^p.

2. Isometries of metric spaces

In this section, we consider isometries of metric spaces and prove certain properties of the sets of recurrent and periodic points peculiar to the isometric action. We begin by giving notation.

Let X be a complete metric space with a distance d, and let $\mathrm{Isom}(X)$ be the group of all isometric automorphisms of X. For a subgroup $G \subset \mathrm{Isom}(X)$, the orbit of $x \in X$ under the action of G is $\mathrm{Orb}(G, x) = \{g(x) \mid g \in G\}$. For an element $g \in \mathrm{Isom}(X)$, the set of fixed points of g is $\mathrm{Fix}(g) = \{x \in X \mid g(x) = x\}$. Furthermore, the stabilizer of a point $x \in X$ in G is $\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}$.

Definition 2.1. For a subgroup $G \subset \mathrm{Isom}(X)$ and for a point $x \in X$, it is said that $y \in X$ is a *limit point* of x for G if there exists a sequence $\{g_n\}$ of distinct elements of G such that $\lim_{n\to\infty} d(g_n(x), y) = 0$. The set of all limit points of x for G is called the limit set of x for G and is denoted by $\Lambda(G, x)$. It is said that $x \in X$ is a *recurrent point* for G if $x \in \Lambda(G, x)$. The set of all recurrent points for G is called the recurrent set for G and is denoted by $\mathrm{Rec}(G)$. The *limit set* for G is defined by $\Lambda(G) = \bigcup_{x\in X} \Lambda(G, x)$. Furthermore, $x \in X$ is a *periodic point* for G if $\mathrm{Stab}_G(x)$ consists of infinitely many elements. The set of all periodic points for G is denoted by $\mathrm{Per}(G)$.

Clearly $\mathrm{Per}(G) \subset \mathrm{Rec}(G) \subset \Lambda(G)$ and these sets are G-invariant. If G is a cyclic group $\langle g\rangle$ generated by $g \in \mathrm{Isom}(X)$, we denote $\mathrm{Orb}(\langle g\rangle, x)$, $\mathrm{Per}(\langle g\rangle)$ and $\mathrm{Rec}(\langle g\rangle)$ simply by $\mathrm{Orb}(g, x)$,

$\mathrm{Per}(g)$ and $\mathrm{Rec}(g)$ respectively. If g is of infinite order, then $\mathrm{Per}(g) = \bigcup_{n\in\mathbb{N}} \mathrm{Fix}(g^n)$ and $\mathrm{Per}(g) = \mathrm{Per}(g^k)$ for every $k \in \mathbb{N}$.

In general, for a continuous map $g : X \to X$, we can also define the recurrent set and the limit set, which will appear later. However their definitions are slightly different from those for an isometry. Actually the following properties make the definitions for an isometry simple.

Proposition 2.2. *For a subgroup $G \subset \mathrm{Isom}(X)$, the recurrent set $\mathrm{Rec}(G)$ is coincident with the limit set $\Lambda(G)$. Moreover $\mathrm{Rec}(G)$ is closed, and so is $\Lambda(G)$.*

A fundamental nature of isometries concerning these properties can be summarized as follows.

Lemma 2.3. *For a subgroup $G \subset \mathrm{Isom}(X)$, the closure of the orbit and the limit set satisfy the following properties.*

1. *Symmetry: $x \in \overline{\mathrm{Orb}(G,y)}$ if and only if $y \in \overline{\mathrm{Orb}(G,x)}$; $x \in \Lambda(G,y)$ if and only if $y \in \Lambda(G,x)$.*
2. *Transitivity: $x \in \overline{\mathrm{Orb}(G,y)}$ and $y \in \overline{\mathrm{Orb}(G,z)}$ imply $x \in \overline{\mathrm{Orb}(G,z)}$; $x \in \Lambda(G,y)$ and $y \in \Lambda(G,z)$ imply $x \in \Lambda(G,z)$.*

In particular, if $x \in \overline{\mathrm{Orb}(G,y)}$, then $\overline{\mathrm{Orb}(G,x)} = \overline{\mathrm{Orb}(G,y)}$; if $x \in \Lambda(G,y)$, then $\Lambda(G,x) = \Lambda(G,y)$.

Proof. We prove this lemma only for the limit set, since the proof for the closure of the orbit is similar and easier.

First we prove statement (1). If $x \in \Lambda(G,y)$, then there exists a sequence $\{g_n\}$ of distinct elements of G satisfying $d(g_n(y),x) \to 0$. Since g_n is isometric, we have $d(g_n^{-1}(x),y) \to 0$, which implies $y \in \Lambda(G,x)$.

Next we prove statement (2). If $x \in \Lambda(G,y)$ and $y \in \Lambda(G,z)$, then there exist sequences $\{g_n\}$ and $\{g'_m\}$ of distinct elements of G such that $d(g_n(y),x) \to 0$ $(n \to \infty)$ and $d(g'_m(z),y) \to 0$ $(m \to \infty)$. If $g(y) = x$ for some $g \in G$, then we have a sequence $\{g \circ g'_m\}$ of distinct elements that satisfies $d(g \circ g'_m(z),x) = d(g'_m(z),y) \to 0$. Thus we have only to consider the case where $g_n(y) \neq x$ for all n. By triangle inequality, we have

$$d(g_n(y),x) - d(g'_m(z),y) \leq d(g_n \circ g'_m(z),x) \leq d(g_n(y),x) + d(g'_m(z),y).$$

If we choose $m = m(n)$ so that $d(g'_{m(n)}(z),y) \leq d(g_n(y),x)/2$, then $g_n \circ g'_{m(n)}(z) \neq x$ and $d(g_n \circ g'_{m(n)}(z),x) \to 0$ $(n \to \infty)$. This implies that $x \in \Lambda(G,z)$. □

Proof of Proposition 2.2: First we prove that $\Lambda(G) \subset \mathrm{Rec}(G)$. If $x \in \Lambda(G)$, then there exists a point $y \in X$ such that $x \in \Lambda(G,y)$. By Lemma 2.3 (1) we have $y \in \Lambda(G,x)$ and by Lemma 2.3 (2) we have $x \in \Lambda(G,x)$. Hence $x \in \mathrm{Rec}(G)$.

Next we prove that $\mathrm{Rec}(G)$ is closed. Let $\{x_n\}$ be a sequence of points in $\mathrm{Rec}(G)$ that converges to $x \in X$. For each $x_n \in \mathrm{Rec}(G)$, we can take a sequence $\{g_{n,i}\}_{i=1}^{\infty}$ of distinct elements of G such that $d(g_{n,i}(x_n),x_n) \to 0$ $(i \to \infty)$. Choose $i(n)$ for each n so that $d(g_{n,i(n)}(x_n),x_n) \leq 1/n$ and that $\{g_{n,i(n)}\}$ are mutually distinct. Then we have

$$\begin{aligned} d(g_{n,i(n)}(x),x) &\leq d(g_{n,i(n)}(x),g_{n,i(n)}(x_n)) + d(g_{n,i(n)}(x_n),x_n) + d(x_n,x) \\ &\leq 2d(x,x_n) + 1/n, \end{aligned}$$

which implies that $d(g_{n,i(n)}(x),x) \to 0$ $(n \to \infty)$. Hence $x \in \mathrm{Rec}(G)$. □

For a continuous map $g : X \to X$ in general, the sets $\{x \in X \mid x \in \Lambda(g,x)\}$ and $\bigcup_{x\in X} \Lambda(g,x)$ are not necessarily closed, whereas $\Lambda(g,x)$ is closed for all $x \in X$. Thus we define the recurrent set and the limit set for g by taking their closure, namely, $\mathrm{Rec}(g) = \overline{\{x \in X \mid x \in \Lambda(g,x)\}}$ and $\Lambda(g) = \overline{\bigcup_{x\in X} \Lambda(g,x)}$. In this case, we have $\mathrm{Rec}(g) \subset \Lambda(g)$, however they are not coincident, in general. This recurrent set $\mathrm{Rec}(g)$ is also called the *Birkhoff center* (see [19]).

Since $\mathrm{Rec}(G)$ is closed by Proposition 2.2, the inclusion relation $\mathrm{Rec}(G) \supset \overline{\mathrm{Per}(G)}$ is satisfied for every $G \subset \mathrm{Isom}(X)$. In this paper, we investigate the properness of this inclusion relation. The following lemma gives a sufficient condition for a point not to belong to $\overline{\mathrm{Per}(G)}$, which is a basis of our consideration.

Lemma 2.4. *Let x_0 be a point of X and G a subgroup of $\mathrm{Isom}(X)$. Suppose that there exists a constant $d_0 > 0$ satisfying a property that, for every $g \in G - \{id\}$, there exists an integer $k \in \mathbb{Z}$ such that $d(g^k(x_0), x_0) \geq d_0$. Then $x_0 \notin \overline{\mathrm{Per}(G)}$.*

Proof. We will prove that $d(x_0, \mathrm{Fix}(g)) \geq d_0/2$ for all $g \in G - \{id\}$. Suppose to the contrary that there exist an element $g \neq id$ and a point $x \in \mathrm{Fix}(g)$ such that $d(x_0, x) < d_0/2$. Since $x \in \mathrm{Fix}(g)$, we have

$$d(g^k(x_0), x) = d(g^k(x_0), g^k(x)) = d(x_0, x) < d_0/2$$

for every $k \in \mathbb{Z}$, and hence

$$d(g^k(x_0), x_0) \leq d(g^k(x_0), x) + d(x_0, x) < d_0.$$

However this contradicts the assumption. Thus $d(x_0, \mathrm{Fix}(g)) \geq d_0/2$ for all $g \in G - \{id\}$. Since d_0 is independent of g, we conclude that $x_0 \notin \overline{\bigcup_{g \in G-\{id\}} \mathrm{Fix}(g)}$. Since $\mathrm{Per}(G) \subset \bigcup_{g \in G-\{id\}} \mathrm{Fix}(g)$, we have the assertion. □

In the remainder of this section, we collect a couple of claims concerning the orbit $\mathrm{Orb}(G, x)$, which will be used later.

Proposition 2.5. *If a subgroup $G \subset \mathrm{Isom}(X)$ is countable, then for every point $x \in \mathrm{Rec}(G) - \mathrm{Per}(G)$, the orbit $\mathrm{Orb}(G, x)$ is not closed.*

Proof. Since $\mathrm{Orb}(G, x)$ is G-invariant, the closure $\overline{\mathrm{Orb}(G, x)}$ is a perfect closed set. In a complete metric space, every non-empty perfect closed set is an uncountable set (cf. p.156 in [10]). Thus $\overline{\mathrm{Orb}(G, x)}$ is uncountable. On the other hand, $\mathrm{Orb}(G, x)$ is countable. Hence $\overline{\mathrm{Orb}(G, x)} - \mathrm{Orb}(G, x) \neq \emptyset$. □

Proposition 2.6. *For a subgroup $G \subset \mathrm{Isom}(X)$, suppose that there exists a point $x_0 \in \mathrm{Rec}(G)$ such that the orbit $\mathrm{Orb}(G, x_0)$ is not dense in $\mathrm{Rec}(G)$. Then for every point $x \in \mathrm{Rec}(G)$, the orbit $\mathrm{Orb}(G, x)$ is not dense in $\mathrm{Rec}(G)$.*

Proof. Suppose to the contrary that there exists a point $x \in \mathrm{Rec}(G)$ such that $\overline{\mathrm{Orb}(G, x)} = \mathrm{Rec}(G)$. Since $x_0 \in \overline{\mathrm{Orb}(G, x)}$, Lemma 2.3 says that $\overline{\mathrm{Orb}(G, x_0)} = \overline{\mathrm{Orb}(G, x)} = \mathrm{Rec}(G)$. This contradicts the assumption. □

Proposition 2.7. *Let G be a subgroup of $\mathrm{Isom}(X)$. If $\mathrm{Rec}(G) - \overline{\mathrm{Per}(G)} \neq \emptyset$ and $\mathrm{Per}(G) \neq \emptyset$, then for every point $x \in \mathrm{Rec}(G)$, the orbit $\mathrm{Orb}(G, x)$ is not dense in $\mathrm{Rec}(G)$.*

Proof. For a point $x \in \mathrm{Per}(G)$, the orbit $\mathrm{Orb}(G, x)$ is contained in $\mathrm{Per}(G)$ because $\mathrm{Per}(G)$ is G-invariant. Then $\overline{\mathrm{Orb}(G, x)} \subset \overline{\mathrm{Per}(G)}$, which is a proper subset of $\mathrm{Rec}(G)$ by the assumption. By Proposition 2.6, we have the assertion. □

3. Isometric linear operators

This section is devoted to an investigation of isometric linear automorphisms of Banach spaces. Let B be a real or complex Banach space and G a group of isometric linear automorphisms of B. We mainly deal with the case where G is an infinite cyclic group $\langle g \rangle$. In this case, recall that the set of periodic points is represented by $\mathrm{Per}(g) = \bigcup_{n \in \mathbb{N}} \mathrm{Fix}(g^n)$.

Proposition 3.1. *For an isometric linear automorphism g of B of infinite order, $\mathrm{Per}(g)$ is a g-invariant linear subspace.*

Proof. For every positive integer $n \in \mathbb{N}$, the set $\mathrm{Fix}(g^n)$ is a g-invariant linear subspace of B. For any n_1 and n_2, there exists a common majorant n, which satisfies $\mathrm{Fix}(g^{n_i}) \subset \mathrm{Fix}(g^n)$ for $i = 1, 2$. Since $\mathrm{Per}(g) = \bigcup_{n\in\mathbb{N}} \mathrm{Fix}(g^n)$, we see that $\mathrm{Per}(g)$ is a g-invariant linear subspace. □

The closure $\overline{\mathrm{Per}(g)}$ is a g-invariant Banach subspace. For the relationship between $\mathrm{Per}(g)$ and $\overline{\mathrm{Per}(g)}$, we have the following.

Proposition 3.2. *For an isometric linear automorphism g of B of infinite order,* $\mathrm{Per}(g)$ *is a proper subspace of B. Assume further that, for each $n \in \mathbb{N}$, there is an integer $k \in \mathbb{N}$ such that* $\mathrm{Fix}(g^n)$ *is properly contained in* $\mathrm{Fix}(g^{kn})$. *Then* $\mathrm{Per}(g)$ *is a proper subspace of the closure* $\overline{\mathrm{Per}(g)}$ *and hence*

$$\mathrm{Rec}(g) - \mathrm{Per}(g) \supset \overline{\mathrm{Per}(g)} - \mathrm{Per}(g) \neq \emptyset.$$

Proof. Suppose that $\mathrm{Per}(g)$ is coincident with the entire space B. Then the Baire category theorem says that $\mathrm{Fix}(g^n) = B$ for some n. However, this implies that $g^n = id$, which contradicts the assumption that g has infinite order. Suppose that $\mathrm{Per}(g)$ is coincident with $\overline{\mathrm{Per}(g)}$. Again the Baire category theorem says that $\mathrm{Fix}(g^n) = \overline{\mathrm{Per}(g)}$ for some n. However, this contradicts the assumption that $\mathrm{Fix}(g^n)$ is properly contained in $\mathrm{Fix}(g^{kn})$ for some k. □

Next, we projectify a Banach space B and isometric linear automorphisms of B. Let $\hat{B} = (B - \{0\})/\mathbb{C}^\times$ be the quotient space and $\pi : B - \{0\} \to \hat{B}$ the projection. The norm on B restricted to the unit sphere induces a distance $\hat{d}$ on $\hat{B}$. Hence we can regard $(\hat{B}, \hat{d})$ as a complete metric space. Also the action of an isometric linear automorphism g of B projects to $\hat{B}$ and induces an isometric automorphism $\hat{g}$ with respect to $\hat{d}$. The sets $\mathrm{Orb}(\hat{g}, \hat{x})$, $\mathrm{Per}(\hat{g})$ and $\mathrm{Rec}(\hat{g})$ are defined in the same way, where $\hat{x} = \pi(x)$ for $x \in B - \{0\}$. These sets are coincident with the projections of $\mathrm{Orb}(g, x)$, $\mathrm{Per}(g)$ and $\mathrm{Rec}(g)$ respectively.

An isometric linear automorphism g of B is called *supercyclic* if there exists a point $\hat{x} \in \hat{B}$ such that the orbit $\mathrm{Orb}(\hat{g}, \hat{x})$ is dense in $\hat{B}$. The density of the orbit is equivalent to topological transitivity of $\hat{g}$ on $\hat{B}$. If B is not separable, these conditions are never satisfied. In this case, we consider the density of the orbit restricted to $\mathrm{Rec}(\hat{g})$ instead.

Proposition 3.3. *Let B be a Banach space and g an isometric linear automorphism of B of infinite order such that* $\mathrm{Per}(g) \neq \{0\}$. *If* $\mathrm{Rec}(\hat{g})$ *consists of infinitely many points, then for every* $\hat{x} \in \mathrm{Rec}(\hat{g})$, *the orbit* $\mathrm{Orb}(\hat{g}, \hat{x})$ *is not dense in* $\mathrm{Rec}(\hat{g})$. *In particular, if* $\mathrm{Rec}(g) - \mathrm{Per}(g) \neq \emptyset$ *or if* $\dim \mathrm{Per}(g) \geq 2$, *then the assumption* $\#\mathrm{Rec}(\hat{g}) = \infty$ *is satisfied.*

Proof. Take a point $\hat{x}_0 \in \mathrm{Per}(\hat{g}) \subset \mathrm{Rec}(\hat{g})$. Since $\mathrm{Orb}(\hat{g}, \hat{x}_0)$ consists of finitely many points, it is not dense in $\mathrm{Rec}(\hat{g})$. Then Proposition 2.6 yields the assertion. If there exists a point $x \in \mathrm{Rec}(g) - \mathrm{Per}(g)$, then the orbit $\mathrm{Orb}(\hat{g}, \hat{x}) \subset \mathrm{Rec}(\hat{g})$ is an infinite set. If $\dim \mathrm{Per}(g) \geq 2$, then $\mathrm{Per}(\hat{g})$ is a non-degenerate continuum, and thus $\mathrm{Rec}(\hat{g})$ is an infinite set. □

Corollary 3.4. *Let B be a Banach space and g an isometric linear automorphism of B having a periodic point besides the origin. Then g is not supercyclic (even if B is separable).*

Proof. Suppose that g is supercyclic; $\mathrm{Orb}(\hat{g}, \hat{x})$ is dense in $\hat{B}$ for some $\hat{x} \in \hat{B}$. Clearly the corresponding point x belongs to $\mathrm{Rec}(g) - \mathrm{Per}(g)$. Then by Proposition 3.3, the orbit $\mathrm{Orb}(\hat{g}, \hat{x})$ is not dense in $\mathrm{Rec}(\hat{g}) \subset \hat{B}$. This is a contradiction. □

4. The bilateral shift operator on $L^\infty(\mathbb{Z})$

On the non-separable Banach space of all bounded bilateral infinite sequences of real numbers

$$L^\infty(\mathbb{Z}) = \{x = (x_n)_{n\in\mathbb{Z}} \mid \|x\|_\infty = \sup_{n\in\mathbb{Z}} |x_n| < \infty\},$$

consider the bilateral shift operator $\sigma : L^\infty(\mathbb{Z}) \to L^\infty(\mathbb{Z})$ defined by $(x_n) \mapsto (x_{n+1})$, which is an isometric linear automorphism. We consider the recurrent set $\mathrm{Rec}(\sigma)$ and the set of periodic points $\mathrm{Per}(\sigma)$ for this σ.

Theorem 4.1. *In the relations $L^\infty(\mathbb{Z}) \supset \mathrm{Rec}(\sigma) \supset \overline{\mathrm{Per}(\sigma)} \supset \mathrm{Per}(\sigma)$, each inclusion is proper.*

We will prove this theorem by dividing the claim into three parts. The properness of the first inclusion is easily seen from the following.

Proposition 4.2. *For $x = (x_n)_{n\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$, suppose that there exist a constant $r > 0$ and an integer $n_0 \in \mathbb{Z}$ such that $|x_n - x_{n_0}| \geq r$ for all $n \neq n_0$. Then $x \notin \mathrm{Rec}(\sigma)$.*

Next we will construct a recurrent point which is not in $\overline{\mathrm{Per}(\sigma)}$.

Definition 4.3. We define a point $\xi = (\xi_n)_{n\in\mathbb{Z}}$ in $L^\infty(\mathbb{Z})$ as follows. Set $\xi_0 = 1$ and $\xi_1 = \xi_{-1} = (1/2)\xi_0 = 1/2$. We proceed as $\xi_n = \xi_{n-6} = (2/3)\xi_{n-3}$ for $n = 2, 3, 4$ and $\xi_n = \xi_{n-18} = (3/4)\xi_{n-9}$ for $n = 5, \ldots, 13$. Inductively, set

$$\xi_n = \xi_{n-2\cdot 3^\ell} = \frac{\ell+1}{\ell+2} \cdot \xi_{n-3^\ell}$$

for $\sum_{i=0}^{\ell-1} 3^i + 1 \leq n \leq \sum_{i=0}^{\ell} 3^i$ stratified with the indices $\ell \in \mathbb{N}$. This is equivalent to the following direct definition by using 3-adic expansion. Every integer $n \in \mathbb{Z}$ is uniquely written as $n = \sum_{i=0}^\infty \varepsilon_i(n) \cdot 3^i$, where $\varepsilon_i(n)$ is either -1, 0 or 1. Then ξ_n is defined by

$$\xi_n = \prod_{\varepsilon_i(n)\neq 0} \frac{i+1}{i+2},$$

where the product is taken over all $i \in \mathbb{N}$ satisfying $\varepsilon_i(n) \neq 0$.

Lemma 4.4. *The point $\xi = (\xi_n)_{n\in\mathbb{Z}} \in L^\infty(\mathbb{Z})$ satisfies the following properties:*

(i) $\lim_{\ell\to\infty} \|\sigma^{3^\ell}(\xi) - \xi\|_\infty = 0$;

(ii) *For every $m \in \mathbb{Z} - \{0\}$, there exists $k \in \mathbb{Z}$ such that $\|\sigma^{km}(\xi) - \xi\|_\infty \geq 1/2$.*

Proof. If $n = \sum_{i=0}^\infty \varepsilon_i(n) \cdot 3^i$, then $n + 3^\ell = \sum_{i\neq\ell} \varepsilon_i(n) \cdot 3^i + (\varepsilon_\ell(n) + 1) \cdot 3^\ell$. In case $\varepsilon_\ell(n)$ is either -1 or 0, the definition of ξ_n yields

$$|\xi_{n+3^\ell} - \xi_n| = \left(1 - \frac{\ell+1}{\ell+2}\right) \prod_{\varepsilon_i(n)\neq 0,\ i\neq\ell} \frac{i+1}{i+2} \leq \frac{1}{\ell+2}.$$

If $\varepsilon_\ell(n) = 1$, then $\varepsilon_\ell(n + 3^\ell) = \varepsilon_\ell(n) + 1 \equiv -1 \pmod 3$ and $\varepsilon_{\ell+1}(n + 3^\ell) = \varepsilon_{\ell+1}(n) + 1$. Again we consider the two cases according to the value of $\varepsilon_{\ell+1}(n)$ above and continue this process. Eventually we have an integer $\ell' \geq \ell$ such that precisely one of $\varepsilon_{\ell'}(n)$ or $\varepsilon_{\ell'}(n + 3^\ell)$ is 0. Then

$$|\xi_{n+3^\ell} - \xi_n| = \left(1 - \frac{\ell'+1}{\ell'+2}\right) \prod_{\varepsilon_i(n)\neq 0,\ i\neq\ell'} \frac{i+1}{i+2} \leq \frac{1}{\ell'+2} \leq \frac{1}{\ell+2}.$$

Since

$$\|\sigma^{3^\ell}(\xi) - \xi\|_\infty = \sup_{n\in\mathbb{Z}} |\xi_{n+3^\ell} - \xi_n|,$$

we have statement (i).

For statement (ii), we will prove that there exists $k \in \mathbb{Z}$ such that $\xi_{km} \leq 1/2$. Since $\xi_0 = 1$, this implies that $\|\sigma^{km}(\xi) - \xi\|_\infty \geq |\xi_{km} - \xi_0| \geq q1/2$. Express $m \in \mathbb{Z} - \{0\}$ as $m = \sum_{i=0}^\infty \varepsilon_i(m) \cdot 3^i$. Let $I(m) \geq 1$ be the largest integer i such that $\varepsilon_{i-1}(m) \neq 0$. For each $\ell \in \mathbb{N}$, set $k_\ell = \sum_{j=0}^{\ell-1} 3^{jI(m)}$. Then

$$k_\ell m = (\sum_{j=0}^{\ell-1} 3^{jI(m)})(\sum_{i=0}^{I(m)-1} \varepsilon_i(m) \cdot 3^i) = \sum_{j=0}^{\ell-1} \sum_{i=0}^{I(m)-1} \varepsilon_i(m) \cdot 3^{jI(m)+i}.$$

From this, we have an estimate

$$\xi_{k_\ell m} \leq \frac{I(m)}{I(m)+1} \cdot \frac{2I(m)}{2I(m)+1} \cdots \frac{\ell I(m)}{\ell I(m)+1}.$$

Since the product in the right-hand side converges to 0 as ℓ tends to ∞, we can take such an integer k that ξ_{km} is arbitrarily small. □

The properness of the second inclusion in Theorem 4.1 immediately follows from Lemma 4.4.

Proposition 4.5. *The point* $\xi \in L^\infty(\mathbb{Z})$ *belongs to* $\mathrm{Rec}(\sigma) - \overline{\mathrm{Per}(\sigma)}$.

Proof. Lemma 4.4 (i) implies $\xi \in \mathrm{Rec}(\sigma)$ and Lemma 4.4 (ii) with Lemma 2.4 imply $\xi \notin \overline{\mathrm{Per}(\sigma)}$. □

Finally, we also see that the third inclusion in Theorem 4.1 is proper.

Proposition 4.6. *The set of periodic points* $\mathrm{Per}(\sigma)$ *is not closed, namely,* $\overline{\mathrm{Per}(\sigma)} - \mathrm{Per}(\sigma) \neq \emptyset$.

Proof. For each n, $\mathrm{Fix}(\sigma^n)$ is a proper subset of $\mathrm{Fix}(\sigma^{2n})$. Thus Proposition 3.2 yields the assertion. □

5. Tight groups

We generalize the observation in the previous section to the Banach space $L^\infty(G)$ of all L^∞-functions on a discrete group G in general. Here a function $x : G \to \mathbb{R}$ belongs to $L^\infty(G)$, by definition, if the supremum norm $\|x\|_\infty = \sup_{\gamma \in G} |x(\gamma)|$ is finite. For each element $g \in G$, its action on $L^\infty(G)$ is defined by $g \cdot x(\gamma) := x(\gamma \circ g)$ $(\gamma \in G)$. Then G acts on $L^\infty(G)$ as a group of isometric linear automorphisms. The Banach space $L^\infty(\mathbb{Z})$ considered in the previous section is nothing but the case where G is the infinite cyclic group $\mathbb{Z}$.

We can define the recurrent set $\mathrm{Rec}(G)$ and the set of periodic points $\mathrm{Per}(G)$ in the same way, and the inclusion relation $\mathrm{Rec}(G) \supset \overline{\mathrm{Per}(G)}$ is satisfied also in this case. For $L^\infty(\mathbb{Z})$, we have seen in Theorem 4.1 that $\mathrm{Rec}(\mathbb{Z})$ properly contains $\overline{\mathrm{Per}(\mathbb{Z})}$. Our standpoint in this section is to understand that this property comes from a nature of the infinite cyclic group. For a general G, the properness $\mathrm{Rec}(G) - \overline{\mathrm{Per}(G)} \neq \emptyset$ may fail to be satisfied. Hence, in the contrary, we define a class of infinite groups for which the properness is not satisfied.

Definition 5.1. An infinite group G is called *tight* if $\mathrm{Rec}(G) = \overline{\mathrm{Per}(G)}$.

We propose a problem to find non-trivial examples of tight groups and then characterize all such groups. The following theorem gives a necessary condition for G to be tight.

Theorem 5.2. *If* G *contains an element* γ *of infinite order, in other words, if* G *contains* $\mathbb{Z} \cong \langle \gamma \rangle$, *then* G *is not tight.*

Proof. Consider a coset decomposition of G by the subgroup $\langle \gamma \rangle$:

$$G = \langle \gamma \rangle \cup \langle \gamma \rangle g_1 \cup \langle \gamma \rangle g_2 \cup \cdots .$$

Let $(\xi_n)_{n \in \mathbb{Z}}$ be the sequence in Definition 4.3. Take a function $x \in L^\infty(G)$ so that $x(\gamma^n) = \xi_n$ on $\langle \gamma \rangle$ and $x(g) = 0$ for all $g \in G - \langle \gamma \rangle$. Then $x \in \mathrm{Rec}(G) - \overline{\mathrm{Per}(G)}$. Indeed, x belongs to $\mathrm{Rec}(\gamma) \subset \mathrm{Rec}(G)$ by Proposition 4.5. For every $g \in G - \langle \gamma \rangle$, we have $\|g \cdot x - x\|_\infty = 1$ because the coset $\langle \gamma \rangle$ is mapped to the different coset $\langle \gamma \rangle g$. For $g = \gamma^n \in \langle \gamma \rangle$, there exists an integer k such that $\|g^k \cdot x - x\|_\infty \geq 1/2$ as is seen in Lemma 4.4. Then, by Lemma 2.4, we conclude that $x \notin \overline{\mathrm{Per}(G)}$. □

By this theorem, there remains a problem on determining the tightness only for infinite groups all of whose elements have finite order. We remark that such a group G actually exists, even if we impose an extra condition that G is finitely generated. This is known as a counterexample for the Burnside problem.

Finally in this section, we just mention another familiar property which might be related to the tightness. We say that G is *amenable* if there exists an invariant mean $\mu : L^\infty(G) \to \mathbb{R}$. Here a bounded linear functional μ on $L^\infty(G)$ is called an invariant mean if the following conditions are satisfied:

(a) $\inf_{\gamma \in G} x(\gamma) \leq \mu(x) \leq \sup_{\gamma \in G} x(\gamma)$ for every $x \in L^\infty(G)$;
(b) $\mu(g \cdot x) = \mu(x)$ for every $g \in G$.

It is known that, if G contains a free group $\mathbb{Z} * \mathbb{Z}$ of rank 2 as a subgroup, then G is not amenable.

6. Application to Teichmüller spaces

In this section, we investigate the action of Teichmüller modular groups on Teichmüller spaces by applying our results obtained in previous sections. Concerning Teichmüller spaces, consult [13] and [17] for instance.

First we recall the following definitions. Fix a Riemann surface S. We say that two quasiconformal homeomorphisms f_1 and f_2 of S are *equivalent* if there exists a conformal homeomorphism $h : f_1(S) \to f_2(S)$ such that $f_2^{-1} \circ h \circ f_1 : S \to S$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The *Teichmüller space* $T(S)$ with the base Riemann surface S is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of S. A distance between two points $[f_1]$ and $[f_2]$ of $T(S)$ is defined by $d_T([f_1], [f_2]) = \log K(f)$, where $f : f_1(S) \to f_2(S)$ is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(S)$, which is called the Teichmüller distance.

We say that two quasiconformal automorphisms h_1 and h_2 of S are *equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The quasiconformal mapping class group $\mathrm{MCG}(S)$ of S is the set of all equivalence classes $[h]$ of quasiconformal automorphisms h of S. Every element $\chi = [h] \in \mathrm{MCG}(S)$ induces an automorphism χ_* of $T(S)$ by $[f] \mapsto [f \circ h^{-1}]$, which is an isometry with respect to d_T. Let $\mathrm{Isom}(T(S))$ be the group of all isometric automorphisms of $T(S)$. Then we have a homomorphism $\iota : \mathrm{MCG}(S) \to \mathrm{Isom}(T(S))$ by $\chi \mapsto \chi_*$, and we define the *Teichmüller modular group* by $\mathrm{Mod}(S) = \mathrm{Im}\, \iota$. With a few exceptional surfaces which we do not treat in this paper, it is known that ι is faithful. See [4], [6] and [14]. Hence we may identify $\mathrm{Mod}(S)$ with $\mathrm{MCG}(S)$.

We say that a Riemann surface is of analytically finite type if it is obtained as a compact surface from which at most finitely many points are removed, and otherwise analytically infinite. Let S be a Riemann surface of analytically infinite type. Then the Teichmüller space $T(S)$ is infinite dimensional, and the orbit $\mathrm{Orb}(G, p)$ of a point $p \in T(S)$ under the action of a subgroup $G \subset \mathrm{Mod}(S)$ is not necessarily discrete. This phenomenon is peculiar to infinite dimensional cases and hence the investigation concerning recurrent and periodic points for G makes sense. The sets $\mathrm{Rec}(G)$ and $\mathrm{Per}(G)$ are coincident with the limit set $\Lambda(G)$ and its subset $\Lambda_\infty(G)$ respectively which were studied in [7].

We will show that there exists a recurrent point which does not belong to the closure of the set of periodic points. For given two constants a, b $(-\infty \leq a < b \leq \infty)$, set

$$R_{a,b} = \{\zeta \in \mathbb{C} \mid a < \mathrm{Im}\, \zeta < b\} - \{n + 2i \mid n \in \mathbb{Z}\} - \{n + 5i \mid n \in \mathbb{Z}\}.$$

Hereafter, we mainly consider a Riemann surface $R_{0,\infty}$, which is denoted by R in brief. Let $\omega(\zeta) = \zeta + 1$ be a conformal automorphism of R. We denote the mapping class $[\omega] \in \mathrm{MCG}(R)$ also by ω. The stabilizer of the base point $o = [id] \in T(R)$ in $\mathrm{Mod}(R)$, which is identified with the conformal automorphism group $\mathrm{Aut}(R)$, is then $\langle \omega_* \rangle$.

We define a particular quasiconformal homeomorphism $f : R \to \mathbb{C}$ respecting to the sequence $(\xi_n)_{n \in \mathbb{Z}}$ defined in Definition 4.3. Remark that every quasiconformal homeomorphism of R into $\mathbb{C}$ extends quasiconformally to the punctures of R. Let the f satisfy $f(\zeta) = \zeta$ for $\zeta = u + iv \in R$

with $0 \le v \le 1$, $4 \le v$ or $u \in \mathbb{Z}+1/2$; and

$$f(n+2i) = n + (2+\xi_n)i$$

for every $n \in \mathbb{Z}$. In the rest of R, we interpolate piecewise linearly. More precisely,

$$f(u+iv) = \begin{cases} u + i\{v + \xi_n(2u-2n+1)(v-1)\} & (n-\frac{1}{2} \le u \le n,\ 1 \le v \le 2) \\ u + i\{v + \xi_n(2n-2u+1)(v-1)\} & (n \le u \le n+\frac{1}{2},\ 1 \le v \le 2) \\ u + i\{v + \xi_n(2u-2n+1)(4-v)/2\} & (n-\frac{1}{2} \le u \le n,\ 2 \le v \le 4) \\ u + i\{v + \xi_n(2n-2u+1)(4-v)/2\} & (n \le u \le n+\frac{1}{2},\ 2 \le v \le 4). \end{cases}$$

Setting $q := [f] \in T(R)$, we will prove that q is the required point.

Theorem 6.1. *The point $q \in T(R)$ satisfies the following properties:*

(i) $\lim_{\ell\to\infty} d_T(\omega_*^{3^\ell}(q), q) = 0$*;*

(ii) *There exists a constant $c > 0$ such that, for every $n \in \mathbb{Z} - \{0\}$, there is $k \in \mathbb{Z}$ satisfying $d_T(\omega_*^{nk}(q), q) \ge c$.*

Proof. First we will prove statement (i). For each integer j, the Teichmüller distance satisfies $d_T(\omega_*^j(q), q) \le \log K(f\circ\omega^j\circ f^{-1})$. Since the maximal dilatation is invariant under the composition of a conformal map, we consider quasiconformal homeomorphisms $g_j(\zeta) = f\circ\omega^j\circ f^{-1}(\zeta) - j$ of $f(R)$ instead, which satisfy $g_j(\zeta_n) = n + (2+\xi_{n+j})\,i$ for $\zeta_n := f(n+2i) = n + (2+\xi_n)\,i$. By the construction of f, we see that

$$K(g_j) \le 1 + 4\sup_{n\in\mathbb{Z}} |g_j(\zeta_n) - \zeta_n| = 1 + 4\sup_{n\in\mathbb{Z}} |\xi_{n+j} - \xi_n|.$$

Since $\sup_{n\in\mathbb{Z}} |\xi_{n+3^\ell} - \xi_n| \to 0$ by Lemma 4.4 (i), we conclude that $K(g_{3^\ell}) \to 1$ as $\ell\to\infty$.

Next we will prove statement (ii). Take an arbitrary quasiconformal homeomorphism g'_j of $f(R)$ that is homotopic to g_j keeping every point of $\mathbb{R}$ fixed. By the proof of Lemma 4.4 (ii), for every $n \in \mathbb{Z}-\{0\}$, there is $k \in \mathbb{Z}$ such that $|g'_{nk}(\zeta_0) - \zeta_0| = |\xi_{nk} - 1| \ge 1/2$, where $\zeta_0 = 2i$. Here, for every ζ with $|\zeta-\zeta_0| \le 1$, the Euclidean distance $|\zeta-\zeta_0|$ is comparable with the hyperbolic distance $d_{5i,1+5i}(\zeta,\zeta_0)$ on the three-punctured sphere $\mathbb{C} - \{5i, 1+5i\}$. In particular, there exists a constant $c > 0$ independent of n such that $d_{5i,1+5i}(g'_{nk}(\zeta_0), \zeta_0) \ge c$. Then, by the Teichmüller theorem (see [12]), we see that $\log K(g'_{nk}) \ge c$. This implies that $d_T(\omega_*^{nk}(q), q) \ge c$. □

By Theorem 6.1, we have the following consequence, which gives a contrast to the dynamics of Kleinian groups. Note that $\mathrm{Per}(\omega_*)$ contains not only the base point o but also a continuum in $T(R)$. See the proof of Proposition 6.6 below.

Corollary 6.2. *The set $\mathrm{Rec}(\omega_*) - \overline{\mathrm{Per}(\omega_*)}$ is not empty.*

Proof. Theorem 6.1 (i) implies $q \in \mathrm{Rec}(\omega_*)$ and Theorem 6.1 (ii) with Lemma 2.4 implies $q \notin \overline{\mathrm{Per}(\omega_*)}$. □

The Bers embedding provides the manifold structure for the Teichmüller space and we can regard it as a domain in a Banach space as follows. For an arbitrary hyperbolic Riemann surface S, take a Fuchsian group Γ acting on the lower half-plane $L = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ such that $L/\Gamma = S$. For $p = [f] \in T(S)$, we lift the quasiconformal homeomorphism f to L so that it extends to a quasiconformal automorphism F of the Riemann sphere $\hat{\mathbb{C}}$ mapping the upper half-plane U conformally. Then the Schwarzian derivative $\tilde\varphi(z) = \mathcal{S}_F(z)$ of the restriction of F to U is a holomorphic function satisfying an automorphic condition

$$(\gamma_*\tilde\varphi)(z) := \tilde\varphi(\gamma^{-1}(z))\cdot(\gamma^{-1})'(z)^2 = \tilde\varphi(z)$$

for every $\gamma \in \Gamma$ and having the hyperbolic L^∞-norm

$$\|\tilde\varphi\|_B := \sup \tilde\rho(z)^{-2}|\tilde\varphi(z)|$$

less than 3/2 for the hyperbolic density $\tilde\rho(z)$ of U.

Let $B(\Gamma)$ is the Banach space of all holomorphic functions $\tilde{\varphi}$ on U satisfying the automorphic condition for Γ and $\|\tilde{\varphi}\|_B < \infty$. Then the correspondence $\beta : T(S) \to B(\Gamma)$ by $[f] \mapsto \mathcal{S}_F$ gives a homeomorphism of $T(S)$ onto a bounded domain of $B(\Gamma)$, which is called the *Bers embedding.* For a conformal automorphism of $S = L/\Gamma$, its lift to L is the restriction of a Möbius transformation h, which also induces a conformal automorphism of U. Thus it acts on $B(\Gamma)$ as an isometry by

$$\tilde{\varphi}(z) \mapsto (h_*\tilde{\varphi})(z) = \tilde{\varphi}(h^{-1}(z)) \cdot (h^{-1})'(z)^2.$$

We assume that our Riemann surface R is represented by $R = L/\Gamma_R$ for a Fuchsian group Γ_R. Let $\tilde{\psi} = \beta(q)$ be the image of the point $q \in \mathrm{Rec}(\omega_*) - \overline{\mathrm{Per}(\omega_*)}$ under the Bers embedding and let $\tilde{\omega} \in \mathrm{Aut}(U)$ be a lift of ω. Then Theorem 6.1 and Corollary 6.2 are equivalent to the following assertion.

Corollary 6.3. *The holomorphic function $\tilde{\psi} \in B(\Gamma_R)$ belongs to $\mathrm{Rec}(\tilde{\omega_*}) - \overline{\mathrm{Per}(\tilde{\omega_*})}$ for the isometric linear automorphism $\tilde{\omega_*}$ of $B(\Gamma_R)$. In fact, $\tilde{\psi}$ satisfies the following properties:*

(i) $\lim_{\ell\to\infty} \|\tilde{\omega_*}^{3^\ell}\tilde{\psi} - \tilde{\psi}\|_B = 0$;

(ii) *There exists a constant $\delta > 0$ such that, for every integer $n \in \mathbb{Z} - \{0\}$, there is some integer $k \in \mathbb{Z}$ satisfying $\|\tilde{\omega_*}^{nk}\tilde{\psi} - \tilde{\psi}\|_B \geq \delta$.*

The Banach space $B(\Gamma)$ is a closed subspace of the Banach space $B(1)$ of all holomorphic functions $\tilde{\varphi}$ on U with $\|\tilde{\varphi}\|_B < \infty$. Every conformal automorphism $h \in \mathrm{Aut}(U)$ induces an isometric linear automorphism of $B(1)$ by $\tilde{\varphi}(z) \mapsto (h_*\tilde{\varphi})(z)$. Note that this is a sort of a weighted composition operator acting on the function space $B(1)$. See [16] for a review on this kind of operators. A conformal automorphism $h \in \mathrm{Aut}(U)$ comes from a conformal automorphism of $S = L/\Gamma$ if and only if h belongs to the normalizer $N(\Gamma)$ of Γ in $\mathrm{Aut}(U)$. Moreover, $h \in \mathrm{Aut}(U)$ belongs to $N(\Gamma)$ if and only if $h_* : B(1) \to B(1)$ preserves $B(\Gamma)$ invariant. (For the "if part" of the latter statement, Γ is assumed to be non-exceptional. See [14].)

We take an element h from a coset $\tilde{\omega}\Gamma_R$, which is the set of all lifts of the conformal automorphism ω of R. If we choose $h \in \tilde{\omega}\Gamma_R$ so that it corresponds to a loop around a cusp of $R/\langle\omega\rangle$, then h is parabolic, and otherwise h is hyperbolic. Moreover, by considering a Riemann surface R of different proportion from the beginning and taking an appropriate conjugate of the Fuchsian group Γ_R, we may assume the lift h to be an arbitrary hyperbolic or parabolic transformation in $\mathrm{Aut}(U)$.

Theorem 6.4. *For every Möbius transformation $h \in \mathrm{Aut}(U)$ of infinite order, the isometric linear automorphism h_* of $B(1)$ satisfies $\mathrm{Rec}(h_*) - \overline{\mathrm{Per}(h_*)} \neq \emptyset$.*

Proof. If h is hyperbolic or parabolic, then we can assume that h comes from a lift of the conformal automorphism ω of the Riemann surface R. Then properties (i) and (ii) in Corollary 6.3 are also valid if we regard h_* as acting on $B(1)$. Thus, relying on Lemma 2.4 again, we conclude that the element $\tilde{\psi} \in B(1)$ belongs to $\mathrm{Rec}(h_*) - \overline{\mathrm{Per}(h_*)}$. In the case where h is elliptic of infinite order, every element of $\mathrm{Per}(h_*)$ is a constant function by the theorem of identity. Hence so is every element of $\overline{\mathrm{Per}(h_*)}$. On the other hand, $\mathrm{Rec}(h_*)$ contains other elements from constants, say $\tilde{\varphi}(z) = z$. □

Here we translate our results to the projective space $\hat{B}(\Gamma) = (B(\Gamma) - \{0\})/\mathbb{C}^\times$ via the projection $\pi : B(\Gamma) - \{0\} \to \hat{B}(\Gamma)$, which were considered in Section 3.

Proposition 6.5. *For the isometric linear automorphism $\hat{\omega_*}$ of $\hat{B}(\Gamma_R)$ induced by $\tilde{\omega_*}$, the orbit $\mathrm{Orb}(\hat{\omega}_*, \hat{\varphi})$ in the projective space $\hat{B}(\Gamma_R)$ is not closed for every point $\hat{\varphi} \in \mathrm{Rec}(\hat{\omega}_*) - \mathrm{Per}(\hat{\omega}_*)$, and $\mathrm{Orb}(\hat{\omega}_*, \hat{\varphi})$ is not dense in $\mathrm{Rec}(\hat{\omega}_*)$ for every $\hat{\varphi} \in \mathrm{Rec}(\hat{\omega}_*)$.*

Proof. The first statement follows from Proposition 2.5, and the second one follows from Proposition 2.7 and Corollary 6.3. □

Actually, the second statement in Proposition 6.5 is valid even in the following general situation.

Proposition 6.6. *If the Banach space $B(\Gamma)$ for an arbitrary Fuchsian group Γ admits an isometric linear automorphism h_* of infinite order induced by $h \in \mathrm{Aut}(U)$, then $\mathrm{Orb}(\hat{h}_*, \hat{\varphi})$ is not dense in $\mathrm{Rec}(\hat{h}_*) \subset \hat{B}(\Gamma)$ for every $\hat{\varphi} \in \mathrm{Rec}(\hat{h}_*)$.*

Proof. Recall that $\mathrm{Per}(h_*) = \bigcup_{n\in\mathbb{N}} \mathrm{Fix}(h_*^n)$ where $\mathrm{Fix}(h_*^n) = B(\langle \Gamma, h^n\rangle)$. Since $\dim B(\langle \Gamma, h^n\rangle) \geq 2$ for a sufficiently large n, we have $\dim \mathrm{Per}(h_*) \geq 2$. Hence the assertion follows from Proposition 3.3. □

Every element $\tilde{\varphi}$ in $B(\Gamma)$ projects to a holomorphic quadratic differential $\varphi = \varphi(\zeta)d\zeta^2$ on the Riemann surface $S' = U/\Gamma$ such that the hyperbolic L^∞-norm $\sup_{\zeta\in S'} \rho(\zeta)^{-2}|\varphi(\zeta)|$ is finite for the hyperbolic density $\rho(\zeta)$ on S'. This is called a *bounded* holomorphic quadratic differential on S'. The Banach space of all bounded holomorphic quadratic differentials φ on S' is denoted by $B(S')$ and the norm is denoted by $\|\varphi\|_{B(S')}$. In this way, $B(\Gamma)$ is identified with $B(S')$. Moreover, since S' is the complex conjugate of the Riemann surface $S = L/\Gamma$, the Banach spaces $B(S')$ and $B(S)$ are isomorphic by the anti-holomorphic involution.

We investigate extendability of the bounded holomorphic quadratic differential $\psi \in B(R')$ that is identified with $\tilde{\psi} = \beta(q) \in B(\Gamma_R)$. Here $R' = U/\Gamma_R$ is the complex conjugate of the Riemann surface $R = L/\Gamma_R$. The double DR' of the Riemann surface R' with respect to the boundary curve $\partial R'$ is obtained by gluing R' and R along their corresponding boundaries. This is coincident with the quotient Riemann surface $\Omega(\Gamma_R)/\Gamma_R$, where $\Omega(\Gamma_R)$ is the region of discontinuity for the Fuchsian group Γ_R. For the quasiconformal homeomorphism f of R representing the point $q \in T(R)$, let F be a quasiconformal automorphism of $\hat{\mathbb{C}}$ that is a lift of f on L and conformal on U. Since f is conformal in $R_{0,1}$, the conformal homeomorphism $F|_U$ extends conformally beyond $\mathbb{R} \cap \Omega(\Gamma_R)$ to a domain $\hat{U}$. Hence the Schwarzian derivative $\tilde{\psi} = \mathcal{S}_F$ on U extends holomorphically to $\hat{U}$ and so does the holomorphic quadratic differential ψ on R'. More precisely, ψ extends holomorphically to $\hat{R}' = \hat{U}/\Gamma_R \subset DR'$. By analytic continuation, we can realize $\hat{R}'$ in $\mathbb{C}$ as the union of $R' = (R_{0,\infty})'$ and $R_{0,1}$, namely $\hat{R}' = (R_{-1,\infty})'$.

We consider the Riemann surface $\hat{R}'$ and $\check{R} = R_{1,\infty} = \check{L}/\Gamma_R$, where $\check{L}$ is the complement of the closure of $\hat{U}$ in $\hat{\mathbb{C}}$. Since $\hat{R}'$ is quasiconformally equivalent to R', we see that $\hat{U}$ and $\check{L}$ are quasidisks. Similarly to the Bers embedding $\beta : T(R) \to B(\Gamma_R) = B(R')$, a generalized Bers embedding of the Teichmüller space $T(\check{R})$ is defined by using the bounded holomorphic automorphic functions on $\hat{U}$ for Γ_R, or equivalently, the bounded holomorphic quadratic differentials on $\hat{R}' = \hat{U}/\Gamma_R$, where the norm is given by using the hyperbolic density $\hat{\rho}(\zeta)$ on $\hat{R}'$. Then $T(\check{R})$ is embedded as a bounded domain in $B(\hat{R}')$. See Section 3.4 of [17] for example. The holomorphic quadratic differential ψ extended to $\hat{R}'$ belongs to $B(\hat{R}')$. Moreover, if we investigate the action of ω_* on $T(\check{R})$, we have a similar consequence to Corollary 6.3 for the ψ as an element of $B(\hat{R}')$.

Replacing the role of R with that of the complex conjugate R' in the above consideration, we have obtained the following result, which will be used in the next section.

Lemma 6.7. *Let $R = R_{0,\infty}$ be embedded in the complex plane $\mathbb{C}$ and $\omega(\zeta) = \zeta + 1$ be acting as a conformal automorphism of R. The bounded holomorphic quadratic differential ψ extends to a bounded holomorphic quadratic differential on $\hat{R} = R_{-1,\infty}$ and it satisfies $\lim_{\ell\to\infty} \|\omega_*^{3^\ell}\psi - \psi\|_{B(\hat{R})} = 0$.*

7. Application to the Hardy space H^∞

In this section, we consider the dynamics of a composition operator on the Hardy space of all bounded holomorphic functions. First we investigate it on $H^\infty(R)$, the Banach space of all bounded holomorphic functions on the Riemann surface $R = R_{0,\infty}$ with the supremum norm $\|\cdot\|_{H^\infty(R)}$. We will transfer our results on $B(R)$ into $H^\infty(R)$ by comparing the norms of $B(R)$ and $H^\infty(R)$.

Since R is embedded in the complex plane $\mathbb{C}$, we use the coordinate ζ as a global coordinate on R hereafter.

Let $\hat{\rho}(\zeta)$ be the hyperbolic density on $\hat{R} = R_{-1,\infty}$. By Lemma 6.7, the holomorphic quadratic differential ψ extends to a bounded holomorphic quadratic differential on $\hat{R}$. From Corollary 6.3 and Lemma 6.7, we see that ψ satisfies the following properties for the conformal automorphism $\omega \in \mathrm{Aut}(R)$ and for the constant $\delta > 0$:

(1) $\lim_{\ell\to\infty} \|\omega_*^{3^\ell}\psi - \psi\|_{B(\hat{R})} = 0$;
(2) For every integer $n \in \mathbb{Z} - \{0\}$, there exists some integer $k \in \mathbb{Z}$ satisfying $\|\omega_*^{nk}\psi - \psi\|_{B(R)} \geq \delta$.

Furthermore, we can replace this property(2) with a stronger one by using the following fact.

Lemma 7.1. *For $\delta > 0$, there exists a constant $b > 0$ such that $\rho(\zeta)^{-2}|\psi(\zeta)| < \delta/3$ for all $\zeta \in R_{b,\infty}$.*

Fix this constant b hereafter. By this lemma, we have

$$\sup_{\zeta\in R_{b,\infty}} \rho(\zeta)^{-2}|\omega_*^{nk}\psi(\zeta) - \psi(\zeta)| \leq 2\delta/3 < \delta.$$

Then property (2) above turns out to be the following:

(2′) For every integer $n \in \mathbb{Z}-\{0\}$, there exists some integer $k \in \mathbb{Z}$ satisfying $\sup_{\zeta\in R_{0,b}} \rho(\zeta)^{-2}|\omega_*^{nk}\psi(\zeta) - \psi(\zeta)| \geq \delta$.

Our proof of Lemma 7.1 is based on an argument in the proof of Theorem 4 in [5]. However, we may also apply Lemma 3.1 in [18].

Proof of Lemma 7.1. First, we choose a universal covering projection of L onto R as follows. Let $\pi_\omega : R \to R/\langle\omega\rangle$ be the projection of R onto the quotient Riemann surface $R/\langle\omega\rangle$, which has three punctures j_1, j_2 and j_∞ and one border. Here j_∞ is corresponding to ∞ of R. Let $\hat{\pi} : L \to R/\langle\omega\rangle$ be the universal cover such that j_∞ corresponds to ∞ of L. More precisely, a horoball $L_\infty \subset L$ at ∞ can be taken as the connected component of the preimage of a cusp neighborhood of j_∞. Then we define the projection $\pi : L \to R$ so that $\pi_\omega \circ \pi = \hat{\pi}$. This is determined up to the composition of ω. This projection π has a property that $\mathrm{Im}\,\overline{z} \to -\infty$ in L if and only if $\mathrm{Im}\,\pi(\overline{z}) \to \infty$ in R.

Suppose to the contrary that we cannot take such a constant b as in the statement of this lemma. Then there exists a sequence $\{\zeta_n\} \subset R$ such that $\mathrm{Im}\,\zeta_n \to \infty$ and $\rho(\zeta_n)^{-2}|\psi(\zeta_n)| \geq \delta/3$. Take $\overline{z_n} \in L_\infty$ so that $\pi(\overline{z_n}) = \zeta_n$. Then $\tilde{\rho}(z_n)^{-2}|\tilde{\psi}(z_n)| \geq \delta/3$.

For each n, we take $\gamma_n \in \mathrm{Aut}(U)$ such that $\gamma_n(z_n) = i\ (= \sqrt{-1})$ and $\gamma_n(\infty) = \infty$. Let F be the quasiconformal automorphism of the Riemann sphere $\hat{\mathbb{C}}$ that is conformal on U and corresponding to f on L. Let $\mu(z)$ be the Beltrami differential of F. From the construction of f and F, we see that $\mu(\overline{z}) = 0$ for $\overline{z} \in \pi^{-1}(R_{6,\infty})$. Since the Möbius transformation γ_n extends to $\hat{\mathbb{C}}$, we define Beltrami coefficients

$$\mu_n = (\gamma_n)_*\,\mu := \mu \circ \gamma_n^{-1} \cdot \frac{\overline{(\gamma_n^{-1})'}}{\gamma_n^{-1}}$$

on $\hat{\mathbb{C}}$. Let F_n be a quasiconformal automorphism of $\hat{\mathbb{C}}$ fixing 0, 1 and ∞ whose Beltrami differential is μ_n.

For an arbitrarily given $r > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $\pi(D(\overline{z_n}, r)) \subset R_{6,\infty}$ for all $n \geq N$. Here $D(\overline{z_n}, r)$ is a hyperbolic disk centered at $\overline{z_n}$ with radius r. Since $\gamma_n^{-1}(D(-i, r)) \subset D(\overline{z_n}, r)$, we have $|\mu_n(\overline{z})| = |\mu(\gamma_n^{-1}(\overline{z}))| \to 0$ as $n \to \infty$ for every $\overline{z} \in D(-i, r)$. This implies that μ_n converge to 0. Then by Theorem I.4.5 in [13], F_n converge to the identity uniformly on $\hat{\mathbb{C}}$ and thus the Schwarzian derivatives $\tilde{\psi}_n = \mathcal{S}_{F_n}$ of F_n converge to 0. In particular, $\lim_{n\to\infty} |\tilde{\psi}_n(i)| = 0$. Since $\tilde{\psi}_n = (\gamma_n)_*\,\tilde{\psi}$ and $|(\gamma_n^{-1})'(i)| = |z_n - \overline{z_n}|/2 = \tilde{\rho}(z_n)^{-1}$, this is equivalent to saying that $\lim_{n\to\infty} \tilde{\rho}(z_n)^{-2}|\tilde{\psi}(z_n)| = 0$. However this contradicts the condition in the second paragraph. □

We fix the following holomorphic quadratic differential on the Riemann surface R:

$$\alpha(\zeta)d\zeta^2 = \frac{1}{(e^{2\pi i\zeta} - e^{-4\pi})^2(e^{2\pi i\zeta} - e^{-10\pi})^2}\, d\zeta^2.$$

This extends to the punctures of R to be a meromorphic quadratic differential having poles of second order. Remark that $\alpha(\zeta)$ is uniformly bounded away from zero on R as well as invariant under the conformal automorphism $\omega(\zeta) = \zeta + 1$.

Let $Q(R)$ be the linear space of all holomorphic quadratic differentials on R having at most a simple pole at each puncture of R. In particular $B(R) \subset Q(R)$. Using the quadratic differential $\alpha(\zeta)d\zeta^2$, we define a linear map from $Q(R)$ to $H(R)$, the linear space of all holomorphic functions on R, by

$$\eta : Q(R) \to H(R), \quad \varphi = \varphi(\zeta)d\zeta^2 \mapsto \frac{\varphi(\zeta)}{\alpha(\zeta)}.$$

Then we have the following two estimates, which prove the boundedness of the linear operator η and its inverse restricted to certain subspaces with appropriate norms.

Proposition 7.2. *There exists a positive constant M_1 such that*

$$\|\eta(\varphi)\|_{H^\infty(R)} \le M_1 \|\varphi\|_{B(\hat{R})}$$

for every $\varphi \in B(\hat{R})$.

Proof. By the equality

$$|\eta(\varphi)(\zeta)| = \left|\frac{\hat{\rho}(\zeta)^2}{\alpha(\zeta)}\right| \cdot |\hat{\rho}(\zeta)^{-2}\varphi(\zeta)|,$$

we have only to estimate $|\hat{\rho}(\zeta)^2/\alpha(\zeta)|$ for $\zeta \in R$. Note that this function is invariant under ω. Since $\hat{\rho}(\zeta)$ is uniformly bounded on R except the puncture neighborhoods and $\alpha(\zeta)$ is uniformly bounded away from zero, an estimate near a puncture gives the assertion. We assume that the puncture is at $\zeta = 0$ by a parallel translation of R. Then $\hat{\rho}(\zeta) \asymp 1/(-|\zeta|\log|\zeta|)$ and $|\alpha(\zeta)| \asymp |\zeta|^{-2}$ asymptotically near 0. Therefore $|\hat{\rho}(\zeta)^2/\alpha(\zeta)| \asymp (\log|\zeta|)^{-2} \to 0$ as $\zeta \to 0$. □

Proposition 7.3. *There exists a positive constant M_2 such that*

$$\sup_{z \in R_{0,b}} \rho(z)^{-2}|\varphi(\zeta)| \le M_2 \|\eta(\varphi)\|_{H^\infty(R)}$$

for every $\varphi \in B(R)$.

Proof. We take an arbitrary puncture of R and assume it to be $\zeta = 0$. Then $\zeta\varphi(\zeta)$ is holomorphic near 0 for every $\varphi \in B(R)$. Here we consider a function $r(\log r)^2$ for $r \ge 0$ and choose a constant $r_0 > 0$ so that it increases in the interval $[0, r_0]$. Since the hyperbolic density $\rho(\zeta)$ is asymptotically $1/(-|\zeta|\log|\zeta|)$ near $\zeta = 0$, the function

$$\rho(\zeta)^{-2}|\varphi(\zeta)| \asymp |\zeta|(\log|\zeta|)^2|\zeta\varphi(\zeta)|$$

then holds the maximum principle in $|\zeta| \le r_1$ by replacing r_0 with a smaller constant $r_1 > 0$ if necessary. Moreover, the invariance of $\rho(\zeta)$ under ω guarantees the existence of a uniform constant $r_2 > 0$ such that, at every puncture of R and for every $\varphi \in B(R)$, the function $\rho(\zeta)^{-2}|\varphi(\zeta)|$ holds the maximum principle in the disk D of radius r_2.

Let E be the union of all such disks D around the punctures. Due to the choice of the constant r_2 as above, we have

$$\sup_{\zeta \in R_{0,b}} \rho(\zeta)^{-2}|\varphi(\zeta)| = \sup_{\zeta \in R_{0,b}-E} \rho(\zeta)^{-2}|\varphi(\zeta)| = \sup_{\zeta \in R_{0,b}-E} \left|\frac{\alpha(\zeta)}{\rho(\zeta)^2}\right| \cdot |\eta(\varphi)(\zeta)|.$$

Since $\alpha(\zeta)$ is uniformly bounded on $R_{0,b} - E$, there exists a constant M_2 such that $|\alpha(\zeta)/\rho(\zeta)^2| \le M_2$ for $\zeta \in R_{0,b} - E$. Hence the right-hand side is bounded by $M_2\|\eta(\varphi)\|_{H^\infty(R)}$. □

We apply these estimates to $\eta(\psi)$ and $\eta(\omega_*^n\psi)$, which belong to $H^\infty(R)$ by Proposition 7.2.

Lemma 7.4. *The bounded holomorphic function $\eta(\psi) \in H^\infty(R)$ satisfies the following properties:*

(i) $\lim_{\ell\to\infty} \|\eta(\psi)\circ\omega^{-3^\ell} - \eta(\psi)\|_{H^\infty(R)} = 0$;

(ii) *There exists a constant $\delta' > 0$ such that, for every integer $n \in \mathbb{Z} - \{0\}$, there is some integer $k \in \mathbb{Z}$ satisfying $\|\eta(\psi)\circ\omega^{-kn} - \eta(\psi)\|_{H^\infty(R)} \geq \delta'$.*

Proof. The invariance under ω gives $\eta(\omega_*\psi)(\zeta) = \eta(\psi)\circ\omega^{-1}(\zeta)$. Hence

$$\|\eta(\psi)\circ\omega^{-n} - \eta(\psi)\|_{H^\infty(R)} \leq M_1\|\omega_*^n\psi - \psi\|_{B(\hat{R})}$$

by Proposition 7.2 and

$$\sup_{\zeta\in R_{0,b}} \rho(\zeta)^{-2}|\omega_*^{nk}\psi(\zeta) - \psi(\zeta)| \leq M_2\|\eta(\psi)\circ\omega^{-n} - \eta(\psi)\|_{H^\infty(R)}$$

by Proposition 7.3. Then statements (i) and (ii) follow from properties (1) and (2′) above. □

For the conformal automorphism ω of the Riemann surface R, consider the composition operator $C_\omega : H^\infty(R) \to H^\infty(R)$ defined by $C_\omega(\phi) = \phi\circ\omega$ for every bounded holomorphic function ϕ on R. This is an isometric linear automorphism of the Banach space $H^\infty(R)$. Then Lemma 7.4 concludes the following.

Theorem 7.5. *The closure $\overline{\mathrm{Per}(C_\omega)}$ is a proper subset of the recurrent set $\mathrm{Rec}(C_\omega)$ for the isometric linear automorphism $C_\omega : H^\infty(R) \to H^\infty(R)$.*

Proof. Property (i) in Lemma 7.4 asserts that $\eta(\psi) \in H^\infty(R)$ belongs to $\mathrm{Rec}(C_\omega)$. On the other hand, property (ii) combined with Lemma 2.4 asserts that $\eta(\psi)$ does not belong to $\overline{\mathrm{Per}(C_\omega)}$. □

Now we lift the holomorphic functions on the Riemann surface R to the unit disk Δ by a universal cover $\pi : \Delta \to R$. The conformal automorphism ω of R lifts to a conformal automorphism h of Δ, which is either a hyperbolic or a parabolic Möbius transformation. Let H^∞ be the Banach space of all holomorphic functions f on Δ with the supremum norm $\|f\|_\infty = \sup_{z\in\Delta}|f(z)|$ finite. Any bounded holomorphic function $\phi \in H^\infty(R)$ lifts to $f \in H^\infty$ satisfying $\|f\|_\infty = \|\phi\|_{H^\infty(R)}$.

For every Möbius transformation $h \in \mathrm{Aut}(\Delta)$, the composition operator $C_h : H^\infty \to H^\infty$ is defined by $C_h(f) = f\circ h$. This is an isometric linear automorphism of H^∞. For an arbitrary hyperbolic or parabolic Möbius transformation h, we can easily modify the Riemann surface R and the conformal automorphism ω so that h is a lift of ω under a universal cover $\pi : \Delta \to R$. Hence we have the following.

Theorem 7.6. *For every Möbius transformation $h \in \mathrm{Aut}(\Delta)$ of infinite order, the composition operator C_h of H^∞ satisfies $\mathrm{Rec}(C_h) - \overline{\mathrm{Per}(C_h)} \neq \emptyset$.*

Proof. For a hyperbolic or parabolic Möbius transformation h, the statement follows from Theorem 7.5. For an elliptic Möbius transformation h of infinite order, $\mathrm{Per}(C_h)$ coincides with the set of all constant functions by theorem of identity. Hence so is $\overline{\mathrm{Per}(C_h)}$. However, there exists a non-constant function, say $f(z) = z$, that belongs to $\mathrm{Rec}(C_h)$. □

There is a separable Banach subspace A in H^∞ consisting of all bounded holomorphic functions on Δ continuously extendable to the boundary $\partial\Delta$. This is called the *disk algebra*. The Möbius composition operator C_h for $h \in \mathrm{Aut}(\Delta)$ preserves A invariant and we can consider a similar problem concerning the dynamics of C_h acting on A. However, the corresponding results are easily obtained in this case.

Proposition 7.7. *Let $C_h : A \to A$ be the composition operator for $h \in \mathrm{Aut}(\Delta)$ acting on the disk algebra A. If h is hyperbolic or parabolic, then $\mathrm{Rec}(C_h)$, $\overline{\mathrm{Per}(C_h)}$ and $\mathrm{Per}(C_h)$ are all coincident with the set of constant functions. If h is elliptic of infinite order, then $\mathrm{Rec}(C_h)$ and $\overline{\mathrm{Per}(C_h)} = \mathrm{Per}(C_h)$ are different, where the latter is coincident with the set of constant functions.*

Proof. We only show that $\mathrm{Rec}(C_h)$ is coincident with the set of constant functions for hyperbolic or parabolic h, for the other claims are easier. Suppose that $f \in A$ belongs to $\mathrm{Rec}(C_h)$. Then there is an increasing (or a decreasing) sequence $\{n_k\} \subset \mathbb{Z}$ such that $(C_h)^{n_k}(f)$ converge to f in A as $k \to \infty$. Let $a \in \partial\Delta$ be the attracting (or the repelling) fixed point of h. Then $(C_h)^{n_k} f(z) = f(h^{n_k}(z))$ converge to $f(a)$ for every $z \in \Delta$. Hence $f(z) = f(a)$, which means that f is a constant function. □

8. Appendix: Möbius composition operators on the Hardy spaces

The Hardy space H^p for $1 \le p < \infty$ is a separable Banach space consisting of all holomorphic functions f on the unit disk Δ such that

$$\|f\|_{H^p}^p = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

A Möbius composition operator $C_h : H^p \to H^p$ for $h \in \mathrm{Aut}(\Delta)$ is defined by $f \mapsto C_h(f) = f \circ h$. It is known that C_h is a bounded linear operator but not an isometry for every finite p. In this case, the Birkhoff center is defined to be the closed recurrent set $\mathrm{Rec}(C_h)$ as is mentioned in Section 2.

A composition operator C_h is called *hypercyclic* if $\overline{\mathrm{Orb}(C_h, f)} = H^p$ for some $f \in H^p$. By the Birkhoff transitivity theorem (see [19]), hypercyclicity is equivalent to *topological transitivity.* Here the operator C_h is considered to be topological transitivity if for any open subsets U and V of H^p, there exists an integer $n \in \mathbb{Z}$ such that $(C_h)^n(U) \cap V \neq \emptyset$. In more details, if C_h is topologically transitive, then there exists a dense subset of H^p such that $\overline{\mathrm{Orb}(C_h, f)} = H^p$ for every element f in it. Therefore the hypercyclicity of C_h in particular implies $\mathrm{Rec}(C_h) = H^p$. Bourdon and Shapiro [2] proved that a Möbius composition operator C_h for $h \in \mathrm{Aut}(\Delta)$ is hypercyclic if and only if h is either hyperbolic or parabolic. See also [20].

Hosokawa [11] and Taniguchi [21] studied another problem, which is the density of periodic points for the composition operator C_h. They proved that $\mathrm{Per}(C_h)$ is dense in H^p for each $1 \le p < \infty$ if and only if $h \in \mathrm{Aut}(\Delta)$ is either hyperbolic or parabolic. Therefore this result combined with the above mentioned hypercyclicity answers to our problem asking whether $\mathrm{Rec}(C_h) = \overline{\mathrm{Per}(C_h)}$ also in the case of the Hardy spaces H^p $(1 \le p < \infty)$. Summing up all results including the case of $p = \infty$, which is proved in the previous section, we record the following.

Corollary 8.1. (1) *For a hyperbolic or parabolic Möbius transformation* $h \in \mathrm{Aut}(\Delta)$*, the composition operator* C_h *on* H^p $(1 \le p \le \infty)$ *satisfies* $\mathrm{Rec}(C_h) = \overline{\mathrm{Per}(C_h)}$ *if and only if* $p \neq \infty$. (2) *For an elliptic Möbius transformation* h *of infinite order,* C_h *satisfies* $\mathrm{Rec}(C_h) - \overline{\mathrm{Per}(C_h)} \neq \emptyset$ *for every* $1 \le p \le \infty$.

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